Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 170762, 16 pages
doi:10.1155/2010/170762

Research Article

On Subnormal Solutions of Periodic Differential Equations

Zong-Xuan Chen[1] and Kwang Ho Shon[2]

[1] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China*
[2] *Department of Mathematics, College of Natural Sciences, Pusan National University, Pusan 609-735, Republic of Korea*

Correspondence should be addressed to Kwang Ho Shon, khshon@pusan.ac.kr

Received 14 July 2010; Revised 22 October 2010; Accepted 12 November 2010

Academic Editor: Irena Lasiecka

We investigate the existence and the form of subnormal solutions of higher-order linear periodic differential equations, and precisely estimate the growth of all solutions.

1. Introduction and Results

In this paper we use standard notations from the value distribution theory (see [1–3]). In addition, we denote the order of growth of $f(z)$ by $\sigma(f)$ and also use the notation $\sigma_2(f)$ to denote the hyperorder of $f(z)$, which is defined as

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log \log T(r,f)}{\log r}. \tag{1.1}$$

Consider the second-order homogeneous linear periodic differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1.2}$$

where $P(z)$ and $Q(z)$ are polynomials in z, but both are not constants. It is well known that every solution f of (1.2) is an entire function.

Suppose $f(\not\equiv 0)$ is a solution of (1.2), and if f satisfies the condition

$$\overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{r} = 0, \tag{1.3}$$

then we say that f is a nontrivial subnormal solution of (1.2). For convenience, we also say that $f \equiv 0$ is a subnormal solution of (1.2) (see [4, 5]).

It clearly follows that

(i) if

$$0 < \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{r} < \infty, \tag{1.4}$$

then $\sigma_2(f) = 1$;

(ii) if $\sigma_2(f) < 1$, then (1.3) holds.

Wittich [5] investigated the subnormal solution of (1.2), which gives the form of all subnormal solutions of (1.2) in the following theorem.

Theorem A (see [5]). *If $f \not\equiv 0$ is a subnormal solution of* (1.2)*, then f must have the form*

$$f(z) = e^{cz}\left(h_0 + h_1 e^z + \cdots + h_m e^{mz}\right), \tag{1.5}$$

where $m \geq 0$ is an integer and $c, h_0, \ldots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

Gundersen and Steinbart [4] refined Theorem A and got the following theorem.

Theorem B (see [4]). *Under the assumption of Theorem A, the following statements hold.*

(i) *If $\deg P > degQ$ and $Q \not\equiv 0$, then any subnormal solution $f(\not\equiv 0)$ of* (1.2) *must have the form*

$$f(z) = \sum_{k=0}^{m} h_k e^{-kz}, \tag{1.6}$$

where $m \geq 1$ is an integer and $h_0, h_1, \ldots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

(ii) *If $Q \equiv 0$ and $\deg P \geq 1$, then any subnormal solution of* (1.2) *must be a constant.*

(iii) *If $\deg P < degQ$, then the only subnormal solution of* (1.2) *is $f \equiv 0$.*

In [6], Chen and Shon proved that supposing that

$$P(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z, \qquad Q(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z, \tag{1.7}$$

where $a_n(z), \ldots, a_1(z), b_s(z), \ldots, b_1(z)$ are polynomials satisfying $a_n(z)b_s(z) \not\equiv 0$, if $n \neq s$, then every solution $f(\not\equiv 0)$ of (1.2) satisfies $\sigma_2(f) = 1$.

In [6], the condition "all constant terms of $P(e^z)$ and $Q(e^z)$ are equal to zero" plays an important role in the growth of solutions of (1.2). This makes us consider that the condition may be applied to higher-order differential equations.

Gundersen and Steinbart [4] consider a subnormal solution of higher-order linear nonhomogeneous differential equation

$$f^{(k)} + P_{k-1}(e^z)f^{(k-1)} + \cdots + P_0(e^z)f = Q_1(e^z) + Q_2(e^{-z}), \tag{1.8}$$

where $Q_d(z)$ $(d = 1, 2)$, $P_j(z)$ $(j = 0, \ldots, k-1)$ are polynomials in z and obtain the following theorem.

Theorem C. *Suppose that, in* (1.8), *one has* $k \geq 2$ *and*

$$\deg P_0 > degP_j \tag{1.9}$$

for all $1 \leq j \leq k-1$. *Then any subnormal solution* f *of* (1.8) *must have the form*

$$f(z) = S_1(e^z) + S_2(e^{-z}), \tag{1.10}$$

where $S_1(z)$ *and* $S_2(z)$ *are polynomials in* z.

From the proof of Theorem C, we see that the condition (1.9) of Theorem C guarantees that the corresponding homogeneous differential equation of (1.8)

$$f^{(k)} + P_{k-1}(e^z)f^{(k-1)} + \cdots + P_0(e^z)f = 0 \tag{1.11}$$

has no nontrivial subnormal solution.

Thus, a natural question is whether or not (1.11) has a nontrivial subnormal solution if the condition (1.9) is replaced by the condition "there exists some s satisfying $\deg P_s > degP_j$ $(j \neq s)$".

Examples 1.1 and 1.2 show that if $\deg P_s > degP_j$ $(j \neq s)$, (1.11) may have a nontrivial subnormal solution.

Example 1.1. The equation

$$f''' - \left(\frac{1}{4}e^{3z} + 2e^{2z}\right)f'' + \frac{1}{2}e^z f' + (e^z + 8)f = 0 \tag{1.12}$$

has a subnormal solution $f = e^{-2z} + 1$.

