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Research Article

Quasinormality and Numerical Ranges of Certain Classes of Dual Toeplitz Operators

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The study of dual Toeplitz operators was elaborated by Stroethoff and Zheng (2002), where various corresponding algebraic and spectral properties were established. In this paper, we characterize numerical ranges of certain classes of dual Toeplitz operators. Moreover, we introduce the analog of Halmos' fifth classification problem for quasinormal dual Toeplitz operators. In particular, we show that there are no quasinormal dual Toeplitz operators with bounded analytic or coanalytic symbols which are not normal.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$, and let dA be the Lebesgue measure on $\mathbb{D}$. The Lebesgue space of (classes of) square summable complex-valued functions is denoted by $\mathcal{L}^2(\mathbb{D}, dA)$. The Bergman space $\mathcal{L}_a^2$ is the Hilbert subspace of $\mathcal{L}^2(\mathbb{D}, dA)$ consisting of analytic functions. The orthogonal complement of $\mathcal{L}_a^2$ in $\mathcal{L}^2(\mathbb{D}, dA)$ is denoted by $(\mathcal{L}_a^2)^{\perp}$. The Hilbert space $(\mathcal{L}_a^2)^{\perp}$ is readily seen to be not a reproducing kernel Hilbert space. This is one of the major difficulties that occurs when dealing with this space. A second one is the fact that its elements have no standard common qualities such as analyticity harmonicity, while a lesser difficulty is the complicated form of the corresponding basis.

Despite the difficulties just listed, Stroethoff and Zheng in [1, 2] have adopted new techniques to investigate various properties of a class of operators acting on $(\mathcal{L}_a^2)^{\perp}$, namely, dual Toeplitz operators. A dual Toeplitz operator is defined on $(\mathcal{L}_a^2)^{\perp}$ to be a multiplication (by the symbol) followed by a projection onto $(\mathcal{L}_a^2)^{\perp}$. Although dual Toeplitz operators are different from Toeplitz operators in many respects, they do share some properties with them. But surprisingly, dual Toeplitz operators on $(\mathcal{L}_a^2)^{\perp}$ resemble much more Hardy space

Toeplitz operators than Bergman space Toeplitz operators. Lu in [3] and Cheng and Yu in [4] considered dual Toeplitz operators in higher dimensions; while Yu and Wu in [5] considered dual Toeplitz operators in the framework of Dirichlet spaces.

The study of the numerical ranges of Hardy space Toeplitz operators goes back to Brown and Halmos [6]. Subsequent treatment was reconsidered in Halmos' book [7]. Later on, Klein [8] showed that the numerical range depends only on the spectrum of the given Hardy space Toeplitz operator. The Bergman space case was successfully considered only twenty years later by Thukral [9] in case of bounded harmonic symbols. More recently Choe and Lee [10], as well as Gu [11], treat higher-dimensional Bergman space analogs. The case of Bergman space Toeplitz operators with bounded radial symbols has been considered very recently by Wang and Wu [12]. The connection between spectral sets and numerical ranges was considered first by Schreiber [13]. Further investigations had been pursued by Hildebrandt [14] and Clark [15]. The subnormality, and particularly the quasinormality, of Hardy space Toeplitz operators has been discussed chronologically by Itô and Wong [16], Amemiya et al. [17], Abrahamse [18], Cowen and Long [19], Cowen [20, 21], Lee [22], and Yoshino [23]. The case of Bergman space Toeplitz operators has been discussed by Faour in [24] as well as by Jim Gleason in a recent preprint.

Accordingly, in this paper, we mainly investigate qualitative properties of the numerical range of a dual Toeplitz operator. We consider various classes of such operators, such as normal and quasinormal ones. We completely describe the numerical ranges of some of them and establish the main qualitative properties of the numerical ranges of others. We also shed some light on the analog of Halmos' fifth problem on the classification of subnormal Toeplitz operators.

Our paper is organized as follows: in Section 2, we exhibit some preliminary concepts needed in the sequel. Section 3 mainly concerns the description of the numerical ranges of normal dual Toeplitz operators. Section 4 contains the characterization of the numerical ranges of the more general case of nonnormal dual Toeplitz operators with harmonic symbols. In Section 5, we give heuristic proofs of some results based on the concept of lines of support of the numerical range. In Section 6, we briefly discuss the connection between spectral sets and spectra of dual Toeplitz operators. Section 7 is devoted to the quasinormality of dual Toeplitz operators. For the case of dual Toeplitz operators, we introduce and adumbrate the analog of Halmos' fifth problem on the classification of subnormal Toeplitz operators.

2. Preliminaries

Let $\mathbf{T}$ be a bounded operator on a Hilbert space $\mathbb{H}$. Denote its spectrum by $\sigma(\mathbf{T})$. The numerical range is a prototype of the spectrum, and it proves useful whenever information about $\mathbf{T}$ is needed. It is defined by $\mathcal{W}(\mathbf{T}) := \{\langle \mathbf{T}\varphi, \varphi\rangle, \varphi \in \mathbb{H}, \|\varphi\| = 1\}$. The main involved features of the numerical range are as follows. $\mathcal{W}(\mathbf{T})$ is always bounded and convex, (Toeplitz-Hausdorff theorem). Its closure $\overline{\mathcal{W}(\mathbf{T})}$ contains the spectrum $\sigma(\mathbf{T})$. If $\mathcal{W}(\mathbf{T})$ reduces to the singleton $\{\lambda\}$, then $\mathbf{T} = \lambda I$. $\mathcal{W}(\mathbf{T})$ is a linear function of $\mathbf{T}$, that is, $\mathcal{W}(\alpha\mathbf{T} + \beta) = \alpha\mathcal{W}(\mathbf{T}) + \beta$, $\forall \alpha, \beta \in \mathbb{C}$; hence we see that $\omega\mathcal{W}(\mathcal{S}_f) = \mathcal{W}(\mathcal{S}_{\omega f})$, for $\omega \in \mathbb{C}$, and that $\lambda \in \mathcal{W}(\mathcal{S}_f)$ implies that $0 \in \mathcal{W}(\mathcal{S}_{f-\lambda})$. If $\mathcal{W}(\mathbf{T})$ is a subset of the real axis, then $\mathbf{T}$ must be self-adjoint. If $\mathbf{T}$ is normal and $\mathcal{W}(\mathbf{T})$ is closed, then the extreme points of $\mathcal{W}(\mathbf{T})$ are eigenvalues. If $\mathbf{T}$ is normal, the closure of $\mathcal{W}(\mathbf{T})$ is the smallest closed convex set containing the spectrum of $\mathbf{T}$. For further details on the numerical ranges as well as various applications of this pioneering tool in operator theory, see [7, 25–27].

