Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 278962, 19 pages
doi:10.1155/2010/278962

*Research Article*

# Oscillation for Third-Order Nonlinear Differential Equations with Deviating Argument

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Received 21 October 2009; Accepted 5 January 2010

Academic Editor: Paul Eloe



We study necessary and sufficient conditions for the oscillation of the third-order nonlinear ordinary differential equation with damping term and deviating argument $x'''(t) + q(t)x'(t) + r(t)f(x(\varphi(t))) = 0$. Motivated by the work of Kiguradze (1992), the existence and asymptotic properties of nonoscillatory solutions are investigated in case when the differential operator $\mathcal{L}x = x''' + q(t)x'$ is oscillatory.

## 1. Introduction

The aim of this paper is to investigate the third order nonlinear functional differential equation with deviating argument

$$x'''(t) + q(t)x'(t) + r(t)f\left(x\left(\varphi(t)\right)\right) = 0. \tag{1.1}$$

The following assumptions will be made. $q, r, \varphi$ are continuous functions for $t \geq 0$, $r(t) > 0$, $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$, and $f$ is a continuous function, $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(u)u > 0 \quad \text{for } u \neq 0. \tag{1.2}$$

In this paper we will restrict our attention to solutions $x$ of (1.1) which are defined in a neighborhood of infinity and $\sup\{|x(s)| : s > t\} > 0$ for any $t$ of this neighborhood. As usual, a solution of (1.1) is said to be *oscillatory* if it has a sequence of zeros converging to infinity; otherwise it is said to be *nonoscillatory*.

Throughout the paper we assume that the operator $\mathcal{L}x = x''' + q(t)x'$ is oscillatory, that is, the second-order equation

$$h''(t) + q(t)h(t) = 0 \tag{1.3}$$

is oscillatory.

It is well known, see, for example, [1], that if (1.3) is nonoscillatory, then (1.1) can be written as a two-term equation of the form

$$\left(a_1(t)\left(a_2(t)x'\right)'\right)' + r(t)f\left(x\left(\varphi(t)\right)\right) = 0, \tag{1.4}$$

where $a_i, i = 1, 2$, are continuous positive functions for $t \geq 0$.

Asymptotic properties of equations of type (1.4) have been widely investigated in the literature. We refer to [1–7] in case when $\int_0^\infty 1/a_i(t)dt = \infty$ $(i = 1, 2)$, that is, the disconjugate differential operator $\mathcal{L}_1 x = (a_1(t)(a_2(t)x')')'$ is in the so-called canonical form [5, 8, 9] when this property does not occur. Some of these results extend the pioneering works [10, 11], devoted to the equation

$$x'''(t) + q(t)x'(t) + r(t)x^{\mu}(t) = 0, \tag{1.5}$$

where $\mu$ is the quotient of odd positive integers. Other contributions deal with the solvability of certain boundary value problems associated to equations of type (1.1) on compact or noncompact intervals see, for example, [12, 13] or [14, 15], respectively, and references therein.

Recently, oscillation criteria for (1.4) with damping term, that is, for

$$\left(a_1(t)\left(a_2(t)x'\right)'\right)' + q(t)x' + r(t)f\left(x\left(\varphi(t)\right)\right) = 0 \tag{1.6}$$

have been presented in [9] by using a generalized Riccati transformation and an integral averaging technique. Here oscillation means that any solution $x$ of this equation is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Several examples [9, Examples 1–5] concern the case when the second-order equation (1.3) is nonoscillatory and so such an equation can be reduced to a two-term equation of the form (1.4).

If the differential operator $\mathcal{L}x = x''' + q(t)x'$ is oscillatory, then very little is known. According to Kiguradze [16], we say that (1.1) has *property A* if each of its solutions either is oscillatory, or satisfies the condition

$$\lim_{t\to\infty} x^{(i)}(t) = 0 \quad (i = 0, 1, 2). \tag{1.7}$$

If any solution $x$ of (1.1) is either oscillatory, or satisfies the condition (1.7), or admits the asymptotic representation

$$x^{(i)} = c(1+\sin(t-\alpha))^{(i)} + \varepsilon_i(t), \quad (i = 0,1,2,3), \tag{1.8}$$

where $c \neq 0$ and $\alpha$ are constants, the continuous functions $\varepsilon_i (i = 0,1,2,3)$ vanish at infinity and $\varepsilon_0$ satisfies the inequality $c\varepsilon_0(t) > 0$ for large $t$, then we say that (1.1) has *weak property A.*

For $n = 3$, the results in [16] deal with the equation

$$x'''(t) + x'(t) + r(t)f(x(t)) = 0, \tag{1.9}$$

and read as follows.

**Theorem 1.1** (see [16, Theorem 1.5]). *Let $f$ be a nondecreasing function satisfying*

$$\int_1^{\infty} \frac{du}{f(u)} < \infty, \qquad \int_{-\infty}^{-1} \frac{du}{f(u)} < \infty. \tag{1.10}$$

*Then the condition*

$$\int_0^{\infty} r(t)dt = \infty \tag{1.11}$$

*is necessary and sufficient in order that* (1.9) *has weak property A.*

**Theorem 1.2** (see [16, Corollary 1.5]). *Let for some $K > 0$ and $a > 0$*

$$r(t)\left|f(u)\right| \geq Kt^{-1}|u| \quad \text{for } u \in \mathbb{R},\ t \geq a. \tag{1.12}$$

*Then* (1.9) *has property A.*

In our previous paper [1] we have investigated (1.1) without deviating argument (i.e., $\varphi(t) = t$), especially when (1.3) is nonoscillatory. More precisely, the nonexistence of possible types of nonoscillatory solutions is examined, independently on the oscillation of (1.3).

