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Research Article

Hyers-Ulam Stability of the Delay Equation $y'(t) = \lambda y(t - \tau)$

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We investigate the approximate solutions $y : [-\tau, \infty) \to \mathbb{R}$ of the delay differential equation $y'(t) = \lambda y(t - \tau)(t \in [0, \infty))$ with an initial condition, where $\lambda > 0$ and $\tau > 0$ are real constants. We show that they can be "approximated" by solutions of the equation that are constant on the interval $[-\tau, 0]$ and, therefore, have quite simple forms. Our results correspond to the notion of stability introduced by Ulam and Hyers.

1. Introduction

While investigating real world phenomena we very often use equations. In general, it is well known that those equations are satisfied, however, with some error. Sometimes that error is neglected and it is believed that this will have only a minor influence on the final outcome. Since it is not always the case, it seems to be of interest to know when we can neglect the error, why, and to what extent.

One of the tools for systematic treatment of the problem described above seems to be the notion of Hyers-Ulam stability and some ideas inspired by it. That notion has not actually been made very precise so far, and we still seek a better understanding of it (see, e.g., [1, 2]). But, roughly speaking, we might say that it is connected with the investigation of the following question: *when is a function satisfying an equation with some "small" (in some sense) error "close" to a solution of that equation*?

The study of the stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to Ulam's problem (see [3, 4]). Thereafter, Hyers and Ulam (see, e.g., [4–8]), but also several other authors (see, e.g., [9–12]), attempted to study the stability problem for various functional equations. In particular, we should mention here the well-known paper [13] of Th.M. Rassias, in which he actually rediscovered the result of Aoki [9] (cf. [14]), and which has significantly influenced research of numerous mathematicians (see [15–21] and the references therein).

Assume that X and Y are normed spaces and that I is an open subset of X. Let $\mathcal{F}$ be a class of differentiable functions mapping I into Y. If for any real $\varepsilon \geq 0$ and any function $y \in \mathcal{F}$ satisfying the differential inequality

$$\sup_{t\in I}\left\|\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + h(t)\right\| \leq \varepsilon, \tag{1.1}$$

there exists a solution $y_0 : I \to Y$ of the differential equation

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + h(t) = 0 \tag{1.2}$$

such that

$$\sup_{t\in I}\|y(t) - y_0(t)\| \leq K(\varepsilon), \tag{1.3}$$

(where $K(\varepsilon)$ depends on ε only), then we say that the above differential equation is Hyers-Ulam stable in the class of function $\mathcal{F}$. We may use this terminology for other differential equations. For more detailed definitions of the Hyers-Ulam stability and some discussions and critiques of that subject, refer to [1, 2, 4, 13, 15, 16, 18–21].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [22, 23]) in the sense described above. Here, let us recall a result of Alsina and Ger (see [24]):

If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$ for any $t \in I$.

An analogous result for the Banach space valued functions has been proved by Takahasi et al. [25]. For some further examples of investigations of such kind of stability of differential equations see also [26–35].

In what follows, $\lambda > 0$ and $\tau > 0$ stand for fixed real constants, unless clearly stated otherwise. Moreover, $\mathcal{F}$ denotes the family of all continuous functions mapping the real interval $[-\tau, \infty)$ into $\mathbb{R}$ which are continuously differentiable in $[0, \infty)$.

In this paper, motivated by the above-mentioned outcomes on Hyers-Ulam stability, we prove that a somewhat similar type of stability is valid for the delay differential equation

$$y'(t) = \lambda y(t - \tau) \tag{1.4}$$

in the class of functions $\mathcal{F}$ with an initial condition. More precisely, for given real numbers ε_1, ε_2 and continuous functions $\xi_1, \xi_2 : [-\tau, 0] \to \mathbb{R}$ with $\varepsilon_1 < \varepsilon_2$ and $\xi_1(t) < \xi_2(t)$ for $t \in [-\tau, 0]$, we describe a behavior of solutions of the following problem (the delay differential inequality with an initial condition)

$$\varepsilon_1 \le y'(t) - \lambda y(t-\tau) \le \varepsilon_2, \quad t \ge 0, \tag{1.5}$$

$$\xi_1(t) \le y(t) - y(0) \le \xi_2(t), \quad -\tau \le t \le 0, \tag{1.6}$$

and compare them with solutions of the delay equation (1.4) that are constant on the interval $[-\tau, 0]$ (and therefore have quite simple forms). Functions satisfying those two inequalities may be considered to be approximate solutions of (1.4).