For $f \in \mathcal{L}^\infty(\mathbb{D})$, define the dual Toeplitz operator $\mathcal{S}_f$ to be the operator on $(\mathcal{L}_a^2)^\perp$ given by

$$\mathcal{S}_f : u \in \left(\mathcal{L}_a^2\right)^\perp \longmapsto \mathcal{S}_f(u) = Q(fu) \in \left(\mathcal{L}_a^2\right)^\perp. \tag{2.1}$$

Here $Q = I - \mathcal{P}$ is the familiar orthogonal projection from $\mathcal{L}^2(\mathbb{D}, dA)$ onto $(\mathcal{L}_a^2)^\perp$ and $\mathcal{P}$ is the orthogonal projection from $\mathcal{L}^2(\mathbb{D}, dA)$ onto the Bergman space $\mathcal{L}_a^2$. Moreover, for $f \in \mathcal{L}^\infty(\mathbb{D})$ and $g \in \mathfrak{H}^\infty$, (the algebra of bounded analytic functions) we have

$$\mathcal{S}_{fg} = \mathcal{S}_g\mathcal{S}_f, \qquad \mathcal{S}_{f\overline{g}} = \mathcal{S}_f\mathcal{S}_{\overline{g}}, \tag{2.2}$$

$$\mathcal{S}_f\mathcal{S}_g = \mathcal{S}_{fg} - H_f H_{\overline{g}}^*, \tag{2.3}$$

where $H_f : u \in \mathcal{L}_a^2 \longmapsto H_f u = Q(fu) \in (\mathcal{L}_a^2)^\perp$ is the Hankel operator. If E is a subset of $\mathbb{C}$, then the convex hull of E, denoted by $\mathcal{H}(E)$, is the smallest convex set containing E. Useful properties of the convex hull are well known [26, 28]. For instance, we know that in general the convex hull of an open set is open and the convex hull of a compact set is compact. The following easy property will be also used

Remark 2.1. If A is a bounded subset of a finite dimensional normed space, then we have $\overline{\mathcal{H}(A)} = \mathcal{H}(\overline{A})$. Indeed, observe that $A \subset \overline{A}$ implies that $\mathcal{H}(A) \subset \mathcal{H}(\overline{A})$, whence $\overline{\mathcal{H}(A)} \subset \overline{\mathcal{H}(\overline{A})} = \mathcal{H}(\overline{A})$. Conversely, since $\overline{A} \subset \overline{\mathcal{H}(A)}$, we infer that $\mathcal{H}(\overline{A}) \subset \mathcal{H}(\overline{\mathcal{H}(A)}) = \overline{\mathcal{H}(A)}$, as $\overline{\mathcal{H}(A)}$ is convex.

The following main spectral properties of $\mathcal{S}_f$ are due to Stroethoff and Zheng [1].

Theorem 2.2. *(1) If f is in $\mathcal{L}^\infty(\mathbb{D})$, then $\mathcal{R}(f) \subset \sigma_e(\mathcal{S}_f) \subset \sigma(\mathcal{S}_f) \subset \mathcal{H}(\mathcal{R}(f))$.*

(2) Let f be a continuous real-valued function on $\mathbb{D}$, then $\sigma(\mathcal{S}_f) = \sigma_e(\mathcal{S}_f) = \overline{f(\mathbb{D})}$.

(3) If f is a bounded analytic or coanalytic function on $\mathbb{D}$, then $\sigma(\mathcal{S}_f) = \sigma_e(\mathcal{S}_f) = \overline{f(\mathbb{D})}$.

For our purpose, we now prove the following useful facts about dual Toeplitz operators.

Lemma 2.3. *Let f be in $\mathcal{L}^\infty(\mathbb{D})$. Then, $\mathcal{S}_f$ is self-adjoint if and only if f is real.*

Proof. $\mathcal{S}_f$ is self-adjoint means that $\mathcal{S}_f = \mathcal{S}_f^*$. This is equivalent to the fact that $f = \overline{f}$, since $\|\mathcal{S}_g\| = \|g\|_\infty$, for $g \in \mathcal{L}^\infty(\mathbb{D})$, which is equivalent to the fact that f is real-valued. □

Lemma 2.4. *Let f be in $\mathcal{L}^\infty(\mathbb{D})$. Then, $\mathcal{S}_f \geq 0$ if and only if $f \geq 0$.*

Proof. If $f \geq 0$, then $\langle \mathcal{S}_f g, g\rangle = \langle Q(fg), g\rangle = \langle fg, g\rangle = \int_{\mathbb{D}} f(z)|g(z)|^2 dA(z) \geq 0, \forall g \in (\mathcal{L}_a^2)^\perp$. Conversely, suppose that $\mathcal{S}_f \geq 0$, then in particular its spectrum lies in $[0, \infty)$. By part (1) of Theorem 2.2, we obtain $\mathcal{R}(f) \subset \sigma(\mathcal{S}_f) \subset [0, \infty)$, whence $f \geq 0$. □

Corollary 2.5. *(i) A bounded dual Toeplitz operator with a real spectrum must be self-adjoint.*

(ii) A bounded dual Toeplitz operator with spectrum lying in the positive real half-axis must be positive.