Motivated by [1, 16], here we continue such a study, by giving necessary and sufficient conditions in order that all solutions of (1.1) are either oscillatory or satisfy $\liminf_{t\to\infty} x(t) = 0$. The property A for (1.1) is also considered and an extension to (1.1) of Theorem 1.1 is presented.

The role of the deviating argument $\varphi$ and some phenomena for (1.1), which do not occur when (1.3) is nonoscillatory, are presented. Our results depend on a a priori classification of nonoscillatory solutions which is based on the concept of phase function [17] and on a suitable energy function. A fixed point method is also employed and sharp upper and lower estimates for bounded nonoscillatory solutions of (1.1) are established by means of a suitable "cut" function. This approach enables us to assume $r \in L^1[0,\infty)$ instead of $R(t) \in L^1[0,\infty)$, where

$$R(t) = \int_t^\infty r(\tau)d\tau. \tag{1.13}$$

## 2. Classification of Nonoscillatory Solutions

A function $g$, defined in a neighborhood of infinity, is said to change its sign, if there exists a sequence $\{t_k\} \to \infty$ such that $g(t_k)g(t_{k+1}) < 0$.

The following theorem shows the possible types of nonoscillatory solutions for (1.1). It is worth noting that here $q$ can change sign.

**Theorem 2.1.** *Any nonoscillatory solution $x$ of* (1.1) *either satisfies*

$$x(t)x'(t) \leq 0, \quad x(t) \neq 0 \quad \text{for large } t \tag{2.1}$$

*or*

$$x(t) \neq 0 \quad \text{for large } t,\ x'(t) \text{ changes its sign}. \tag{2.2}$$

*Proof.* Without loss of generality suppose that there exists a solution $x$ of (1.1) and $T \geq 0$ such that $x(t) > 0$, $x(\varphi(t)) > 0$, $x'(t) \geq 0$ on $[T,\infty)$.

O. Boruvka [17] proved that if (1.3) is oscillatory, then there exists a continuously differentiable function $\alpha : [T,\infty) \to [T,\infty)$, called a *phase function*, such that $\alpha'(t) > 0$ and

$$\frac{3(\alpha''(t))^2}{4(\alpha'(t))^4} - \frac{\alpha'''(t)}{2(\alpha'(t))^3} + \frac{q(t)}{\alpha'^2(t)} = 1. \tag{2.3}$$

Using this result, we can consider the change of variables

$$s = \alpha(t)\ , \qquad x'(t) = \frac{1}{\sqrt{\alpha'(t)}}\dot{X}(s), \qquad \left(\cdot = \frac{d}{ds}\right) \tag{2.4}$$

for $t \in [T,\infty)$, $s \in [T^*,\infty)$, $T^* = \alpha(T)$. Thus, $t = \alpha^{-1}(s)$ and

$$\begin{aligned} x''(t) &= -\frac{1}{2}(\alpha'(t))^{-3/2}\alpha''(t)\dot{X}(s) + (\alpha'(t))^{1/2}\ddot{X}(s), \\ x'''(t) &= -\frac{1}{2}(\alpha'(t))^{-3/2}\alpha'''(t)\dot{X}(s) + \frac{3}{4}(\alpha'(t))^{-5/2}(\alpha''(t))^2\dot{X}(s) + (\alpha'(t))^{3/2}\dddot{X}(s). \end{aligned} \tag{2.5}$$

Substituting into (1.1), we obtain

$$\dddot{X}(s)+\dot{X}(s)\left(\frac{3}{4}(\alpha'(t))^{-4}(\alpha''(t))^{2}-\frac{1}{2}(\alpha'(t))^{-3}\alpha'''(t)+q(t)(\alpha'(t))^{-2}\right)$$
$$+(\alpha'(t))^{-3/2}r(t)f(x(\varphi(t)))=0. \tag{2.6}$$

From here and (2.3) we obtain

$$\dddot{X}(s)+\dot{X}(s)+(\alpha'(t))^{-3/2}r(t)f(x(\varphi(t)))=0. \tag{2.7}$$

Because $\lim_{t\to\infty}\varphi(t)=\infty$, we have for large $t$

$$\dddot{X}(s)+\dot{X}(s)=-(\alpha'(t))^{-3/2}r(t)f(x(\varphi(t)))<0. \tag{2.8}$$

Since $x'(t)\geq 0$, (2.4) yields $\dot{X}(s)\geq 0$ and so $\dddot{X}(s)<0$, that is, $\ddot{X}$ is decreasing. If there exists $s_1\geq T$ such that $\ddot{X}(s_1)<0$, $\dot{X}$ becomes eventually negative, which is a contradiction. Then $\ddot{X}(s)\geq 0$ and $\dot{X}(s)$ is nondecreasing. Let $T_1\geq T^*$ be such that $\dot{X}(s)>0$ on $[T_1,\infty)$. Thus, using (2.8) we obtain

$$\ddot{X}(s)-\ddot{X}(T_1)=\int_{T_1}^{s}\dddot{X}(u)du\leq-\int_{T_1}^{s}\dot{X}(u)du\leq-\dot{X}(T_1)(s-T_1). \tag{2.9}$$

Hence, $\lim_{s\to\infty}\ddot{X}(s)=-\infty$, which contradicts the positivity of $\ddot{X}(s)$. Finally, the case $\dot{X}(s)\equiv 0$ on $[T^*,\infty)$ cannot occur, because, if $x'(t)\equiv 0$ on $[T,\infty)$, then $x\equiv 0$, which is a contradiction. □

*Remark 2.2.* Theorem 2.1 extends [1, Proposition 2] for (1.1) with $\varphi(t)=t$ and improves [11, Theorem 3.2] for (1.5).