2. An Auxiliary Theorem

Let us recall a description of a class of solutions of (1.4).

Remark 2.1. It is known that if $\lambda \neq 0$ and $\tau > 0$ are real constants, then the general solution $y : [-\tau, \infty) \to \mathbb{R}$ of the delay differential equation (1.4), which is constant on the interval $[-\tau, 0]$, is given by

$$y(t) = \alpha \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!}, \quad t \in [-\tau, \infty), \tag{2.1}$$

where α is an arbitrary real number and $[t/\tau]$ denotes the greatest integer that is less than or equal to t/τ.

The following theorem will be very useful in the sequel.

Theorem 2.2. *Let z, $\phi \in \mathcal{F}$. Assume that*

$$z'(t) - \lambda z(t-\tau) \le \phi'(t) - \lambda \phi(t-\tau), \quad t \ge 0, \tag{2.2}$$

$$z(t) - z(0) \le \phi(t) - \phi(0), \quad t \in [-\tau, 0]. \tag{2.3}$$

There exist unique solutions $\hat{z}, \hat{\phi} \in \mathcal{F}$ of (1.4) *satisfying the initial conditions*

$$\hat{\phi}(t) = \phi(0), \quad t \in [-\tau, 0), \tag{2.4}$$

$$\hat{z}(t) = z(0), \quad t \in [-\tau, 0), \tag{2.5}$$

such that

$$z(t) - \hat{z}(t) \le \phi(t) - \hat{\phi}(t), \quad t \ge 0. \tag{2.6}$$

Proof. Write $y := z - \phi$, $y_0 := z(0) - \phi(0)$ and

$$\hat{y}(t) = y_0 \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1}(t - n\tau)^{n+1}}{(n+1)!}, \quad t \in [-\tau, \infty). \tag{2.7}$$

Clearly,

$$\hat{y}(t) = y_0, \quad t \in [-\tau, 0), \tag{2.8}$$

and by (2.2),

$$y'(t) \le \lambda y(t - \tau), \quad t \ge 0. \tag{2.9}$$

By the induction on $[t/\tau]$ we prove that, for each $t \ge 0$,

$$y(t) \le \hat{y}(t). \tag{2.10}$$

Let $[t/\tau] = 0$ (i.e., $0 \le t < \tau$). It follows from (2.3) and (2.9) that $y'(t) \le \lambda y_0$. If we integrate each term from 0 to t, then we get

$$y(t) - y_0 \le \lambda y_0 t, \tag{2.11}$$

which together with the definition of $\hat{y}$ gives (2.10) (for $t \in [0, \tau)$).

Now, take a nonnegative integer m and assume that inequality (2.10) is true for $[t/\tau] = m$, that is,

$$y(t) \le y_0 \sum_{n=-1}^{m} \frac{\lambda^{n+1}(t - n\tau)^{n+1}}{(n+1)!}, \quad m\tau \le t < m\tau + \tau. \tag{2.12}$$

We are to show that this is also the case for $[t/\tau] = m + 1$ (i.e., for $m\tau + \tau \le t < m\tau + 2\tau$).

Due to (2.9), we have

$$\int_{m\tau+\tau}^{t} y'(u)du \le \lambda \int_{m\tau+\tau}^{t} y(u - \tau)du, \tag{2.13}$$

and hence

$$y(t) \le \lambda \int_{m\tau}^{t-\tau} y(v)dv + y(m\tau + \tau) \tag{2.14}$$

for all $m\tau + \tau \le t < m\tau + 2\tau$.

Substitute $t-\tau$ for t in (2.12) (with $m\tau+\tau \le t < m\tau+2\tau$) and integrate each term of the resulting inequalities from $m\tau+\tau$ to t. Then, we have

$$\int_{m\tau}^{t-\tau} y(v)dv \le y_0 \sum_{n=0}^{m+1} \frac{\lambda^n (t-n\tau)^{n+1}}{(n+1)!} - y_0 \sum_{n=0}^{m+1} \frac{\lambda^n (m\tau - n\tau + \tau)^{n+1}}{(n+1)!} \tag{2.15}$$

for any $m\tau+\tau \le t < m\tau+2\tau$. Moreover, since y is continuous in $[0,\infty)$,

$$\lim_{t\to m\tau+\tau} y(t) = y(m\tau+\tau) \tag{2.16}$$

and consequently (2.12) yields

$$y(m\tau+\tau) \le y_0 \sum_{n=-1}^{m} \frac{\lambda^{n+1} (m\tau - n\tau + \tau)^{n+1}}{(n+1)!}. \tag{2.17}$$