Example 1.2. The equation

$$f''' + \left(e^{3z} + 4\right)f'' + \left(e^{4z} + e^z\right)f' + (3e^z - 9)f = 0 \tag{1.13}$$

has a subnormal solution $f = e^{-3z} + 1$.

Thus, a natural question is what conditions will guarantee that (1.11) has no nontrivial subnormal solution under the condition $\deg P_s > degP_j$ $(j \neq s)$.

In Theorem 1.3, we answer this question. We conclude that if all constant terms of P_j are equal to zero under the conditions $\deg P_s > degP_j$ $(j \neq s)$ and $P_0 \not\equiv 0$, then (1.11) has no nontrivial subnormal solution, and we also prove that all solutions of (1.11) satisfy $\sigma_2(f) = 1$.

Examples 1.1 and 1.2 show that the condition "all constant terms of P_j are equal to zero" cannot be deleted in Theorem 1.3.

In this paper, we firstly investigate the existence of subnormal solutions. It is an important problem in theory of periodic differential equations.

Theorem 1.4 generalizes the result of Theorem C, shows that (1.8) has at most one nontrivial subnormal solution, gives the form of subnormal solution of (1.8), and proves that all other solutions f of (1.8) satisfy $\sigma_2(f) = 1$.

Theorem 1.6 refines Theorem C.

Our method for obtaining the proof is totally different from the method applied in [4, 5].

Theorem 1.3. *Let* $P_j(z)$ $(j = 0, \ldots, k-1)$ *be polynomials in* z *such that all constant terms of* P_j *are equal to zero and* $\deg P_j = m_j$*, that is,*

$$P_j(e^z) = a_{jm_j} e^{m_j z} + a_{j(m_j-1)} e^{(m_j-1)z} + \cdots + a_{j1} e^z, \tag{1.14}$$

where $a_{jm_j}, a_{j(m_j-1)}, \ldots, a_{j1}$ *are constants and* $a_{jm_j} \not\equiv 0$; $m_j \geq 1$ *are integers. Suppose that there exists* m_s $(s \in \{0, \ldots, k-1\})$ *satisfying*

$$m_s > max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\} = m. \tag{1.15}$$

Then, one has the following properties.

(i) *If* $P_0 \not\equiv 0$*, then* (1.11) *has no nontrivial subnormal solution and every solution of* (1.11) *is of hyper order* $\sigma_2(f) = 1$.

(ii) *If* $P_0 \equiv \cdots \equiv P_{d-1} \equiv 0$ *and* $P_d \not\equiv 0$ $(d < s)$*, then any polynomials with degree* $\leq d-1$ *are subnormal solutions of* (1.11) *and all other solutions* f *of* (1.11) *satisfy* $\sigma_2(f) = 1$.

Considering proof of theorems, if the set $e^z = \zeta$, then (1.8) (or (1.11)) becomes an equation with rational coefficients, but the equation with rational coefficients may have nonmeromorphic solution. For example, the equation

$$zf'' + zf' - 2f = 0 \tag{1.16}$$

has a solution $f = \exp\{1/z\}$. This shows that we cannot use the transformation $e^z = \zeta$ to prove that every solution of (1.11) is of $\sigma_2(f) = 1$.

Theorem 1.4. *Let $P_j(e^z)$ $(j = 0, \ldots, k-1)$ satisfy* (1.14) *and* (1.15)*. Let $Q_1(z)$ and $Q_2(z)$ be polynomials in z. If $P_0 \not\equiv 0$, then*

(i) (1.8) *possesses at most one nontrivial subnormal solution f_0, and f_0 is of the form* (1.10)*, where $S_1(z)$ and $S_2(z)$ are polynomials in z;*

(ii) *all other solutions f of* (1.8) *satisfy $\sigma_2(f) = 1$ except the possible subnormal solution in (i).*

Example 1.5 shows the existence of subnormal solution in Theorem 1.4.

Example 1.5. The equation

$$f''' - 2e^{2z} f'' - e^z f' + e^z f = -2e^{3z} - e^{-z} + 2 \tag{1.17}$$

has a subnormal solution $f = e^z + e^{-z} + 1$.

Theorem 1.6. *Under the assumption of Theorem C, the following statements hold.*

(i) *Equation* (1.11) *has no nontrivial subnormal solution, and all solutions of* (1.11) *satisfy $\sigma_2(f) = 1$.*

(ii) *Equation* (1.8) *has at most one nontrivial subnormal solution f_0, and f_0 is of the form* (1.10)*; all other solutions f of* (1.8) *satisfy $\sigma_2(f) = 1$.*

2. Lemmas for the Proofs of Theorems

Lemma 2.1 (see [7, 8]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, and let $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions and satisfy*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

(ii) *when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant;*

(iii) *when $1 \leq j \leq n, 1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E), \tag{2.1}$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 2.2. *Let P_j, m_j, m_s, and m satisfy the hypotheses of Theorem 1.3.*

(i) *If $P_0 \not\equiv 0$, then* (1.11) *has no nonzero polynomial solution.*

(ii) *If $P_0 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0$ $(d < s)$, then all polynomials with degree $\leq d-1$ are solutions of* (1.11)*, and any polynomial with degree $\geq d$ is not solution of* (1.11).