The following lemma is similar to [16, Lemma 2.2].

**Lemma 2.3.** *Any solution $x$ of* (1.1) *satisfies*

$$\liminf_{t\to\infty}|x'(t)|=0, \tag{2.10}$$

$$\liminf_{t\to\infty}|x''(t)|=0. \tag{2.11}$$

*Proof.* In view of Theorem 2.1, it is sufficient to prove (2.10) and (2.11) for solutions $x$ of (1.1) such that $x(t)>0, x'(t)\leq 0$ for large $t$. If (2.10) does not hold, then, in view of Theorem 2.1, a positive constant $C_1$ exists such that $x'(t)\leq -C_1$ for large $t$ and, hence, $\lim_{t\to\infty}x(t)=-\infty$, a contradiction.

Now let us prove (2.11) and, without loss of generality, assume $x(t) > 0$, $x(\varphi(t)) > 0$ for $t \geq t_0$. Suppose, for the sake of contradiction, that $\liminf_{t\to\infty}|x''(t)| = 2C > 0$. Then there exists $t_1 \geq t_0$ such that either

$$x''(t) \geq C \quad \text{or} \quad x''(t) \leq -C \quad \text{for } t \geq t_1. \tag{2.12}$$

If $x''(t) \geq C$, then $\lim_{t\to\infty}x'(t) = \infty$, which is a contradiction with Theorem 2.1. Thus $x''(t) \leq -C$ on $[t_1, \infty)$. From this and the Taylor theorem we obtain

$$\begin{aligned} x(t) &= x(t_1) + x'(t_1)(t-t_1) + \int_{t_1}^{t}(t-\sigma)x''(\sigma)d\sigma \\ &\leq x(t_1) + x'(t_1)(t-t_1) - \frac{C}{2}(t-t_1)^2, \end{aligned} \tag{2.13}$$

which gives $\lim_{t\to\infty}x(t) = -\infty$, a contradiction to the positivity of $x$ and so (2.11) holds. □

**Lemma 2.4.** *Let $q(t) \geq 0$ for large $t$. If $x$ is a solution of* (1.1) *such that $x(t) > 0$ for large $t$ and the function*

$$w(t) = x''(t) + q(t)x(t) \tag{2.14}$$

*is nonincreasing for large $t$, then*

$$\lim_{t\to\infty} w(t) = w_0 \geq 0. \tag{2.15}$$

*Proof.* By contradiction, assume $w_0 < 0$. Then for large $t$

$$x''(t) < -q(t)x(t) \leq 0. \tag{2.16}$$

If there exists $T \geq 0$ such that $x'(T) < 0$, then $x(t) < 0$ for large $t$, which is a contradiction. Thus $x'(t) \geq 0$ for large $t$, which is again a contradiction to Theorem 2.1. □

In view of Theorem 2.1, any nonoscillatory solution $x$ of (1.1) is one of the following types.

Type I: $x$ satisfies for large $t$

$$x(t)x'(t) \leq 0, \quad \lim_{t\to\infty} x(t) = 0. \tag{2.17}$$

Type II: $x$ satisfies for large $t$

$$|x(t)| \geq m_x > 0. \tag{2.18}$$

Type III: $x$ satisfies for large $t$

$$x(t) \neq 0, \quad \liminf_{t\to\infty} |x(t)| = 0, \quad x' \text{ changes its sign.} \tag{2.19}$$

*Remark 2.5.* Nonoscillatory solution $x$ of (1.1), such that $x'$ changes its sign, is usually called *weakly oscillatory solution*. Note that weakly oscillatory solutions can be either of Type II or Type III. When $\varphi(t) \equiv t$, in [1, Theorem 6] conditions are given under which (1.1) does not have weakly oscillatory solutions, especially solutions of Type III. This result will be used later for proving that the only nonoscillatory solutions of (1.1) are of Type I.

## 3. Necessary Condition for Oscillation

Our main result here deals with the existence of solutions of Type II.

**Theorem 3.1.** *Assume $q$ is continuously differentiable and is bounded away from zero, that is,*

$$q(t) \geq q_0 > 0. \tag{3.1}$$

*If*

$$\begin{gathered} \int_0^\infty \left|q'(t)\right| dt < \infty, \\ \int_0^\infty r(t) dt < \infty, \end{gathered} \tag{3.2}$$

*then for any $c \in \mathbb{R} \setminus \{0\}$ there exists a solution $x$ of* (1.1) *satisfying*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 1, 2. \tag{3.3}$$

*Proof.* We prove the existence of solutions of (1.1) satisfying (3.3) for $c = 1$.

Let $u$ and $v$ be two linearly independent solutions of (1.3) with Wronskian $d = 1$. By assumptions on $q$, all solutions of (1.3) and their derivatives are bounded; see, for example, [18, Theorem 2]. Put

$$\beta = \max_{1/2 \leq u \leq 3/2} f(u), \qquad M = \sup_{t \geq t_0} \left\{ |u(t)|, \left|u'(t)\right|, |v(t)|, \left|v'(t)\right| \right\} \tag{3.4}$$

and denote $h(s,t) = u(s)v'(t) - u'(t)v(s)$. Thus, there exists $k_1 > 0$ such that $(1 + 2|h(s,t)|) f(u) \leq k_1$ for any $u \in [1/2, 3/2]$. Hence

$$2(|h(s,t)|) f(u) \leq k_1, \qquad (1 + |h(s,t)|) f(u) \leq k_1. \tag{3.5}$$

Let $t_0$ be large so that

$$\frac{4k_1}{3q_0}\int_{t_0}^{\infty} r(t)dt \leq \frac{1}{2}, \qquad \frac{1}{q_0}\int_{t_0}^{\infty} \left|q'(s)\right| ds \leq \frac{1}{4}. \tag{3.6}$$