Thus, from (2.15) and (2.17), we obtain

$$\lambda \int_{m\tau}^{t-\tau} y(v)dv + y(m\tau+\tau) \le y_0 \sum_{n=-1}^{m+1} \frac{\lambda^{n+1} (t-n\tau)^{n+1}}{(n+1)!} \tag{2.18}$$

for any $m\tau+\tau \le t < m\tau+2\tau$. Hence, by (2.14) and the definition of $\widehat{y}$, (2.10) holds true for $[t/\tau] = m+1$ (i.e., $m\tau+\tau \le t < m\tau+2\tau$), as well.

Thus we have proved that (2.10) is valid for all $t \ge 0$. Define

$$\begin{aligned} \widehat{\phi}(t) &= \phi(0) \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1} (t-n\tau)^{n+1}}{(n+1)!}, \quad t \ge -\tau, \\ \widehat{z}(t) &= z(0) \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1} (t-n\tau)^{n+1}}{(n+1)!}, \quad t \ge -\tau. \end{aligned} \tag{2.19}$$

Clearly, (2.4) and (2.5) are valid. Moreover, $\widehat{y} = \widehat{z} - \widehat{\phi}$ and hence, (2.6) follows from (2.10).

The uniqueness of $\widehat{\phi}$ and $\widehat{z}$ follows from Remark 2.1. □

3. The Main Stability Results

Now, we present some corollaries that are immediate consequences of Theorem 2.2. They contain stability results for (1.4).

Corollary 3.1. *Let $y,\ \psi_1,\ \psi_2 \in \mathcal{F}$. Assume that $\psi_1(0) = \psi_2(0) = 0$,*

$$\psi_1'(t) - \lambda\psi_1(t-\tau) \le y'(t) - \lambda y(t-\tau) \le \psi_2'(t) - \lambda\psi_2(t-\tau), \quad t \ge 0, \tag{3.1}$$

$$\psi_1(t) \le y(t) - y(0) \le \psi_2(t), \quad t \in [-\tau, 0]. \tag{3.2}$$

Then there is a unique solution $\widehat{y} \in \mathcal{F}$ *of* (1.4) *such that* $\widehat{y}(t) = y(0)$ *for* $t \in [-\tau, 0)$ *and*

$$\varphi_1(t) \le y(t) - \widehat{y}(t) \le \varphi_2(t), \quad t \ge 0. \tag{3.3}$$

Proof. Observe that (3.2) implies

$$\varphi_1(t) - \varphi_1(0) \le y(t) - y(0) \le \varphi_2(t) - \varphi_2(0), \quad t \in [-\tau, 0]. \tag{3.4}$$

Hence, it is enough to use Theorem 2.2, first with $z = \varphi_1$, $\phi = y$, and next with $z = y$, $\phi = \varphi_2$, where

$$\widehat{\varphi}_i(t) = \varphi_i(0) \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1}(t - n\tau)^{n+1}}{(n+1)!} = 0, \quad t \in [-\tau, \infty), \tag{3.5}$$

for $i = 1, 2$. □

Remark 3.2. Observe that, for $\lambda > 0$, the inequality in the proof of Corollary 3.1 implies that the estimation (3.3) is actually valid for all $t \ge -\tau$.

It is interesting whether analogues of Corollary 3.1 and our further results can be obtained also for $\lambda < 0$.

The next corollary is a particular case of Corollary 3.1 and corresponds to the classical Hyers-Ulam stability results.

Corollary 3.3. *Let* $y, \varphi_2 \in \mathcal{F}$. *Suppose that* $\varphi_2(0) = 0$,

$$\begin{gathered} \left|y'(t) - \lambda y(t - \tau)\right| \le \varphi_2'(t) - \lambda \varphi_2(t - \tau), \quad t \ge 0, \\ \left|y(t) - y(0)\right| \le \varphi_2(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{3.6}$$

Then there exists a unique solution $\widehat{y} \in \mathcal{F}$ *of* (1.4) *satisfying the initial condition:*

$$\widehat{y}(t) = y(0), \quad t \in [-\tau, 0), \tag{3.7}$$

and such that

$$\left|y(t) - \widehat{y}(t)\right| \le \varphi_2(t), \quad t \ge 0. \tag{3.8}$$

Proof. Clearly we have

$$-\varphi_2(t) + \varphi_2(0) \le y(t) - y(0) \le \varphi_2(t) - \varphi_2(0), \quad t \in [-\tau, 0]. \tag{3.9}$$

Hence from Theorem 2.2 with $\varphi_1 := -\varphi_2$, analogously as in the proof of Corollary 3.1, we get the statement. □

Note that the functions φ_1, φ_2 in Theorem 2.2 and Corollary 3.1 can be constant only if $\varphi_i(t) \equiv 0$, $i = 1, 2$. Therefore the case of problem (1.5), (1.6) is a bit more complicated. It is described by the following theorem (in particular, note that (1.6) implies (3.10)).