Proof. This result follows from part (1) of Theorem 2.2 and Lemmas 2.3 and 2.4. □

Theorem 2.6. *Let f be a nonconstant bounded harmonic real-valued function on $\mathbb{D}$. Then, the operator $\mathcal{S}_f$ has no eigenvalues.*

Proof. Since in general for a constant λ we have $\mathcal{S}_f - \lambda = \mathcal{S}_{f-\lambda}$ and $f - \lambda$ is harmonic if f is, it suffices to show that $\mathcal{S}_f g = 0$ implies that $g = 0$, $\forall g \in (\mathcal{L}_a^2)^{\perp}$. If $\mathcal{S}_f g = Q(fg) = 0$, then $fg \in \mathcal{L}_a^2$. Let $h \in \mathfrak{H}^\infty(\mathbb{D})$, then $\overline{h}g \in (\mathcal{L}_a^2)^{\perp}$. Indeed, $\|\overline{h}g\|_2 \le \|h\|_\infty \|g\|_2$, and $\langle \overline{h}g, u\rangle = \langle g, hu\rangle = 0$, $\forall u \in \mathcal{L}_a^2$, (as $hu \in \mathcal{L}_a^2$ and $g \in (\mathcal{L}_a^2)^{\perp}$). It follows that $\langle fg, \overline{h}g\rangle = 0$. Taking real parts and noticing that f is real, we see that

$$\mathcal{R}e\left(\left\langle fg, \overline{h}g\right\rangle\right) = \mathcal{R}e\int_{\mathbb{D}} fg(h\overline{g})\,dA = \int_{\mathbb{D}} f|g|^2 \mathcal{R}e(h)\,dA = 0. \tag{2.4}$$

Since $\mathcal{R}e(\mathfrak{H}^\infty)$ is weak $*$-dense in the set of bounded real harmonic functions [29], we can replace $\mathcal{R}e(h)$ with f in the latter to obtain $f^2|g|^2 = 0$. This implies that f^2 and $|g|^2$ have disjoint supports. However, the harmonic function f cannot vanish on a set of positive measure without being zero, whence $g = 0$. □

3. Characterization of the Numerical Range

An operator $\mathbf{T} : \mathbf{M} \to \mathbf{M}$ is said to be subnormal if it admits an extension $\mathbf{S} : \mathbf{E} \to \mathbf{E}$, such that $\mathbf{M} \subset \mathbf{E}$, $\mathbf{S}$ is normal, and $\mathbf{M}$ is invariant under $\mathbf{S}$. It is well known that if an operator $\mathbf{T}$ is subnormal, then it is convexoid, that is, $\overline{\mathcal{W}(\mathbf{T})} = \mathcal{H}(\sigma(\mathbf{T}))$. First, the following observation is worth stressing.

Proposition 3.1. *Suppose that $f \in \mathfrak{H}^\infty(\mathbb{D})$. Then $\mathcal{S}_f$ and $\mathcal{S}_{\overline{f}}$ are convexoid, that is, $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\sigma(\mathcal{S}_f))$, and $\overline{\mathcal{W}(\mathcal{S}_{\overline{f}})} = \mathcal{H}(\sigma(\mathcal{S}_{\overline{f}}))$.*

Proof. Let $f \in \mathfrak{H}^\infty(\mathbb{D})$, then the multiplication operator $\mathcal{M}_{\overline{f}} : \mathcal{L}^2(\mathbb{D}) \to \mathcal{L}^2(\mathbb{D})$ is a normal extension of $\mathcal{S}_{\overline{f}}$. Indeed, we have $\mathcal{M}_{\overline{f}}\mathcal{M}_f = \mathcal{M}_f\mathcal{M}_{\overline{f}}$ and $\mathcal{M}_{\overline{f}}((\mathcal{L}_a^2)^{\perp}) = \{\overline{f}g, g \in (\mathcal{L}_a^2)^{\perp}\} \subset (\mathcal{L}_a^2)^{\perp}$. Moreover, it is clear that $\mathcal{S}_{\overline{f}}$ is the restriction of $\mathcal{M}_{\overline{f}}$ to $(\mathcal{L}_a^2)^{\perp}$. Thus $\mathcal{S}_{\overline{f}}$ is subnormal, whence $\overline{\mathcal{W}(\mathcal{S}_{\overline{f}})} = \mathcal{H}(\sigma(\mathcal{S}_{\overline{f}}))$.

Concerning the other part of the Proposition, of course $\mathcal{S}_f$ is not necessarily subnormal, nevertheless we obtain a similar result by exploring the fact that $\mathcal{W}(\mathcal{S}_f) = \{\overline{\lambda}, \lambda \in \mathcal{W}(\mathcal{S}_{\overline{f}})\}$ and $\sigma(\mathcal{S}_f) = \{\overline{\lambda}, \lambda \in \sigma(\mathcal{S}_{\overline{f}})\}$. Hence, we obtain $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\sigma(\mathcal{S}_f))$. □

For bounded analytic or coanalytic symbols, the fact that $\mathcal{S}_f$ is convexoid comes from the subnormality of this operator as Proposition 3.1 asserts. However, the spectral inclusion theorem, (namely, part (1) of Theorem 2.2), refines this result. Indeed, making use of the spectral inclusion property, it turns out that all bounded dual Toeplitz operators are convexoid; this represents the aim of the following assertion.

Proposition 3.2. *The closure of the numerical range of a bounded dual Toeplitz operator is the convex hull of its spectrum, that is,* $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\sigma(\mathcal{S}_f))$, *for* $f \in \mathcal{L}^\infty(\mathbb{D})$.

Proof. Consider the multiplication operator $\mathcal{M}_f$ on $\mathcal{L}^2(\mathbb{D}, dA)$. It is known to be normal, whence convexoid. Thus $\overline{\mathcal{W}(\mathcal{M}_f)} = \mathcal{H}(\sigma(\mathcal{M}_f))$. By Problem 67 of [7], we see that $\sigma(\mathcal{M}_f) = \mathcal{R}(f)$. By the spectral inclusion Theorem, we see that $\sigma(\mathcal{M}_f) = \mathcal{R}(f) \subset \sigma(\mathcal{S}_f)$. This yields the following inclusions $\overline{\mathcal{W}(\mathcal{M}_f)} = \mathcal{H}(\sigma(\mathcal{M}_f)) = \mathcal{H}(\mathcal{R}(f)) \subset \mathcal{H}(\sigma(\mathcal{S}_f))$. Now, since $\mathcal{M}_f$ is the minimal normal dilation of $\mathcal{S}_f$, we see that $\overline{\mathcal{W}(\mathcal{S}_f)} \subset \overline{\mathcal{W}(\mathcal{M}_f)}$, which is clear from the definition of the numerical range and the fact that $\mathcal{S}_f$ is the compression of $\mathcal{M}_f$. Therefore, we obtain the first inclusion $\overline{\mathcal{W}(\mathcal{S}_f)} \subset \mathcal{H}(\sigma(\mathcal{S}_f))$.