Let $\bar{t} \geq t_0$ be such that $\varphi(t) \geq t_0$ for $t \geq \bar{t}$. Define

$$\overline{\varphi}(t) = \begin{cases} \varphi(t), & \text{if } t \geq \bar{t}, \\ \varphi\left(\bar{t}\right), & \text{if } t_0 \leq t \leq \bar{t}. \end{cases} \tag{3.7}$$

Denote by $C[t_0, \infty)$ the Fréchet space of all continuous functions on $[t_0, \infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \left\{ x \in C[t_0, \infty) : \frac{1}{2} \leq x(t) \leq \frac{3}{2} \right\}. \tag{3.8}$$

Let $T \in [t_0, \infty)$ be fixed and let $I = [t_0, T]$. For any $x \in \Omega$, consider the "cut" function

$$G_x(t) = -\int_t^T \int_\tau^\infty \left(r(s) f\left(x\left(\overline{\varphi}(s)\right)\right)\right)\left(u(s)v(\tau) - u(\tau)v(s)\right) ds\, d\tau, \quad t \in I. \tag{3.9}$$

Then

$$\begin{aligned} G_x'(t) &= \int_t^\infty \left(r(s) f\left(x\left(\overline{\varphi}(s)\right)\right)\right)\left(u(s)v(t) - u(t)v(s)\right) ds, \\ G_x''(t) &= \int_t^\infty \left(r(s) f\left(x\left(\overline{\varphi}(s)\right)\right)\right)\left(u(s)v'(t) - u'(t)v(s)\right) ds, \\ G_x'''(t) &= -r(t) f\left(x\left(\overline{\varphi}(t)\right)\right) - q(t)\int_t^\infty \left(r(s) f\left(x\left(\overline{\varphi}(s)\right)\right)\right)\left(u(s)v(t) - u(t)v(s)\right) ds, \\ &= -r(t) f\left(x\left(\overline{\varphi}(t)\right)\right) - G_x'(t) q(t). \end{aligned} \tag{3.10}$$

Consider the function

$$g(t) = G_x''(t) + q(t) G_x(t), \quad t \in I. \tag{3.11}$$

Then $g(T) = G_x''(T)$ and

$$g'(t) = G_x'''(t) + q(t)G_x'(t) + q'(t)G_x(t) = -r(t) f\left(x\left(\overline{\varphi}(t)\right)\right) + q'(t) G_x(t), \quad t \in I. \tag{3.12}$$

Integrating from $t$ to $T$ we have

$$\begin{aligned} g(t) &= g(T) - \int_t^T g'(s)ds = g(T) + \int_t^T \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)ds - \int_t^T q'(s)G_x(s)ds \\ &= G''_x(x)(T) + \int_t^T \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)ds - \int_t^T q'(s)G_x(s)ds. \end{aligned} \tag{3.13}$$

From here and (3.11) we get

$$\begin{aligned} G_x(t) &= \frac{1}{q(t)}\left(G''_x(T) - G''_x(t) + \int_t^T \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)ds - \int_t^T q'(s)G_x(s)ds\right) \\ &= \frac{1}{q(t)}\int_t^T \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)(1-h(s,t))ds \\ &\quad + \frac{1}{q(t)}\int_T^\infty \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)(h(s,T)-h(s,t))ds - \frac{1}{q(t)}\int_t^T q'(s)G_x(s)ds. \end{aligned} \tag{3.14}$$

Thus, in view of (3.5), we obtain

$$|G_x(t)| \le \frac{k_1}{q(t)}\int_t^\infty r(s)ds + \max_{t\le s\le T}|G_x(s)|\ \frac{1}{q(t)}\int_{t_0}^\infty \left|q'(s)\right|ds, \tag{3.15}$$

or, in view of (3.6),

$$|G_x(t)| \le \frac{k_1}{q_0}\int_t^\infty r(s)ds + \frac{1}{4}\max_{t\le s\le T}|G_x(s)|, \tag{3.16}$$

that is,

$$\max_{t\le\sigma\le T}|G_x(\sigma)| \le \frac{k_1}{q_0}\int_t^\infty r(s)ds + \frac{1}{4}\max_{t\le\sigma\le T}|G_x(\sigma)|. \tag{3.17}$$

Hence

$$|G_x(t)| \le \max_{t\le\sigma\le T}|G_x(\sigma)| \le \frac{4k_1}{3q_0}\int_t^\infty r(s)ds, \tag{3.18}$$

that is,

$$\left|\int_t^T\int_\tau^\infty \left(r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)\right)(u(s)v(\tau)-u(\tau)v(s))ds\,d\tau\right| \le \frac{4k_1}{3q_0}\int_t^\infty r(s)ds. \tag{3.19}$$

In view of (3.19), using the Cauchy criterion, the limit

$$\lim_{T\to\infty}\int_t^T\int_\tau^\infty (r(s)f(x(\overline{\varphi}(s))))(u(s)v(\tau)-u(\tau)v(s))ds\,d\tau \tag{3.20}$$

exists finitely for any fixed $t$. This fact means that the operator

$$\mathrm{H}(x)(t)=1-\int_t^\infty\int_\tau^\infty (r(s)f(x(\overline{\varphi}(s))))(u(s)v(\tau)-u(\tau)v(s))ds\,d\tau \tag{3.21}$$

is well defined for any $x\in\Omega$. Moreover, from (3.19) we have for $t\geq t_0$

$$\left|\int_t^\infty\int_\tau^\infty (r(s)f(x(\overline{\varphi}(s))))(u(s)v(\tau)-u(\tau)v(s))ds\,d\tau\right|\leq \frac{4k_1}{3q_0}\int_t^\infty r(s)ds \tag{3.22}$$

and so, in view of (3.6), H maps $\Omega$ into itself.