Proof. (i) Firstly, by $P_0 \not\equiv 0$, we see that all nonzero constants cannot be a solution of (1.11). Now suppose that $f_0 = b_n z^n + \cdots + b_1 z + b_0$ $(n \geq 1$, $b_n, \ldots, b_0$ are constants, $b_n \neq 0)$ is a solution of (1.11). If $n \geq s$, then $f_0^{(s)} \not\equiv 0$. Substituting f_0 into (1.11) and taking $z = r$, we conclude that

$$\begin{aligned}
&|a_{sm_s}|e^{m_s r}|b_n|n(n-1)\cdots(n-s+1)r^{n-s}(1-o(1)) \\
&\quad\leq \left|-P_s(e^z)f_0^{(s)}(z)\right| \\
&\quad\leq \left|f_0^{(k)}(z)\right| + \left|P_{k-1}(e^z)f_0^{(k-1)}(z)\right| + \cdots + \left|P_{s+1}(e^z)f_0^{(s+1)}(z)\right| \\
&\qquad + \left|P_{s-1}(e^z)f_0^{(s-1)}(z)\right| + \cdots + \left|P_0(e^z)f_0(z)\right| \leq Mr^n e^{mr}(1+o(1)),
\end{aligned} \tag{2.2}$$

where $M(>0)$ is some constant. Since $m_s > m$, we see that (2.2) is a contradiction. If $n < s$, then

$$P_n(e^z)f_0^{(n)}(z) + \cdots + P_0(e^z)f_0(z) = 0. \tag{2.3}$$

Set $\max\{\deg P_j : j = 0,\ldots,n\} = h$. If $\deg P_j = m_j < h$, then we can rewrite

$$P_j(e^z) = a_{jh}e^{hz} + \cdots + a_{j(m_j+1)}e^{(m_j+1)z} + a_{jm_j}e^{m_j z} + \cdots + a_{j1}e^z \quad (j = 0,\ldots,n), \tag{2.4}$$

where $a_{jh} = \cdots = a_{j(m_j+1)} = 0$. Thus, we conclude by (2.3) and (2.4) that

$$\begin{aligned}
&\left(a_{nh}f_0^{(n)} + a_{(n-1)h}f_0^{(n-1)} + \cdots + a_{0h}f_0\right)e^{hz} + \cdots \\
&\quad + \left(a_{nj}f_0^{(n)} + a_{(n-1)j}f_0^{(n-1)} + \cdots + a_{0j}f_0\right)e^{jz} + \cdots \\
&\quad + \left(a_{n1}f_0^{(n)} + a_{(n-1)1}f_0^{(n-1)} + \cdots + a_{01}f_0\right)e^{z} = 0.
\end{aligned} \tag{2.5}$$

Set

$$Q_j(z) = a_{nj}f_0^{(n)} + a_{(n-1)j}f_0^{(n-1)} + \cdots + a_{0j}f_0 \quad (j = 1,\ldots,h). \tag{2.6}$$

Since f_0 is the polynomial, we see that

$$m(r, Q_j) = o\left\{m\left(r, e^{(\alpha-\beta)z}\right\}\right. \quad (1 \leq \beta < \alpha \leq h). \tag{2.7}$$

By Lemma 2.1, (2.5)–(2.7), we conclude that

$$Q_1(z) \equiv Q_2(z) \equiv \cdots \equiv Q_h(z) \equiv 0. \tag{2.8}$$

Since $\deg f_0 > deg f_0' > \cdots > \deg f_0^{(n)}$, by (2.6) and (2.8), we see that

$$a_{00} = a_{01} = \cdots = a_{0h} = 0. \tag{2.9}$$

Since $h \geq m_0 = \deg P_0$, we have $P_0 \equiv 0$. This contradicts our assumption that $P_0 \not\equiv 0$.

(ii) Since $P_0 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0$ $(d < s)$, clearly all polynomials with degree $\leq d-1$ are solutions of (1.11). By $P_d \not\equiv 0$ and (i), we see that $f^{(d)}$ cannot be a nonzero polynomial, and hence f cannot be a polynomial with degree $(\geq d)$. □

Lemma 2.3 (see [9]). *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, and let $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$, for $i = 1, \ldots, q$, and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E \subset [-\pi/2, 3\pi/2)$ that has linear measure zero, such that if $\psi \in [-\pi/2, 3\pi/2) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$ and for all $(k, j) \in H$, one has*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.10}$$

Lemma 2.4 (see [10]). *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \leq |z_n|^{k-j}(1 + o(1)) \quad (j = 0, \ldots, k-1). \tag{2.11}$$

Lemma 2.5 (see [11]). *Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Let there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi) \setminus E$, $|f(re^{i\theta_0})| \leq Mr^k$ $(M = M(\theta_0) > 0$ is a constant, and $k(> 0)$ is a constant independent of $\theta_0)$. Then $f(z)$ is a polynomial with $\deg f \leq k$.*

Lemma 2.6 can be obtained from [12, Theorem 4] or [2, Theorem 7.3].

Lemma 2.6. *Let $A_0, \ldots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0, \tag{2.12}$$

then $\sigma_2(f) \leq \max\{\sigma(A_j) : j = 0, \ldots, k-1\}$.