The reverse inclusion is easier: we have $\sigma(\mathcal{S}_f) \subset \overline{\mathcal{W}(\mathcal{S}_f)}$, which implies that $\mathcal{H}(\sigma(\mathcal{S}_f)) \subset \mathcal{H}(\overline{\mathcal{W}(\mathcal{S}_f)}) = \overline{\mathcal{W}(\mathcal{S}_f)}$, since $\overline{\mathcal{W}(\mathcal{S}_f)}$ is convex. □

Remark 3.3. In connection with Proposition 3.2, we ask whether all elements of the dual Toeplitz algebra $\mathfrak{D}\mathfrak{T}$, (the C*-algebra generated by $\{\mathcal{S}_f, f \in \mathfrak{H}^\infty\}$), are convexoid, according to the fact that it is generated by subnormal operators.

Now, we are going to characterize the numerical ranges of dual Toeplitz operators with bounded harmonic symbols. First, we make the following observation.

Remark 3.4. According to part (2) of Theorem 2.2, for a nonconstant bounded continuous real-valued function f on $\mathbb{D}$, we infer that $\sigma(\mathcal{S}_f) = \mathcal{R}(f) = \overline{f(\mathbb{D})}$ is an interval. As f is bounded, we deduce that $\sigma(\mathcal{S}_f) = [\inf f, \sup f]$. Obviously, we have $\mathcal{H}(\sigma(\mathcal{S}_f)) = [\inf f, \sup f]$.

Lemma 3.5. *Suppose that f is a nonconstant bounded harmonic real-valued function on $\mathbb{D}$. Then one has $\mathcal{W}(\mathcal{S}_f) = (\inf f, \sup f)$.*

Proof. By Proposition 3.2 and Remark 3.4, we see that $\overline{\mathcal{W}(\mathcal{S}_f)} = [\inf f, \sup f]$, (the continuity is redundant in the harmonicity). Since $\mathcal{W}(\mathcal{S}_f)$ is a convex set whose closure is $[\inf f, \sup f]$, $\mathcal{W}(\mathcal{S}_f)$ contains all elements of $[\inf f, \sup f]$ except possibly the extreme points $\inf f$ and $\sup f$. Thus $(\inf f, \sup f) \subset \mathcal{W}(\mathcal{S}_f)$.

Now, suppose that either $\inf f$ or $\sup f$ belongs to $\mathcal{W}(\mathcal{S}_f)$. Then it is an extreme point of $\mathcal{W}(\mathcal{S}_f)$, which in fact must be an eigenvalue of $\mathcal{S}_f$. However, Theorem 2.6 tells us that such $\mathcal{S}_f$ has no eigenvalues. Thus, we should have $\mathcal{W}(\mathcal{S}_f) = (\inf f, \sup f)$. □

Parallel to Brown and Halmos [6] characterization of normal Hardy space Toeplitz operators as well as Axler and Čučković [30] one of normal Bergman space Toeplitz operators with bounded harmonic symbols, normal dual Toeplitz operators were characterized in [1] as follows: for a bounded measurable function f on $\mathbb{D}$, the dual Toeplitz operator $\mathcal{S}_f$ is normal if and only if the range of f lies on a line. Accordingly, we are able to characterize the numerical range of normal dual Toeplitz operators with bounded harmonic symbols.

Theorem 3.6. *Let f be a nonconstant bounded harmonic function on $\mathbb{D}$, and suppose that $\mathcal{S}_f$ is normal. Then, there are (complex) numbers a, b such that $\sigma(\mathcal{S}_f)$ is the closed line segment $[a, b]$ and $\mathcal{W}(\mathcal{S}_f)$ is the corresponding open line segment (a, b).*

Proof. Taking into account the assumption that $\mathcal{S}_f$ is normal, we are certain from the existence of (complex) constants α, β and a real-valued function g such that $f = \alpha g + \beta$ whence $\mathcal{S}_f =$

$\alpha\mathcal{S}_g + \beta$. From the harmonicity of f and the linearity of the Laplacian, we see that g must be bounded harmonic and real-valued. Now, Lemma 3.5 asserts, therefore, that $\sigma(\mathcal{S}_g) = [m, M]$ and $\mathcal{W}(\mathcal{S}_g) = (m, M)$, where $m = \inf g$ and $M = \sup g$. Thus $\sigma(\mathcal{S}_f) = [a, b]$ and $\mathcal{W}(\mathcal{S}_f) = (a, b)$, (line segments in $\mathbb{C}$), with $a = \alpha m + \beta$, and $b = \alpha M + \beta$. □

4. The Numerical Range of a Nonnormal Dual Toeplitz Operator

Lemma 2.4 has a nice consequence on the self-adjointness of certain dual Toeplitz operators.