Let us show that $\mathrm{H}(\Omega)$ is relatively compact, that is, $\mathrm{H}(\Omega)$ consists of functions which are equibounded and equicontinuous on every compact interval of $[t_0,\infty)$. Because $\mathrm{H}(\Omega)\subset\Omega$, the equiboundedness follows. Moreover, for any $x\in\Omega$ we have

$$\mathrm{H}'(x)(t)=\int_t^\infty (r(s)f(x(\overline{\varphi}(s))))(u(s)v(t)-u(t)v(s))ds, \tag{3.23}$$

and so

$$\left|\mathrm{H}'(x)(t)\right|\leq 2\beta M^2\int_t^\infty r(s)ds, \tag{3.24}$$

which proves the equicontinuity of the elements of $\mathrm{H}(\Omega)$.

Now we prove the continuity of H on $\Omega$. Let $\{x_n\}, n\in\mathbb{N}$, be a sequence in $\Omega$ which converges uniformly on every compact interval of $[t_0,\infty)$ to $x\in\Omega$. Because $\mathrm{H}(\Omega)$ is relatively compact, the sequence $\{\mathrm{H}(x_n)\}$ admits a subsequence, denoted again by $\{\mathrm{H}(x_n)\}$ for sake of simplicity, which is convergent to $\overline{y}\in\Omega$. From (3.4) we obtain

$$\left|\int_\tau^\infty (r(s)f(x_n(\overline{\varphi}(s))))(u(s)v(\tau)-u(\tau)v(s))ds\right|\leq 2\beta M^2R(\tau), \tag{3.25}$$

where $R$ is defined in (1.13). Hence, in virtue of the Lebesgue dominated convergence theorem, $\{G_{x_n}(t)\}$ converges pointwise to $G_x(t)$ on $[t_0,T]$, that is,

$$\lim_{n\to\infty}G_{x_n}(t)=G_x(t). \tag{3.26}$$

Choosing a sufficiently large $T$ and using (3.22), we obtain

$$\begin{aligned}
&|\mathrm{H}(x_n)(t) - \mathrm{H}(x)(t)| \\
&\quad = |\mathrm{H}(x_n)(t) - G_{x_n}(t) + G_{x_n}(t) - G_x(t) + G_x(t) - \mathrm{H}(x)(t)| \\
&\quad \leq |\mathrm{H}(x_n)(T) - 1| + |G_{x_n}(t) - G_x(t)| + |\mathrm{H}(x)(T) - 1| \\
&\quad \leq |G_{x_n}(t) - G_x(t)| + \frac{8k_1}{3q_0}\int_T^\infty r(s)ds.
\end{aligned} \tag{3.27}$$

Then, from (3.26), $\{\mathrm{H}(x_n)\}$ converges point-wise to $\mathrm{H}(x)$ and, in view of the uniqueness of the limit, $\mathrm{H}(x) = \overline{y}$ is the only cluster point of the compact sequence $\{\mathrm{H}(x_n)\}$, that is, the continuity of H in $\Omega$. Applying the Tychonov fixed point theorem, there exists a solution $x$ of the integral equation

$$x(t) = \mathrm{H}(x)(t), \tag{3.28}$$

which is a solution of (1.1) with the required properties. □

*Remark 3.2.* Theorem 3.1 partially extends [16, Theorem 1.4] for $n = 3$.

## 4. Sufficient Condition for Oscillation

In this section we give a sufficient condition for oscillation of (1.1) in the sense that any solution is either oscillatory or satisfies $\liminf_{t\to\infty} x(t) = 0$.

**Theorem 4.1.** *Assume*

$$\liminf_{u\to\pm\infty} \left| f(u) \right| > 0, \tag{4.1}$$

*and $q$ continuously differentiable for large $t$ satisfying*

$$q(t) > 0, \qquad q'(t) \leq 0. \tag{4.2}$$

*If*

$$\int^\infty \infty r(t)dt = \infty, \tag{4.3}$$

*then any (nonoscillatory) solution $x$ of* (1.1) *satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0. \tag{4.4}$$

*Moreover, any nonoscillatory solution with $x(t)x'(t) \leq 0$ for large $t$ satisfies $\lim_{t\to\infty} x^{(i)}(t) = 0$ for $i = 0, 1, 2$.*

*Proof.* To prove the first assertion, it is sufficient to show that (1.1) does not have solutions of Type II. By contradiction, let $x$ be a solution of (1.1) such that $x(t) > m > 0, x(\varphi(t)) > m$ for $t \geq t_0$.

Consider the function $w$ given by (2.14). Then

$$w'(t) = x'''(t) + q(t)x'(t) + q'(t)x(t) = -r(t)f(x(\varphi(t))) + q'(t)x(t) \leq 0. \tag{4.5}$$

By Lemma 2.4, we have $\lim_{t\to\infty} w(t) = w_\infty, 0 \leq w_\infty < \infty$. Since

$$f(x(\varphi(t))) \geq \inf_{u \geq m} f(u) = K > 0 \quad \text{for } t \geq t_0, \tag{4.6}$$

we get

$$f(x(\varphi(t))) - \frac{q'(t)}{r(t)}x(t) \geq K. \tag{4.7}$$

Consequently, we have from (4.5)

$$w(t_0) \geq w(t_0) - w(t) = \int_{t_0}^{t} r(s)\left(f(x(\varphi(s))) - \frac{q'(s)}{r(s)}x(s)\right)ds \geq K\int_{t_0}^{t} r(s)ds, \tag{4.8}$$

and, as $t \to \infty$, we get a contradiction. Hence, any nonoscillatory solution satisfies (4.4).