Theorem 3.4. *Let $\varepsilon_1, \varepsilon_2$ be real numbers with $\varepsilon_1 \leq \varepsilon_2$ and $\xi_1, \xi_2 \in \mathcal{F}$ be solutions of* (1.4). *Suppose that $y \in \mathcal{F}$ satisfy inequality* (1.5) *and*

$$\xi_1(t) - \xi_1(0) \leq y(t) - y(0) \leq \xi_2(t) - \xi_2(0), \quad t \in [-\tau, 0]. \tag{3.10}$$

Then there exists unique solutions $\hat{y}$, $\hat{\xi}_1$, $\hat{\xi}_2 \in \mathcal{F}$ of (1.4) *such that*

$$\begin{gathered} \hat{y}(t) = y(0), \qquad \hat{\xi}_i(t) = \xi_i(0) - \frac{\varepsilon_i}{\lambda}, \quad t \in [-\tau, 0),\ i = 1, 2, \\ \xi_1(t) - \hat{\xi}_1(t) - \frac{\varepsilon_1}{\lambda} \leq y(t) - \hat{y}(t) \leq \xi_2(t) - \hat{\xi}_2(t) - \frac{\varepsilon_2}{\lambda}, \quad t \geq 0. \end{gathered} \tag{3.11}$$

Proof. Let

$$\varphi_i := \xi_i - \frac{\varepsilon_i}{\lambda}, \quad i = 1, 2. \tag{3.12}$$

Then

$$\varphi_i'(t) - \lambda \varphi_i(t - \tau) = \varepsilon_i, \quad i = 1, 2. \tag{3.13}$$

Moreover (3.10) yields

$$\varphi_1(t) - \varphi_1(0) \leq y(t) - y(0) \leq \varphi_2(t) - \varphi_2(0), \quad t \in [-\tau, 0]. \tag{3.14}$$

Consequently, by Theorem 2.2, there exists unique solutions $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{y} \in \mathcal{F}$ of (1.4) such that $\hat{\varphi}_i(t) = \varphi_i(0)$ and $\hat{y}(t) = y(0)$ for $t \in [-\tau, 0)$, $i = 1, 2$, and

$$\varphi_1(t) - \hat{\varphi}_1(t) \leq y(t) - \hat{y}(t) \leq \varphi_2(t) - \hat{\varphi}_2(t), \quad t \geq 0. \tag{3.15}$$

This and Remark 2.1 yield the statement. □

A particular case of Theorem 3.4 is the subsequent corollary.

Corollary 3.5. *Let $\varepsilon \geq 0$ be a real constant, $y, \xi \in \mathcal{F}$, and ξ be a solution of* (1.4). *Suppose that $|y'(t) - \lambda y(t - \tau)| \leq \varepsilon$ for $t \geq 0$ and there is a real number u_0 with $|y(t) - u_0| \leq \xi(t)$ for $t \in [-\tau, 0]$. Then there are unique solutions $\hat{\xi}_0, \hat{y} \in \mathcal{F}$ of* (1.4) *such that* (3.7) *holds,*

$$\hat{\xi}_0(t) = \xi(0) - \frac{\varepsilon}{\lambda}, \quad t \in [-\tau, 0), \tag{3.16}$$

and, for each $t \geq 0$,

$$\left|y(t) - \widehat{y}(t)\right| \leq \xi(t) - \widehat{\xi}_0(t) - \frac{\varepsilon}{\lambda} = \xi(t) - \xi(0) - \left(\xi(0) - \frac{\varepsilon}{\lambda}\right) \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t - n\tau)^{n+1}}{(n+1)!}. \tag{3.17}$$

Proof. It is enough to use Theorem 3.4, with $\varepsilon_1 := -\varepsilon$ and $\varepsilon_2 := \varepsilon$, and Remark 2.1. □