Theorem 4.1. *Let $f \in \mathcal{L}^\infty(\mathbb{D})$ be harmonic. Suppose that $\mathcal{W}(\mathcal{S}_f)$ lies in the complex upper half-plane and contains some real number. Then $\mathcal{S}_f$ must be self-adjoint.*

Proof. The numerical range lies in the upper half-plane means that $\mathfrak{Im}(\langle \mathcal{S}_f g, g\rangle) \geq 0$, for $\|g\| = 1$. Taking $g/\|g\|_2$ if necessary, we may conclude that $\mathfrak{Im}(\langle \mathcal{S}_f g, g\rangle) \geq 0, \forall g \in (\mathcal{L}_a^2)^\perp$. Since $\langle \mathcal{S}_f g, g\rangle = \int_{\mathbb{D}} f|g|^2 dA, \forall g \in (\mathcal{L}_a^2)^\perp$, taking imaginary parts, we obtain

$$\mathfrak{Im}(\langle \mathcal{S}_f g, g\rangle) = \int_{\mathbb{D}} \mathfrak{Im}(f)\left|g\right|^2 dA = \langle \mathcal{S}_{\mathfrak{Im}(f)} g, g\rangle \geq 0, \quad \forall g \in \left(\mathcal{L}_a^2\right)^\perp. \tag{4.1}$$

This happens only if $\mathfrak{Im}(f) \geq 0$, by Lemma 2.4. Now, suppose that a real number $c \in \mathcal{W}(\mathcal{S}_f)$, then there exists some $h \in (\mathcal{L}_a^2)^\perp$, with $\|h\| = 1$, such that $\langle \mathcal{S}_f h, h\rangle = c$. Writing c in the form $c = c\langle h, h\rangle$, we obtain $\int_{\mathbb{D}} (f - c) h\overline{h} dA = 0$. Again, taking imaginary parts we obtain

$$\int_{\mathbb{D}} \mathfrak{Im}(f)|h|^2 dA = 0. \tag{4.2}$$

As $\mathfrak{Im}(f) \geq 0$, we deduce that $\mathfrak{Im}(f)|h|^2 = 0$ on $\mathbb{D}$. This implies that $\mathfrak{Im}(f)$ and $|h|^2$ have disjoint supports. Since $\|h\| = 1$, we deduce that $h \neq 0$ on $\mathbb{D}$. Thus $\operatorname{supp}(\mathfrak{Im}(f))$ has a positive measure, that is, the harmonic function $\mathfrak{Im}(f)$ must be zero on a set of nonzero measure. It follows that $\mathfrak{Im}(f) = 0$ on $\mathbb{D}$, whence $\mathcal{S}_f$ is self-adjoint by Lemma 2.3. □

Regarding the numerical ranges of a certain class of dual Toeplitz operators, we have the following qualitative characterization.

Theorem 4.2. *Let $f \in \mathcal{L}^\infty(\mathbb{D})$ be harmonic such that $\mathcal{S}_f$ is nonnormal. Then $\mathcal{W}(\mathcal{S}_f)$ is an open convex set.*

Proof. We need only to verify that the convex set $\mathcal{W}(\mathcal{S}_f)$ is open. To see this, we proceed by contradiction and suppose that it is not open. Hence it intersects its boundary $\partial\mathcal{W}(\mathcal{S}_f)$. Let λ be one of such points, that is, $\lambda \in \mathcal{W}(\mathcal{S}_f) \cap \partial\mathcal{W}(\mathcal{S}_f)$, which can be rewritten as $0 \in \mathcal{W}(\mathcal{S}_{f-\lambda}) \cap \partial\mathcal{W}(\mathcal{S}_{f-\lambda})$. Now, the convexity of $\mathcal{W}(\mathcal{S}_{f-\lambda})$ and the fact that $0 \in \partial\mathcal{W}(\mathcal{S}_{f-\lambda})$ enable us to rotate it so that it lies in the upper half-plane. This means that there exists a unimodular complex number $\omega = e^{i\theta}$, for some $\theta \in [0, 2\pi]$, such that $\omega\mathcal{W}(\mathcal{S}_{f-\lambda}) = \mathcal{W}(\mathcal{S}_{\omega f - \omega\lambda})$ lies in the upper half-plane. By Theorem 4.1, $\mathcal{S}_{\omega f - \omega\lambda}$ must be self-adjoint. In other words there exists a real-valued function g on $\mathbb{D}$ such that $\omega f - \omega\lambda = g$. This implies that $f = \alpha g + \beta$, for some constants α and β. So, we infer that $\mathcal{S}_f = \alpha\mathcal{S}_g + \beta$ is normal, which contradicts the original hypothesisand completes the proof. □

Remark 4.3. Note that Theorems 4.1 and 4.2, (as well as their subsequent corollaries), remain still valid if one assumes merely that the symbols have harmonic imaginary parts.

Corollary 4.4. *If f is a bounded analytic or coanalytic function, then $\mathcal{W}(\mathcal{S}_f) = \mathcal{H}(f(\mathbb{D}))$.*

Proof. If f is constant, the fact is trivially satisfied. If f is not constant, then $\mathcal{S}_f$ is not normal (because the range of an analytic or coanalytic function cannot lie on a line). Hence, by Theorem 4.2, $\mathcal{W}(\mathcal{S}_f)$ is an open convex set. On the other hand, by the open mapping theorem $f(\mathbb{D})$ is open whence $\mathcal{H}(f(\mathbb{D}))$ is an open convex set too. Now, by Proposition 3.1 and part (3) of Theorem 2.2 as well as Remark 2.1, we see that $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\overline{f(\mathbb{D})}) = \overline{\mathcal{H}(f(\mathbb{D}))}$. Since an open convex planar set is the interior of its closure, we infer that $\mathcal{W}(\mathcal{S}_f)$ coincides with the convex hull of $f(\mathbb{D})$. □

5. Aesthetic Consequences Using Lines of Support of $\mathcal{W}(\mathcal{S}_f)$

A line L is said to be a line of support of a planar convex set K at a boundary point $P \in \partial K$, if $P \in L$ and the set K is contained in the closure of one of the two half-planes into which L cuts the plane. Clearly, every point on the boundary of a planar convex set lies on a line of support. Based on the concept of lines of support of the numerical range $\mathcal{W}(\mathcal{S}_f)$, several interesting consequences of Theorem 4.1 can be observed. Note that the underlying idea goes back to Brown and Halmos [6].