Now let $x$ be a solution of (1.1) such that $x(t) > 0,\ x(\varphi(t)) > 0,\ x'(t) \leq 0$ for $t \geq t_1 \geq 0$. Hence $x$ is of Type I and

$$\lim_{t\to\infty} x(t) = 0. \tag{4.9}$$

Because $q$ is bounded and $w$ is nonincreasing, (2.11) and (4.9) yield

$$\lim_{t\to\infty} w(t) = 0, \tag{4.10}$$

and so $\lim_{t\to\infty} x''(t) = 0$. Consider the function $F$ defined by

$$F(t) = F(x)(t) = 2x''(t)x(t) - \left(x'(t)\right)^2 + q(t)x^2(t). \tag{4.11}$$

Then we have for $t \geq t_1$

$$F'(t) = -2r(t)f(x(\varphi(t)))x(t) + q'(t)x^2(t) < 0. \tag{4.12}$$

Since $F$ is decreasing and $q$ is bounded, in view of (2.10) we obtain $\lim_{t\to\infty} F(t) = 0$ and so $\lim_{t\to\infty} x'(t) = 0$. $\square$

*Remark 4.2.* Theorem 4.1 generalizes Theorem 1.1. In [11, Theorem 3.8] the first part of Theorem 4.1 is proved, by a different method, for the particular (1.5) under assumptions (4.2) and (4.3).

Applying Theorems 3.1 and 4.1 we get the following result which extends Theorem 1.1 for (1.9).

**Corollary 4.3.** *Assume* (4.1) *and that $q$ is continuously differentiable for large $t$ satisfying* (4.2) *then, condition* (4.3) *is necessary and sufficient in order to every solution of* (1.1) *is either oscillatory or satisfies* (4.4).

From Theorem 4.1 and its proof, we have the following results.

**Corollary 4.4.** *Assume that $q$ is continuously differentiable for large $t$ satisfying* (4.2). *If* (4.1) *and* (4.3) *are satisfied, then for any nonoscillatory solution of* (1.1) *one has*

$$F(t) > 0, \quad F'(t) < 0 \quad \text{for large } t, \tag{4.13}$$

*where $F$ is defined by* (4.11).

*In addition, if $\varphi(t) \equiv t$, then $x$ is a nonoscillatory solution of* (1.1) *if and only if $F(t) > 0$ for large $t$.*

*Proof.* By Theorem 4.1, any nonoscillatory solution of (1.1) is either of Type I or Type III. If $x$ is of Type I, then $\lim_{t\to\infty} x^{(i)}(t) = 0$ for $i = 0, 1, 2$ and so $\lim_{t\to\infty} F(t) = 0$. Hence, by using the argument in the proof of Theorem 4.1, we obtain $F'(t) < 0$ for large $t$.

If $x$ is of Type III, then there exists a sequence $\{t_k\}$ of zeros of $x'$ tending to $\infty$ such that $x''(t_k) \geq 0$ and so $F(t_k) > 0$. Similarly, reasoning as in the proof of Theorem 4.1, one has that $F'(t) < 0$ for this solution. Thus, in both cases, the monotonicity of $F$ gives (4.13).

Finally, if $\varphi(t) \equiv t$ and $x$ is an oscillatory solution, then $f(x(t))x(t) \geq 0$ for any $t$ and so $F'(t) \leq 0$ for large $t$. Since $F(\tau_k) \leq 0$ for some sequence $\{\tau_k\}$, we get $F(t) < 0$ for large $t$. □

**Theorem 4.5.** *Let $\varphi(t) \geq t$ and $q$ is continuously differentiable for $t \geq 0$ satisfying* (4.2). *If* (4.1) *and* (4.3) *are satisfied, for any nonoscillatory solution $x$ of* (1.1) *defined for $t \geq t_x$, one has*

$$x(t) \neq 0 \quad \text{for } t \geq t_x. \tag{4.14}$$

*In particular, any continuable solution with zero is oscillatory.*

*Proof.* Assume that $x(t) > 0$ for $t > T > t_x$ and $x(T) = 0$. Since $\varphi(t) \geq t$, we have $f(x(\varphi(t)))x(t) > 0$ for $t > T$. For the function $F$ defined by (4.11), from (4.12) we obtain $F'(t) \leq 0$ for $t > T$. Because $F(T) \leq 0$, we obtain $F(t) \leq 0$ for $t > T$. This is a contradiction with Corollary 4.4 and the assertion follows. □

We conclude this section with the following result on the continuability of solutions of (1.1).

**Proposition 4.6.** *Assume $q$ is continuously differentiable for $t \geq 0$ satisfying* (4.2). *Then any solution of* (1.1) *which is not continuable at infinity has an infinite number of zeros.*

*Proof.* Let $x$ be a solution of (1.1) defined on $[t_x, \tau), \tau < \infty$. If $x$ has a finite number of zeros on $[t_x, \tau)$, then there exists $T, t_x < T < \tau$, such that $x(t) \neq 0$, $x(\varphi(t)) \neq 0$ on $J = [T, \tau)$. Suppose $x(t) > 0$ on $J$. Consider the function $w$ given by (2.14). Then $w'(t) \leq 0$.