4. Some Immediate Consequences

Clearly, if $\xi(0) = 0$, then (3.17) has the following simpler form:

$$\left|y(t) - \widehat{y}(t)\right| \leq \xi(t) + \frac{\varepsilon}{\lambda} \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t - n\tau)^{n+1}}{(n+1)!}. \tag{4.1}$$

If $\xi(t) \equiv 0$ (i.e., $y(t) = u_0$ for $t \in [-\tau, 0]$) and $\lambda = \tau = 1$, then the inequality (3.17) takes the following form:

$$\left|y(t) - y_0 \sum_{n=-1}^{[t]} \frac{(t-n)^{n+1}}{(n+1)!}\right| \leq \varepsilon \sum_{n=0}^{[t]} \frac{(t-n)^{n+1}}{(n+1)!}, \quad t \geq 0. \tag{4.2}$$

From Theorem 3.4 we can derive an estimation of solutions of the equation

$$y'(t) = \lambda\left(y(t-\tau) + d(t-\tau)\right), \quad t \in [0, \infty), \tag{4.3}$$

where $d \in \mathcal{F}$ is fixed. Namely we have the following result.

Corollary 4.1. *Let* $\xi_1, \xi_2 \in \mathcal{F}$ *be solutions of* (1.4) *and* $y \in \mathcal{F}$ *be a solution of* (4.3) *satisfying condition* (3.10)*. Let* $\varepsilon_1 := \inf_{t \in [0,\infty)} d'(t)$ *and* $\varepsilon_2 := \sup_{t \in [0,\infty)} d'(t)$*. Then, for each* $t \geq 0$,

$$\begin{aligned} &\xi_1(t) - \xi_1(0) + \left(y(0) + d(0) + \frac{\varepsilon_1}{\lambda} - \xi_1(0)\right) \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!} \\ &\quad \leq y(t) + d(t) \leq \xi_2(t) - \xi_2(0) + \left(y(0) + d(0) + \frac{\varepsilon_2}{\lambda} - \xi_2(0)\right) \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!}. \end{aligned} \tag{4.4}$$

Proof. Assume that $\varepsilon_1 > -\infty$. Write $\phi := y + d$ and $z := \xi_1 - (\varepsilon_1/\lambda)$. Then $\phi'(t) - \lambda\phi(t-\tau) = \phi'(t)$ for $t \geq 0$. Consequently (2.2) holds true. Further, (3.10) implies (2.3). Now, it is enough to use Theorem 2.2.

If $\varepsilon_2 < \infty$, we use Theorem 2.2 with $z := y + d$ and $\phi := \xi_2 - (\varepsilon_2/\lambda)$. □

We end the paper with some estimation of solutions of a more general equation

$$y'(t) = \lambda y(t-\tau) + b(\mathcal{T} y(t)), \quad t \in [-\tau, \infty), \tag{4.5}$$

where Z is a nonempty set, $I = [-\tau, \infty)$, and $\mathcal{T}$ is an operator that maps the family of function $\mathbb{R}^I$ into the family of function Z^I, and $b : Z \to \mathbb{R}$.

Corollary 4.2. *Let $\xi_1, \xi_2 \in \mathcal{F}$ be solutions of* (1.4) *and $y \in \mathcal{F}$ be a solution of* (4.5) *satisfying condition* (3.10). *Let $\varepsilon_1 = \inf_{t\in\mathbb{R}} b(t)$ and $\varepsilon_2 = \sup_{t\in\mathbb{R}} b(t)$. Then, for each $t \geq 0$,*

$$\begin{aligned} &\xi_1(t) - \xi_1(0) + \left(y(0) + \frac{\varepsilon_1}{\lambda} - \xi_1(0)\right) \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!} \\ &\quad \leq y(t) \leq \xi_2(t) - \xi_2(0) + \left(y(0) + \frac{\varepsilon_2}{\lambda} - \xi_2(0)\right) \sum_{n=0}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!}. \end{aligned} \tag{4.6}$$

Proof. Note that $\varepsilon_1 \leq y'(t) - \lambda y(t-\tau) = b(\mathcal{T} y(t)) \leq \varepsilon_2$ for $t \geq 0$. So it is enough to use Theorem 2.2 analogously as in the proof of Corollary 4.1, first with $\phi = y$ and $z := \xi_1 - (\varepsilon_1/\lambda)$, and next with $z := y$ and $\phi := \xi_2 - (\varepsilon_2/\lambda)$. □

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