Corollary 5.1. *Let $f \in \mathcal{L}^\infty(\mathbb{D})$ be harmonic. If a line of support of the numerical range $\mathcal{W}(\mathcal{S}_f)$ of the dual Toeplitz operator $\mathcal{S}_f$ contains a point of $\mathcal{W}(\mathcal{S}_f)$, then it contains its whole spectrum $\sigma(\mathcal{S}_f)$ (and hence its entire numerical range $\mathcal{W}(\mathcal{S}_f)$).*

Corollary 5.2. *Let $f \in \mathcal{L}^\infty(\mathbb{D})$ be harmonic. If the spectrum $\sigma(\mathcal{S}_f)$ consists of merely a finite number of eigenvalues, then the operator $\mathcal{S}_f$ is scalar.*

Proofs

Proof of Corollary 5.1

With regard to Corollary 5.1, clearly the line of support can be rotated along with the numerical range till the line will be horizontal and the numerical range above it. Then we translate both in such a way that the line of support will be the real axis; (these two operations correspond to a linear function of the form $z \to e^{i\theta}z+\omega$, for a fixed complex number ω and a fixed $\theta \in [0, 2\pi]$). Then one can apply Theorem 4.1 to conclude that $e^{i\theta}\mathcal{S}_f + \omega$ is self-adjoint, whence $e^{i\theta}\sigma(\mathcal{S}_f) + \omega \subset \mathbb{R}$, and therefore $\sigma(\mathcal{S}_f)$ lies on the original line of support. Since $\mathcal{W}(\mathcal{S}_f) \subset \overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\sigma(\mathcal{S}_f))$, by Proposition 3.2, we infer that $\mathcal{W}(\mathcal{S}_f)$ lies on the relevant line of support.

Proof of Corollary 5.2

Concerning Corollary 5.2, if $\sigma(\mathcal{S}_f)$ consists of a finite number of eigenvalues, then $\mathcal{H}(\sigma(\mathcal{S}_f))$ is a polygon with vertices of some of such values. Thus, one can find at least a line of support passing through one of such vertices, which must be in fact an extreme point of $\mathcal{W}(\mathcal{S}_f)$. Hence, making use of Corollary 5.1, we infer that all those eigenvalues of $\mathcal{S}_f$ lie on such line. Consequently, the numerical range is a segment on this line with two extreme points. Taking the line of support perpendicular to the above line, and passing by an endpoint of the segment, and repeating the same procedure, we infer that the numerical range lies on the new line of support as well. A segment lying in two perpendicular lines must be a single point. So, $\sigma(\mathcal{S}_f)$ must be a singleton. By Proposition 3.2 and the properties of the numerical range, we infer that $\mathcal{S}_f$ must be scalar.

Remark 5.3. Suppose that $\mathcal{S}_f$ is compact. We know that its spectrum consists of at most a countable number of eigenvalues, including 0. Arguing in the same manner as in the proof of Corollary 5.2, we can show that there are no nonzero compact dual Toeplitz operators with bounded harmonic symbols, which is a particular case of [1, Theorem 7.5].

6. Connection with Spectral Sets

A compact subset $\mathbb{E} \subset \mathbb{C}$ containing the spectrum $\sigma(T)$ of a bounded linear operator T acting on a given Hilbert space is called a k-spectral set for T if

$$\|\mathcal{F}(T)\| \leq k\|\mathcal{F}\|_{\mathbb{E}} \tag{6.1}$$

holds for every function $\mathcal{F}(z)$ analytic in a neighborhood of $\mathbb{E}$ (in [13], rational functions are used instead), where the first norm in the above inequality is the operator norm and the second one is the sup norm over $\mathbb{E}$. In particular, if $k = 1$, 1-spectral sets are simply called spectral sets. For more details on spectral sets, we refer to [13–15, 31] and the references therein. For instance, the spectrum of any subnormal operator is a spectral set. The closed unit disk is a spectral set for any contraction.

Our main concern in this section is to describe the dual Toeplitz operators analog of an interesting connection, pointed out by Schreiber [13], between spectral sets and numerical ranges of Toeplitz operators. Let us start with some trivial situations, where $\sigma(\mathcal{S}_\varphi)$ is spectral

(1) If $\mathcal{S}_\varphi$ is a bounded normal dual Toeplitz operator, (i.e., $\varphi(\mathbb{D})$ lies on a line in $\mathbb{C}$ [1]), then $\sigma(\mathcal{S}_\varphi)$ is a spectral set.

(2) If φ is a bounded coanalytic function on the unit disk $\mathbb{D}$, then $\sigma(\mathcal{S}_\varphi)$ is a spectral set, (since it is subnormal by the proof of Proposition 3.1).

(3) If the spectrum of a bounded dual Toeplitz operator $\mathcal{S}_\varphi$ is a disk, then $\sigma(\mathcal{S}_\varphi)$ is a spectral set too.

However, we can observe that the spectra of dual Toeplitz operators with analytic symbols are also spectral sets for their corresponding operators. This follows from the following observation. If $\mathbb{E} \subset \mathbb{C}$ is a planar subset, set $\mathbb{E}^* = \{\overline{\omega}, \omega \in \mathbb{E}\}$, in particular it can be easily verified that $\sigma(T^*) = \sigma(T)^*$. Also we adopt the notation $f^*(z) = \overline{f(\overline{z})}$ which is an analytic function on $\mathbb{E}^*$ whenever f is analytic on $\mathbb{E}$. Then we have the following.

Lemma 6.1. *Let T be a bounded linear operator on a given Hilbert space. Then, $\sigma(T)$ is a k-spectral set for T if and only if $\sigma(T)^*$ is a k-spectral set for its adjoint T^*.*

Proof. The fact that $\sigma(T)$ is a k-spectral set of T means that Inequality (6.1) holds, for any $\mathcal{F}$ holomorphic in a neighborhood of $\sigma(T)$. Notice that from the definition of the holomorphic functional calculus we have

$$\|\mathcal{F}^*(T^*)\| = \left\|(\mathcal{F}(T))^*\right\| = \|\mathcal{F}(T)\|. \tag{6.2}$$

On the other hand, we have

$$\begin{aligned} \|\mathcal{F}^*\|_{\sigma(T)^*} &= \sup\left\{|\mathcal{F}^*(w)|,\ w \in \sigma(T)^*\right\} \\ &= \sup\left\{\left|\overline{\mathcal{F}}(z)\right|,\ z \in \sigma(T)\right\} \\ &= \|\mathcal{F}\|_{\sigma(T)}. \end{aligned} \tag{6.3}$$

Combining Inequality (6.1) and the last two identities, we infer that

$$\|\mathcal{F}^*(T^*)\| \leq k\|\mathcal{F}^*\|_{\sigma(T^*)}. \tag{6.4}$$

In order to establish the equivalence, it suffices to observe that if $\mathcal{F}$ is holomorphic in a neighborhood of $\sigma(T)$, then $\mathcal{F}^*$ is holomorphic in the conjugate of the same neighborhood which contains $\sigma(T)^*$ and conversely. □

Therefore, from the above discussion, we deduce the following.