Now consider these two cases. (a) Let $x''(t) > 0$ in the left neighborhood of $\tau$ or $x''$ changes sign in this neighborhood. Then $0 \leq w(t) \leq w(T)$ for $t \in J$. Integrating this inequality on $J$ we obtain

$$x'(t) \leq x'(T) + w(T)(\tau - T) - \int_T^t q(s)x(s)ds \leq x'(T) + w(T)(\tau - T) = c, \tag{4.15}$$

Thus $x(t) \leq x(T) + c(t - T)$ and $x$ is bounded on $J$. From here and the boundedness of $w$ we have that $x''$ is bounded on $J$. Since $x^{(i)}$ $(i = 0, 1, 2)$ are bounded, the solution $x$ can be extended beyond $\tau$, which is a contradiction.

(b) Let $x''(t) < 0$ in the left neighborhood of $\tau$, say $J_1 \subset J$. Then $x'$ is decreasing and $x$ is bounded on $J_1$. We claim that $x'(t)$ does not tend to $-\infty$ as $t \to \tau-$. Indeed, if $x'(\tau-) = -\infty$, then there exists a sequence $\{t_k\}, t_k \to \tau-$, such that $\lim_k x'''(t_k) = -\infty$, which is a contradiction with the boundedness of $x$. Thus $x'''$ is bounded and so $x''$ is bounded on $J_1$ and $x$ is continuable, which is again a contradiction. □

*Remark 4.7.* Theorem 4.5 and Proposition 4.6 improve [11, Corollary 3.4] and [11, Theorem 1.2] for (1.5), respectively.

## 5. Property A

In this section we study Property A for (1.1), that is, any solution either is oscillatory or tends to zero as $t \to \infty$.

We start with the following result on the boundedness of nonoscillatory solutions, which extends [11, Theorem 3.12] for (1.5).

**Theorem 5.1.** *Assume that $q$ is continuously differentiable for large $t$ and there exists a constant $q_0$ such that*

$$q(t) \geq q_0 > 0, \quad q'(t) \leq 0 \quad \text{for large } t. \tag{5.1}$$

*Then every nonoscillatory solution $x$ of* (1.1) *satisfies*

$$\limsup_{t \to \infty} \left| x^{(i)}(t) \right| < \infty, \quad i = 0, 1, 2. \tag{5.2}$$

*Proof.* Without loss of generality, assume that $x$ is a solution of (1.1) such that $x(t) > 0$, $x(\varphi(t)) > 0$ for $t \geq T \geq 0$. Consider the function $w$ defined by (2.14). Then

$$w'(t) = -r(t)f(x(\varphi(t))) + q'(t)x(t) \leq q'(t)x(t) \leq 0, \tag{5.3}$$

that is, $w$ is nonincreasing. By Lemma 2.4, $\lim_{t\to\infty} w(t) = w_0 \geq 0$. Moreover, from (5.3) we have

$$\int_T^\infty |q'(t)|x(t)dt \leq -\int_T^\infty w'(t)dt = w(T) - w_0, \tag{5.4}$$

or

$$\lim_{t\to\infty}\int_t^\infty |q'(t)|x(t)dt = 0\ , \qquad \lim_{t\to\infty}\int_t^\infty |w'(t)|dt = 0. \tag{5.5}$$

Let $u, v$ be two linearly independent solutions of (1.3) with Wronskian $d$. Because all solutions of (1.3) together with their derivatives are bounded, see, for example, [19, Ch. XIV, Theorem 3.1], from (5.5) we get

$$\lim_{t\to\infty}\int_t^\infty (w'(s) - q'(s)x(s))(u(s)v(t) - u(t)v(s))ds = 0. \tag{5.6}$$

Moreover, from the equality

$$x'''(t) + q(t)x'(t) = w'(t) - q'(t)x(t), \tag{5.7}$$

using the variation of constants formula, we have

$$x'(t) = c_1u(t) + c_2v(t) - \frac{1}{d}\int_t^\infty (w'(s) - q'(s)x(s))(u(s)v(t) - u(t)v(s))ds, \tag{5.8}$$

where $c_1$ and $c_2$ are real constants. Thus $x'$ is bounded. Moreover,

$$x''(t) = c_1u'(t) + c_2v'(t) - \int_t^\infty (w'(s) - q'(s)x(s))(u(s)v'(t) - u'(t)v(s))ds. \tag{5.9}$$

Since $w$ is bounded, $u, v, u', v'$ are bounded, and (5.5) holds, $x''$ is bounded, too. From here and (2.14) the boundedness of $x$ follows. □

The next result describes the asymptotic properties of nonoscillatory solutions and will be used later.

**Theorem 5.2.** *Assume* (4.1), (4.3), *and q is continuously differentiable satisfying* (5.1). *If*

$$\liminf_{u\to 0}\frac{f(u)}{u} > 0, \tag{5.10}$$

$$\int_0^\infty r(t)|t - \varphi(t)|dt < \infty, \tag{5.11}$$

*then any nonoscillatory solution $x$ of* (1.1) *defined on* $[t_x, \infty)$ *satisfies*

$$\int_{t_x}^{\infty} r(t)x^2(t)dt < \infty. \tag{5.12}$$

*If, moreover,*

$$\frac{q(t)}{r(t)} \le C < \infty \quad \text{for large } t, \tag{5.13}$$

*then $x$ satisfies*

$$\int_{t_x}^{\infty} \left(x'(t)\right)^2 dt < \infty. \tag{5.14}$$

*Proof.* Theorem 4.1 yields that $x$ is not of Type II. Then there exists $T \ge t_x$ such that for $t \ge T$ either

$$x(t) > 0, \qquad x(\varphi(t)) > 0, \qquad x'(t) \le 0, \quad \lim_{t\to\infty} x(t) = 0, \tag{5.15}$$

or

$$x(t) > 0, \qquad x(\varphi(t)) > 0, \quad x' \text{ changes its sign}. \tag{5.16}$$