Lemma 2.7 (see [13]). *Let $g(z)$ be an entire function of infinite order with the hyperorder $\sigma_2(g) = \sigma$, and let $\nu(r)$ be the central index of g. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log \log \nu(r)}{\log r} = \sigma_2(g) = \sigma. \tag{2.13}$$

Lemma 2.8 (see [6]). *Let $f(z)$ be an entire function of infinite order with $\sigma_2(f) = \alpha$ $(0 \leq \alpha < \infty)$, and let a set $E \subset [1, \infty)$ have finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such* *that* $|f(z_k)| = M(r_k, f), \theta_k \in [-\pi/2, 3\pi/2), \lim_{k\to\infty}\theta_k = \theta_0 \in [-\pi/2, 3\pi/2)$, $r_k \notin E, r_k \to \infty$ *such that*

(i) *if* $\sigma_2(f) = \alpha(0 < \alpha < \infty)$, *then for any given* ε_1 $(0 < \varepsilon_1 < \alpha)$,

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\}; \tag{2.14}$$

(ii) *if* $\sigma(f) = \infty$ *and* $\sigma_2(f) = 0$, *then for any given* ε_2 $(0 < \varepsilon_2 < 1/2)$ *and for any large* $M(> 0)$, *one has as* r_k *sufficiently large*

$$r_k^M < \nu(r_k) < \exp\{r_k^{\varepsilon_2}\}. \tag{2.15}$$

Lemma 2.9 (see [9]). *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* i, j $(i < j$ $(i, j \in \mathbb{N}))$, *such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E$, *one has*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{j-i}. \tag{2.16}$$

Remark 2.10. From the proof of Lemma 2.9 (i.e., Theorem 3 in [9]), we can see that exceptional set E satisfies that if a_n and b_m $(n, m = 1, 2, \ldots)$ denote all zeros and poles of f, respectively, $O(a_n)$ and $O(b_m)$ denote sufficiently small neighborhoods of a_n and b_m, respectively, then

$$E = \left\{|z| : z \in \left(\bigcup_{n=1}^{+\infty} O(a_n)\right) \bigcup \left(\bigcup_{m=1}^{+\infty} O(b_m)\right)\right\}. \tag{2.17}$$

Hence, if $f(z)$ is a transcendental entire function and z is a point such that it satisfies that $|f(z)|$ is sufficiently large, then (2.16) holds.

Lemma 2.11 (see [4]). *Consider an nth-order linear differential equation of the form*

$$P_0(e^z, e^{-z})f^{(n)} + P_1(e^z, e^{-z})f^{(n-1)} + \cdots + P_n(e^z, e^{-z})f = P_{n+1}(e^z, e^{-z}), \tag{2.18}$$

where each $P_j(z, w)$ *is a polynomial in* z *and* w *with* $P_0(z, w) \not\equiv 0$. *Suppose that* $f = \phi(z)$ *is an entire subnormal solution of* (2.18), *that is, an entire solution of* (2.18) *that also satisfies* (1.3). *If* ϕ *is periodic with period* $2\pi i$, *then*

$$\phi(z) = S_1(e^z) + S_2(e^{-z}), \tag{2.19}$$

where $S_1(z)$ *and* $S_2(z)$ *are polynomials in* z.

3. Proof of Theorem 1.3

(i) Suppose that $P_0 \not\equiv 0$ and $f(\not\equiv 0)$ are the solution of (1.11). Then f is an entire function. By Lemma 2.2 (i), we see that f is transcendental.

Step 1. We prove that $\sigma(f) = \infty$. Suppose to the contrary that $\sigma(f) = \sigma < \infty$. By Lemma 2.3, we know that for any given $\varepsilon > 0$, there exists a set $E \subset [-\pi/2, 3\pi/2)$ of linear measure zero, such that if $\psi \in [-\pi/2, 3\pi/2) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| = r > R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq r^{(\sigma-1+\varepsilon)(j-s)}, \quad j = s+1, \ldots, k. \tag{3.1}$$

Now we take a ray $\arg z = \theta \in (-\pi/2, \pi/2) \setminus E$, then $\cos\theta > 0$. We assert that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_t = r_t e^{i\theta}$ $(t = 1, 2, \ldots)$ such that, as $r_t \to \infty$, $f'(z_t) \to \infty$ and

$$\left|\frac{f^{(i)}(z_t)}{f^{(s)}(z_t)}\right| \leq r_t^{s-i}(1 + o(1)) \quad (i = 0, \ldots, s-1). \tag{3.2}$$

By (1.11), we get that

$$\begin{aligned} -P_s(e^{z_t}) = {} & \frac{f^{(k)}(z_t)}{f^{(s)}(z_t)} + P_{k-1}(e^{z_t})\frac{f^{(k-1)}(z_t)}{f^{(s)}(z_t)} + \cdots \\ & + P_{s+1}(e^{z_t})\frac{f^{(s+1)}(z_t)}{f^{(s)}(z_t)} + P_{s-1}(e^{z_t})\frac{f^{(s-1)}(z_t)}{f^{(s)}(z_t)} + \cdots + P_0(e^{z_t})\frac{f(z_t)}{f^{(s)}(z_t)}. \end{aligned} \tag{3.3}$$

Since $\cos\theta > 0$ and (1.14), we have

$$\begin{aligned} |P_s(e^{z_t})| &= |a_{sm_s}| e^{m_s r_t \cos\theta}(1 + o(1)), \\ \left|P_j(e^{z_t})\right| &\leq M e^{m r_t \cos\theta}(1 + o(1)) \quad (j = 0, \ldots, s-1, s+1, \ldots, k-1), \end{aligned} \tag{3.4}$$

where $M(> 0)$ is some constant. Substituting (3.1), (3.2), and (3.4) into (3.3), we get that

$$|a_{sm_s}| e^{m_s r_t \cos\theta}(1 + o(1)) \leq k M r_t^{(\sigma+1)k} e^{m r_t \cos\theta}(1 + o(1)). \tag{3.5}$$

By $m_s > m$, we know that when $r_t \to \infty$, (3.5) is a contradiction. So,

$$\left|f\left(re^{i\theta}\right)\right| \leq M_1 r^s \tag{3.6}$$

on the ray $\arg z = \theta \in (-\pi/2, \pi/2) \setminus E$, where $M_1(> 0)$ is some constant.