Corollary 6.2. *If φ is a bounded analytic function on the unit disk $\mathbb{D}$, then $\sigma(\mathcal{S}_\varphi)$ is a spectral set for $\mathcal{S}_\varphi$.*

Similar results for coanalytic Toeplitz operators on both Hardy and Bergman spaces can also be inferred.

Corollary 6.3. *(i) If $\overline{\varphi} \in \mathfrak{H}^\infty(\partial\mathbb{D})$, then $\sigma(T_\varphi)$ is a spectral set for T_φ defined on $\mathfrak{H}^2(\partial\mathbb{D})$.*

(ii) If φ is a bounded coanalytic function on the unit disk $\mathbb{D}$, then $\sigma(T_\varphi)$ is a spectral set for T_φ defined on $\mathcal{L}_a^2(\mathbb{D})$.

7. Some Thoughts on Quasinormal Dual Toeplitz Operators

The Bergman space $\mathcal{L}_a^2$ has normalized reproducing kernel k_w given by

$$k_w(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)^2}. \tag{7.1}$$

Recall that for $w \in \mathbb{D}$, the involutive disk automorphism φ_w is defined by

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad \text{for } z \in \mathbb{D}. \tag{7.2}$$

For a linear operator $\mathbf{T}$ on $(\mathcal{L}_a^2)^{\perp}$ and $w \in \mathbb{D}$, define the operator $\mathbb{S}_w(\mathbf{T}) := \mathbf{T} - \mathcal{S}_{\varphi_w}\mathbf{T}\mathcal{S}_{\overline{\varphi}_w}$. A second application of it gives

$$\mathbb{S}_w^2(T) = \mathbf{T} - 2\mathcal{S}_{\varphi_w}\mathbf{T}\mathcal{S}_{\overline{\varphi}_w} + \mathcal{S}_{\varphi_w}^2\mathbf{T}\mathcal{S}_{\overline{\varphi}_w}^2. \tag{7.3}$$

For $f, g \in \mathcal{L}^2(\mathbb{D}, dA)$, define the rank one operator $(f \otimes g)$ by

$$(f \otimes g) : h \in \mathcal{L}^2(D, dA) \longrightarrow (f \otimes g)(h) = \langle h, g\rangle f \in \mathcal{L}^2(D, dA). \tag{7.4}$$

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are bounded linear operators on $\mathcal{L}^2(\mathbb{D}, dA)$, then for $f, g \in \mathcal{L}^2(\mathbb{D}, dA)$, we have

$$\mathbf{T}_1(f \otimes g)\mathbf{T}_2^* = (\mathbf{T}_1 f) \otimes (\mathbf{T}_2 g). \tag{7.5}$$

In the sequel, we will need a formula relating the image of the product $H_f H_{\overline{g}}^*$ under the action of the operator $\mathbb{S}_w^2$ and the functions $H_f(k_w)$ and $H_{\overline{g}}(k_w)$. For $f, g \in \mathcal{L}^\infty(\mathbb{D})$, combining (2.3), (7.3), and (7.5), (for a detailed proof see [1, 2]), we obtain

$$\mathbb{S}_w^2\left(H_f H_{\overline{g}}^*\right) = H_f(k_w \otimes k_w)H_{\overline{g}}^* = (H_f(k_w)) \otimes (H_{\overline{g}}(k_w)). \tag{7.6}$$

An operator $\mathbf{T}$ on a Hilbert space is called quasinormal if it commutes with $\mathbf{T}^*\mathbf{T}$. It is well known that quasinormal operators are subnormal. In what follows we are going to show that there are no quasinormal dual Toeplitz operators with bounded analytic or coanalytic symbols that are not normal.

Theorem 7.1. *Let f be in $\mathfrak{H}^\infty(\mathbb{D})$, and suppose that $\mathcal{S}_f$ is quasinormal. Then, the symbol f must be constant.*

Proof. If $f = 0$, the conclusion is obvious. So, suppose that $f \neq 0$ and that $\mathcal{S}_f$ is quasinormal, then we have

$$\mathcal{S}_f\mathcal{S}_f^*\mathcal{S}_f = \mathcal{S}_f^*\mathcal{S}_f\mathcal{S}_f. \tag{7.7}$$

Since f is analytic, using Relations (2.2), we obtain

$$\mathcal{S}_f\mathcal{S}_f^*\mathcal{S}_f = \mathcal{S}_{|f|^2}\mathcal{S}_f = \mathcal{S}_{|f|^2 f} - H_{|f|^2}H_{\overline{f}}^*, \tag{7.8}$$

$$\mathcal{S}_f^*\mathcal{S}_f\mathcal{S}_f = \mathcal{S}_{\overline{f}}\mathcal{S}_{f^2} = \mathcal{S}_{\overline{f}f^2} - H_{\overline{f}}H_{\overline{f}^2}^*. \tag{7.9}$$

Now, (7.7)–(7.9) yield

$$H_{|f|^2}H_{\overline{f}}^* = H_{\overline{f}}H_{\overline{f}^2}^*. \tag{7.10}$$

Introducing the operator $\mathbb{S}_\omega$, by (7.6), we see that

$$\mathbb{S}_\omega^2\left(H_{|f|^2}H_{\overline{f}}^*\right) = H_{|f|^2}k_\omega \otimes H_{\overline{f}}k_\omega, \qquad \mathbb{S}_\omega^2\left(H_{\overline{f}}H_{\overline{f}^2}^*\right) = H_{\overline{f}}k_\omega \otimes H_{\overline{f}^2}k_\omega. \tag{7.11}$$