By Theorem 5.1, $x$ and $x'$ are bounded, that is, there exists $m > 0$ such that $|x(t)| \le m, |x'(t)| \le m$ for $t \ge t_x$. By Corollary 4.4, the function $F$, given by (4.11), is positive for large $t$, say $t \ge T$, and has a finite limit as $t \to \infty$. Moreover

$$\begin{aligned} \infty > F(T) - F(\infty) = \int_T^{\infty} F'(t)dt &\ge 2\int_T^{\infty} r(t)x(t)f(x(\varphi(t)))dt \\ &\ge 2K\int_T^{\infty} r(t)x(t)x(\varphi(t))dt. \end{aligned} \tag{5.17}$$

From the boundedness of $x'$ we get

$$x(\varphi(t)) = x(t) - x'(\xi)(t - \varphi(t)) \ge x(t) - m|t - \varphi(t)|, \tag{5.18}$$

and so

$$\int_T^{\infty} r(t)x(t)x(\varphi(t))dt \ge \int_T^{\infty} r(t)x^2(t)dt - m^2\int_{T_1}^{\infty} r(t)|t - \varphi(t)|dt. \tag{5.19}$$

Hence, in view of (5.11) and (5.17), we obtain (5.12).

In order to complete the proof, define for $t \geq T$

$$z(t) = 2x'(t)x(t) - 3\int_T^t \left(x'(s)\right)^2 ds + \int_T^t q(s)x^2(s)ds. \tag{5.20}$$

Then $z'(t) = F(t) > 0$ for $t \geq T$ and so $z$ is increasing. If $x$ satisfies either (5.15) or (5.16), there exists a sequence $\{\tau_k\}$ tending to $\infty$ such that

$$x(\tau_k)x'(\tau_k) \leq 0\,, \quad k = 1, 2, \ldots. \tag{5.21}$$

Moreover,

$$\begin{aligned}\int_T^\infty q(s)x^2(s)ds &= \int_T^\infty \frac{q(s)}{r(s)} r(s)x^2(s)ds \\ &\leq C\int_T^\infty r(s)x^2(s)ds = C_1 < \infty.\end{aligned} \tag{5.22}$$

Now, in view of (5.20) and (5.21), we obtain

$$\begin{aligned}3\int_T^{\tau_k} \left(x'(t)\right)^2 dt &= -z(\tau_k) + 2x'(\tau_k)x(\tau_k) + \int_T^{\tau_k} q(s)x^2(s)ds \\ &\leq -z(\tau_1) + C_1,\end{aligned} \tag{5.23}$$

and so (5.14) is satisfied. □

Using the previous results, we obtain a sufficient condition for property A.

**Theorem 5.3.** *Assume* (4.1), (5.10), (5.11), *and $q$ is continuously differentiable satisfying* (5.1). *If there exists $M_0 > 0$ such that*

$$r(t) \geq M_0 \quad \textit{for large } t, \tag{5.24}$$

*then* (1.1) *has property A, that is, any nonoscillatory solution $x$ of* (1.1) *satisfies*

$$\lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 0, 1, 2. \tag{5.25}$$

*Proof.* Let $x$ be a nonoscillatory solution of (1.1) defined for $t \geq t_x \geq 0$. Because (5.24) implies (4.3), by Theorem 4.1, $x$ is of Type I or Type III. If $x$ is of Type I, the assertion follows applying again Theorem 4.1. Now let $x$ be of Type III and assume $x(t) > 0$, $x(\varphi(t)) > 0$ for $t \geq T \geq t_x$. By applying Theorem 5.2, in view of (5.24), we obtain

$$\int_{t_x}^\infty x^2(t)dt < \infty, \qquad \int_{t_x}^\infty \left(x'(t)\right)^2 dt < \infty. \tag{5.26}$$

According to the inequality of Nagy, see, for example, [20, V, § 2, Theorem 1], we get

$$\lim_{t\to\infty} x(t) = 0. \tag{5.27}$$

Now consider the function $w$ given by (2.14). Then $w'(t) = -r(t)f(x(\varphi(t))) + q'(t)x(t) < 0$ for $t \geq T$. For a sequence $\{t_k\}$ of zeros of $x''$ tending to $\infty$, we have $\lim_{k\to\infty} w(t_k) = 0$. Hence, $\lim_{t\to\infty} w(t) = 0$ and

$$\lim_{t\to\infty} x''(t) = 0. \tag{5.28}$$

By Corollary 4.4, the function $F$, given by (4.11), is positive decreasing for large $t$. From here, (2.10), (5.27), and (5.28) we obtain $\lim_{t\to\infty} x'(t) = 0$. □

**Corollary 5.4.** *Suppose assumptions of Theorem 5.3, $\varphi(t) = t$, and for some $K > 0$*

$$\left|f(u)\right| \geq K|u| \quad \textit{for } u \in \mathbb{R}. \tag{5.29}$$

*then any nonoscillatory solution $x$ of* (1.1) *satisfies $x(t)x'(t) < 0$ for large $t$.*

*Proof.* The assertion follows from Theorem 5.3 and [1, Theorem 6]. □

## Acknowledgments

The first and third authors are supported by the Research Project 0021622409 of the Ministry of Education of the Czech Republic and Grant 201/08/0469 of the Czech Grant Agency. The fourth author is supported by the Research Project PRIN07-Area 01, no. 37 of the Italian Ministry of Education.

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