Now we take a ray $\arg z = \theta \in (\pi/2, 3\pi/2) \setminus E$. Then, $\cos\theta < 0$. If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_t' = r_t' e^{i\theta}$ $(t = 1, 2, \ldots)$ such that, as $r_t' \to \infty$, $f^{(k)}(z_t') \to \infty$ and

$$\left|\frac{f^{(i)}(z_t')}{f^{(k)}(z_t')}\right| \leq \left(r_t'\right)^{k-i}(1 + o(1)) \quad (i = 0, \ldots, k-1). \tag{3.7}$$

By (1.11), we get that

$$|-1| \le \left|P_{k-1}\left(e^{z_t'}\right)\frac{f^{(k-1)}(z_t')}{f^{(k)}(z_t')}\right| + \cdots + \left|P_0\left(e^{z_t'}\right)\frac{f(z_t')}{f^{(k)}(z_t')}\right|. \tag{3.8}$$

Since $\cos\theta < 0$ and (3.7), for $j = 0,\dots,k-1$ as $r_t' \to \infty$,

$$\left|P_j\left(e^{z_t'}\right)\frac{f^{(j)}(z_t')}{f^{(k)}(z_t')}\right| \le \left(\left|a_{jm_j}\right|e^{m_j r_t'\cos\theta} + \cdots + \left|a_{j1}\right|e^{r_t'\cos\theta}\right)\left(r_t'\right)^{k-j}(1+o(1)) \longrightarrow 0. \tag{3.9}$$

By (3.8) and (3.9), we get that $1 \le 0$; this is a contradiction. So,

$$\left|f\left(re^{i\theta}\right)\right| \le M_1 r^k \tag{3.10}$$

on the ray $\arg z = \theta \in (\pi/2, 3\pi/2) \setminus E$.

By Lemma 2.5, (3.6), and (3.10), we know that $f(z)$ is a polynomial, which contradicts the above assertion that $f(z)$ is transcendental. Therefore $\sigma(f) = \infty$.

Step 2. We prove that (1.11) has no nontrivial subnormal solution. Now suppose that (1.11) has a nontrivial subnormal solution f_0, and we will deduce a contradiction. By the conclusion in Step 1, f_0 satisfies (1.3) and and $\sigma(f_0) = \infty$. By Lemma 2.6, we see that $\sigma_2(f) \le 1$. Set $\sigma_2(f) = \alpha \le 1$. By Lemma 2.9, we see that there exist a subset $E_1 \subset (1,\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f_0^{(j)}(z)}{f_0(z)}\right| \le B\left[T(2r, f_0)\right]^{k+1} \quad (j = 1,\dots,k). \tag{3.11}$$

From the Wiman-Valiron theory (see [2, page 51]), there is a set $E_2 \subset (1,\infty)$ having finite logarithmic measure, so we can choose z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f_0(z)| = M(r, f_0)$. Thus, we get

$$\frac{f_0^{(j)}(z)}{f_0(z)} = \left(\frac{\nu(r)}{z}\right)^j (1+o(1)), \quad j = 1,\dots,k, \tag{3.12}$$

where $\nu(r)$ is the central index of $f_0(z)$.

By Lemma 2.8, we see that there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f), \theta_n \in [-\pi/2, 3\pi/2), \lim\theta_n = \theta_0 \in [-\pi/2, 3\pi/2), r_n \notin [0,1] \cup E_1 \cup E_2, r_n \to \infty$, and if $\alpha > 0$, then by (2.14), we see that for any given ε_1 $(0 < \varepsilon_1 < \alpha)$, and for sufficiently large r_n,

$$\exp\{r_n^{\alpha-\varepsilon_1}\} < \nu(r_n) < \exp\{r_n^{\alpha+\varepsilon_1}\}, \tag{3.13}$$

and if $\alpha = 0$, then by $\sigma(f) = \infty$ and (2.15), we see that for any given ε_2 $(0 < \varepsilon_2 < 1/2)$ and for any sufficiently large $M_2 > 2k + 3$, as r_n is sufficiently large,

$$r_n^{M_2} < \nu(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{3.14}$$

Since θ_0 may belong to $(-\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$, or $\{-\pi/2, \pi/2\}$, we divide this proof into three cases to prove.

Case 1. Suppose that $\theta_0 \in (-\pi/2, \pi/2)$, then $\cos\theta_0 > 0$. If we take $\delta = (1/4)(\pi/2 - |\theta_0|)$, then $[\theta_0 - \delta, \theta_0 + \delta] \subset (-\pi/2, \pi/2)$. By $\theta_n \to \theta_0$, we see that there is a constant $N(> 0)$ such that, as $n > N$, $\theta_n \in [\theta_0 - \delta, \theta_0 + \delta]$, and $0 < \cos(|\theta_0| + \delta) \le \cos\theta_n$. By (3.11), we see that for any given ε_3 satisfying $0 < \varepsilon_3 < (1/(4(k+1)))\cos(|\theta_0| + \delta)$,

$$\left[T(2r_n, f_0)\right]^{k+1} \le e^{\varepsilon_3 (k+1) 2 r_n} \le e^{(1/2)\cos(|\theta_0|+\delta) r_n} \le e^{(1/2)\cos\theta_n r_n} \tag{3.15}$$

holds for $n > N$. By (3.11), (3.12), and (3.15), we see that

$$\left|\frac{f_0^{(k-s)}(z_n)}{f_0(z_n)}\right| = \left(\frac{\nu(r_n)}{r_n}\right)^{k-s}(1 + o(1)) \le B\left[T(2r_n, f_0)\right]^{k+1} \le Be^{(1/2)\cos\theta_n r_n}. \tag{3.16}$$

By (1.11), we get

$$\begin{aligned} -\frac{f_0^{(s)}(z_n)}{f_0(z_n)}\left(a_{sm_s}e^{m_s z_n} + \cdots + a_{s1}e^{z_n}\right) = {} & \frac{f_0^{(k)}(z_n)}{f_0(z_n)} + P_{k-1}(e^{z_n})\frac{f_0^{(k-1)}(z_n)}{f_0(z_n)} + \cdots \\ & + P_{s+1}(e^{z_n})\frac{f_0^{(s+1)}(z_n)}{f_0(z_n)} P_{s-1}(e^{z_n})\frac{f_0^{(s-1)}(z_n)}{f_0(z_n)} \\ & + \cdots + P_1(e^{z_n})\frac{f_0'(z_n)}{f_0(z_n)} + P_0(e^{z_n}). \end{aligned} \tag{3.17}$$

Since $\cos\theta_n > 0$ and (1.14), we get that

$$\begin{gathered} \left|a_{sm_s}e^{m_s z_n} + \cdots + a_{s1}e^{z_n}\right| = |a_{sm_s}| e^{m_s r_n \cos\theta_n}(1 + o(1)), \\ \left|P_j(e^{z_n})\right| \le M_3 e^{m r_n \cos\theta_n} \quad (j = 0, \ldots, s-1, s+1, \ldots k-1), \end{gathered} \tag{3.18}$$

where $M_3(> 0)$ is some constant. Substituting (3.12) and (3.18) into (3.17), we deduce that for sufficiently large r_n,

$$\begin{aligned} &\left(\frac{\nu(r_n)}{r_n}\right)^{s} |a_{sm_s}| e^{m_s r_n \cos\theta_n}(1 + o(1)) \\ &\quad \le \left(\frac{\nu(r_n)}{r_n}\right)^{k}(1 + o(1)) + M_3 e^{m r_n \cos\theta_n} \sum_{j=0, j\ne s}^{k-1}\left(\frac{\nu(r_n)}{r_n}\right)^{j}(1 + o(1)). \end{aligned} \tag{3.19}$$

From (3.13) or (3.14), we have

$$\nu(r_n) > r_n^{M_2} > r_n^{2k+3} > r_n. \tag{3.20}$$

By (3.16), (3.19), and (3.20), we get

$$\left(\frac{\nu(r_n)}{r_n}\right)^s |a_{sm_s}| e^{m_s r_n \cos\theta_n}(1+o(1)) \leq kM_3 e^{mr_n\cos\theta_n}\left(\frac{\nu(r_n)}{r_n}\right)^k (1+o(1)), \tag{3.21}$$

$$|a_{sm_s}| e^{(m_s-m)r_n\cos\theta_n}(1+o(1)) \leq kM_3 B e^{(1/2)r_n\cos\theta_n}. \tag{3.22}$$

Since $m_s - m \geq 1 > 1/2$, we see that (3.22) is a contradiction.

Case 2. Suppose that $\theta_0 \in (\pi/2, 3\pi/2)$. By $\cos\theta_0 < 0$ and $\theta_n \to \theta_0$, we see that for sufficiently large n, $\cos\theta_n < 0$. By (1.11), we get

$$-\frac{f_0^{(k)}(z_n)}{f_0(z_n)} = P_{k-1}(e^{z_n})\frac{f_0^{(k-1)}(z_n)}{f_0(z_n)} + \cdots + P_1(e^{z_n})\frac{f_0'(z_n)}{f_0(z_n)} + P_0(e^{z_n}). \tag{3.23}$$

Since $\cos\theta_n < 0$ and (1.14), we get that for $j = 0, \ldots, k-1$

$$\left|P_j(e^{z_n})\right| \leq \left|a_{jm_j}\right| e^{m_j r_n\cos\theta_n} + \cdots + \left|a_{j1}\right| e^{r_n\cos\theta_n} \leq C_j, \tag{3.24}$$

where C_j $(j = 0, \ldots, k-1)$ are constants. By (3.12), (3.20), (3.23), and (3.24), we get that

$$\left(\frac{\nu(r_n)}{r_n}\right)^k (1+o(1)) = \left|\frac{f_0^{(k)}(z_n)}{f_0(z_n)}\right| \leq [C_{k-1} + \cdots + C_0]\left(\frac{\nu(r_n)}{r_n}\right)^{k-1}(1+o(1)). \tag{3.25}$$

Thus, we have

$$\nu(r_n)(1+o(1)) \leq [C_{k-1} + \cdots + C_0] r_n (1+o(1)). \tag{3.26}$$

By (3.20), we see that (3.26) is also a contradiction.

Case 3. Suppose that $\theta_0 = \pi/2$ or $\theta_0 = -\pi/2$. Since the proof of $\theta_0 = -\pi/2$ is the same as the proof of $\theta_0 = \pi/2$, we only prove the case that $\theta_0 = \pi/2$. Since $\theta_n \to \theta_0$, for any given ε_4 $(0 < \varepsilon_4 < 1/10)$, we see that there is an integer $N(> 0)$, as $n > N$, $\theta_n \in [\pi/2 - \varepsilon_4, \pi/2 + \varepsilon_4]$, and

$$z_n = r_n e^{i\theta_n} \in \overline{\Omega} = \left\{z : \frac{\pi}{2} - \varepsilon_4 \leq \arg z \leq \frac{\pi}{2} + \varepsilon_4\right\}. \tag{3.27}$$

By Lemma 2.9, there exist a subset $E_3 \subset (1,\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, we have

$$\left|\frac{f_0^{(d)}(z)}{f_0^{(s)}(z)}\right| \le B\left[T\left(2r, f_0^{(s)}\right)\right]^{k+1} \quad (d = s+1,\dots,k). \tag{3.28}$$

Now we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega} \setminus \{\pi/2\}$. If $\theta \in [\pi/2 - \varepsilon_4, \pi/2)$, then $\cos\theta > 0$. By (1.3), for any given ε_5 satisfying $0 < \varepsilon_5 < (1/(4(k+1)))\cos\theta$,

$$\left[T\left(2r, f_0^{(s)}\right)\right]^{k+1} \le e^{\varepsilon_5(k+1)2r} \le e^{(1/2)\cos\theta r}. \tag{3.29}$$

If $|f_0^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists a sequence $\{y_j = R_j e^{i\theta}\}$ such that, as $R_j \to \infty$, $f_0^{(s)}(y_j) \to \infty$ and

$$\left|\frac{f_0^{(i)}(y_j)}{f_0^{(s)}(y_j)}\right| \le R_j^{s-i}(1+o(1)) \quad (i = 0,\dots,s-1). \tag{3.30}$$

By Remark 2.10 and $f_0^{(s)}(y_j) \to \infty$, we know that y_j satisfies (3.28). By (3.28) and (3.29), we see that for sufficiently large j,

$$\left|\frac{f_0^{(d)}(y_j)}{f_0^{(s)}(y_j)}\right| \le B\left[T\left(2R_j, f_0^{(s)}\right)\right]^{k+1} \le e^{(1/2)\cos\theta R_j} \quad (d = s+1,\dots,k). \tag{3.31}$$

By (1.11), (3.18), (3.30), and (3.31), we deduce that

$$|a_{sm_s}| e^{m_s R_j \cos\theta}(1+o(1)) = |-P_s(e^{y_j})| \le kM_3Be^{(m+(1/2))R_j\cos\theta} R_j^s(1+o(1)). \tag{3.32}$$

Since $m_s > m + 1/2$, we know that when $R_j \to \infty$, (3.32) is a contradiction. Hence

$$\left|f_0\left(re^{i\theta}\right)\right| \le Mr^s \tag{3.33}$$

on the ray $\arg z = \theta \in [\pi/2 - \varepsilon_4, \pi/2)$.

If $\theta \in (\pi/2, \pi/2 + \varepsilon_4]$, then $\cos\theta < 0$. We assert that $|f_0^{(k)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f_0^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists a sequence $\{y_j^* = R_j^* e^{i\theta}\}$ such that, as $R_j^* \to \infty$, $f_0^{(k)}(y_j^*) \to \infty$ and

$$\left|\frac{f_0^{(i)}\left(y_j^*\right)}{f_0^{(k)}\left(y_j^*\right)}\right| \le \left(R_j^*\right)^{k-i}(1+o(1)) \quad (i = 0,\dots,k-1). \tag{3.34}$$

Since $\cos\theta < 0$, for fixed $t \in \{0, 1, \ldots, k-1\}$, we deduce that as $R_j^* \to \infty$

$$\left|P_t\left(e^{y_j^*}\right)\right|\left(R_j^*\right)^k \leq |a_{tm_t}|e^{m_t R_j^* \cos\theta}\left(R_j^*\right)^k + \cdots + |a_{t1}|e^{R_j^* \cos\theta}\left(R_j^*\right)^k \longrightarrow 0. \tag{3.35}$$

By (1.11), (3.34), and (3.35), we deduce that as $R_j^* \to \infty$

$$\begin{aligned} 1 &\leq \left|P_{k-1}\left(e^{y_j^*}\right)\frac{f_0^{(k-1)}\left(y_j^*\right)}{f_0^{(k)}\left(y_j^*\right)}\right| + \cdots + \left|P_0\left(e^{y_j^*}\right)\frac{f_0\left(y_j^*\right)}{f_0^{(k)}\left(y_j^*\right)}\right| \\ &\leq \left|P_{k-1}\left(e^{y_j^*}\right)\right|\left(R_j^*\right)^k(1+o(1)) + \cdots + \left|P_0\left(e^{y_j^*}\right)\right|\left(R_j^*\right)^k(1+o(1)) \longrightarrow 0. \end{aligned} \tag{3.36}$$

This, (3.36) is a contradiction. Hence

$$\left|f_0\left(re^{i\theta}\right)\right| \leq Mr^k \tag{3.37}$$

on the ray $\arg z = \theta \in (\pi/2, \pi/2 + \varepsilon_4]$.

By (3.33) and (3.37), we see that $|f_0(re^{i\theta})|$ satisfies

$$\left|f_0\left(re^{i\theta}\right)\right| \leq Mr^k \tag{3.38}$$

on the ray $\arg z = \theta \in \overline{\Omega} \setminus \{\pi/2\}$.

However, since $f_0(z)$ is of infinite order and $\{z_n = r_n e^{i\theta_n}\}$ satisfies $|f_0(z_n)| = M(r_n, f_0)$, we see that for any large $N(> k)$, as n is sufficiently large

$$|f_0(z_n)| = \left|f_0\left(r_n e^{i\theta_n}\right)\right| \geq \exp\left\{r_n^N\right\}. \tag{3.39}$$

Since $z_n \in \overline{\Omega}$, by (3.38) and (3.39), we see that for sufficiently large n

$$\theta_n = \frac{\pi}{2}. \tag{3.40}$$

Thus $\cos\theta_n = 0$ and for sufficiently large n

$$|P_j(e^{z_n})| = \left|a_{jm_j}e^{m_j z_n} + \cdots + a_{j1}e^{z_n}\right| \leq C \quad (j = 0, \ldots, k-1), \tag{3.41}$$

where $C(> 0)$ is some constant. By (1.11) and (3.12), we get that

$$\begin{aligned} -\left(\frac{\nu(r_n)}{z_n}\right)^k(1+o(1)) &= P_{k-1}(e^{z_n})\left(\frac{\nu(r_n)}{z_n}\right)^{k-1}(1+o(1)) \\ &\quad + \cdots + P_1(e^{z_n})\frac{\nu(r_n)}{z_n}(1+o(1)) + P_0(e^{z_n}). \end{aligned} \tag{3.42}$$

By (3.20), (3.41) and (3.42), we get that

$$\nu(r_n) \leq kCr_n^k. \tag{3.43}$$

By (3.13) (or (3.14)), we see that (3.43) is a contradiction. Hence (1.11) has no nontrivial subnormal solution.

Step 3. We prove that all solutions of (1.11) satisfy $\sigma_2(f) = 1$. If there is a solution f_1 satisfying $\sigma_2(f_1) < 1$, then f_1 satisfies (1.3), that is, f_1 is subnormal, but this contradicts the conclusion in Step 2. Hence every solution f satisfies $\sigma_2(f) \geq 1$. By this and $\sigma_2(f) \leq 1$, we get $\sigma_2(f) = 1$. Theorem 1.3(i) is thus proved.

(ii) Since $P_0 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0$, we clearly see that all polynomials with degree $\leq d-1$ are subnormal solutions of (1.11). By (i), we see that every $f^{(d)}$ satisfies $\sigma_2(f^{(d)}) = 1$ or $f^{(d)} \equiv 0$. Hence $\sigma_2(f) = 1$ or f is a polynomial with degree $\leq d-1$.

4. Proof of Theorem 1.4

Suppose that f_1 and $f_2(\not\equiv f_1)$ are nontrivial subnormal solutions of (1.8), then $f_1 - f_2(\not\equiv 0)$ is a subnormal solution of the corresponding homogeneous equation (1.11) of (1.8). This contradicts the assertion of Theorem 1.3(i). Hence (1.8) possesses at most one nontrivial subnormal solution.

Now suppose that f_0 is a nontrivial subnormal solution of (1.8), then $f_0(z + 2\pi i)$ is also nontrivial subnormal solution, so, $f_0(z) = f_0(z + 2\pi i)$ by the above assertion. Thus, by Lemma 2.11, we see that f_0 satisfies (1.10).

By Theorem 1.3(i), we see that all solutions of the corresponding homogeneous equation (1.11) of (1.8) are of $\sigma_2(f) = 1$. By variation of parameters, we see that all solutions of (1.8) satisfy $\sigma_2(f) \leq 1$. If $\sigma_2(f) < 1$, then f clearly satisfies (1.3); that is, f is subnormal. Hence all other solutions f of (1.8) satisfy $\sigma_2(f) = 1$ except at most one nontrivial subnormal solution.

5. Proof of Theorem 1.6

(i) By Lemma 2.6 and $\sigma(P_j) = 1$ $(j = 0, \ldots, k-1)$, we see that $\sigma_2(f) \leq 1$. By Lemma 2.9, we see that there exist a subset $E \subset (1, \infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E$,

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B\left[T(2r, f)\right]^{k+1} \quad (j = 1, \ldots, k). \tag{5.1}$$

Taking $z = r$, by (1.11) and (5.1), we deduce that

$$|a_{0m_0}| e^{m_0 r}(1 + o(1)) = |-P_0(e^{y_j})| \leq kB\left[T(2r, f)\right]^{k+1} M e^{mr}(1 + o(1)). \tag{5.2}$$

Since $m_0 > m$, by (5.2), we get that $\sigma_2(f) \geq 1$. Hence $\sigma_2(f) = 1$.

(ii) Using a similar method as in the proof of Theorem 1.4, we can prove (ii).

Acknowledgments

The authors are grateful to referees for a number of helpful suggestions to improve the paper. Z-X. Chen was supported by the National Natural Science Foundation of China (no.: 10871076). K. H. Shon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0009646).

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