Combining the latter with (7.10), we obtain $H_{|f|^2}k_\omega \otimes H_{\overline{f}}k_\omega = H_{\overline{f}}k_\omega \otimes H_{\overline{f}^2}k_\omega$. Since $k_0 = 1$, taking $\omega = 0$, we obtain $H_{|f|^2}1 \otimes H_{\overline{f}}1 = H_{\overline{f}}1 \otimes H_{\overline{f}^2}1$. In other words, by the definition of the rank one operator, we have

$$\left\langle u, H_{\overline{f}}1\right\rangle H_{|f|^2}1 = \left\langle u, H_{\overline{f}^2}1\right\rangle H_{\overline{f}}1, \quad \forall u \in \left(\mathcal{L}_a^2\right)^\perp. \tag{7.12}$$

If $H_{\overline{f}}1 = 0$, then $\overline{f} \in \mathcal{L}_a^2$. Thus $\overline{f}$ must be constant, whence the result follows immediately. If $H_{\overline{f}}1 \neq 0$, we distinguish several cases as follows.

(i) The case $H_{|f|^2}1 = 0$ and $H_{\overline{f}^2}1 = 0$ cannot happen (because $H_{\overline{f}}1 = Q(\overline{f}) = \overline{f} - \overline{f}(0)$ and $H_{\overline{f}^2}1 = Q(\overline{f}^2) = \overline{f}^2 - \overline{f}^2(0)$ vanish simultaneously).

(ii) The similar case $H_{|f|^2}1 \neq 0$ and $H_{\overline{f}^2}1 = 0$ is impossible too, for the same reason.

(iii) The case $H_{|f|^2}1 = 0$ and $H_{\overline{f}^2}1 \neq 0$ is impossible too, otherwise $\langle u, H_{\overline{f}^2}1\rangle = 0$, for all $u \in (\mathcal{L}_a^2)^\perp$; thus $H_{\overline{f}^2}1 \in \mathcal{L}_a^2$ too, whence $H_{\overline{f}^2}1 = 0$ contradicting the assumption.

(iv) If $H_{|f|^2}1 \neq 0$ and $H_{\overline{f}^2}1 \neq 0$, then clearly from (7.12) there exists some nonzero constant $\lambda \in \mathbb{C}$, such that

$$H_{\overline{f}}1 = \lambda H_{|f|^2}1, \qquad H_{\overline{f}}1 = \overline{\lambda} H_{\overline{f}^2}1. \tag{7.13}$$

Rephrasing (7.13), we see that $Q(\overline{f} - \lambda|f|^2) = 0$ and $Q(\overline{f} - \overline{\lambda}\,\overline{f}^2) = 0$. Thus, $\overline{f} - \lambda|f|^2$ and $\overline{f} - \overline{\lambda}\,\overline{f}^2$ are in $\mathcal{L}_a^2$, (they are analytic in particular). But, since f is analytic, we see that $f(1 - \lambda f) = f - \lambda f^2$ is analytic too, whence it is constant. So, set $f(1 - \lambda f) = \mu$, for some nonzero complex constant μ. On the other hand, since $\overline{f} - \lambda|f|^2 = \overline{f}(1 - \lambda f)$ is analytic, multiplying by the analytic function f^2, we obtain an analytic function, namely, $\overline{f}(1 - \lambda f)f^2 = |f|^2(1 - \lambda f)f = \mu|f|^2$. Now, the function $\mu|f|^2$, (whose range lies on a line, as $|f|^2$ is real-valued and μ is constant), can be analytic only if it is constant; whence we infer that $|f|^2$ is constant. Finally, it is well known that an analytic function with a constant modulus must be constant, whence f must be constant. □

For bounded conjugate analytic symbols the matter is much more simpler and it uses the Brown-Halmos type Theorem (namely, [1, Theorem 3.1]). Indeed, we have the following.

Theorem 7.2. *Let f be in $\overline{\mathfrak{H}^\infty(\mathbb{D})}$, and suppose that $\mathcal{S}_f$ is quasinormal. Then, f must be constant.*

Proof. If $f = 0$, the matter is obvious; so suppose that $f \neq 0$. Since f is coanalytic, using Relations (2.2) and (2.3), we obtain

$$\mathcal{S}_f \mathcal{S}_f^* \mathcal{S}_f = \mathcal{S}_f \mathcal{S}_{|f|^2}, \tag{7.14}$$

$$\mathcal{S}_f^* \mathcal{S}_f \mathcal{S}_f = \mathcal{S}_{|f|^2} \mathcal{S}_f = \mathcal{S}_{f|f|^2}. \tag{7.15}$$

Suppose that $\mathcal{S}_f$ is quasinormal, then from (7.7), (7.14), and (7.15), we see that $\mathcal{S}_f \mathcal{S}_{|f|^2} = \mathcal{S}_{f|f|^2}$. Hence $\mathcal{S}_f \mathcal{S}_{|f|^2}$ is a dual Toeplitz operator. By the Brown-Halmos type Theorem ([1, Theorem 3.1]), we infer that either f is analytic or $|f|^2$ is coanalytic. If f is analytic, then it is constant since it is coanalytic by hypothesis. If the real function $|f|^2$ is coanalytic, then it is constant; whence f is constant as well, (as a coanalytic function with constant modulus). Thus, in all cases, we infer that f is constant. □

Remark 7.3. Clearly if f is constant, then $\mathcal{S}_f$ is normal and then it is quasinormal. So, Theorems 7.1 and 7.2 can be expressed as follows. `Let f be a bounded analytic, (or coanalytic), function. Then,` $\mathcal{S}_f$ `is quasinormal if and only if f is constant,` that is, there are no quasinormal dual Toeplitz operators with bounded analytic or coanalytic symbols that are not normal.

Corollary 7.4. *Let f be a bounded analytic or coanalytic function, and suppose that $\mathcal{S}_f$ is quasinormal. Then, the numerical range of $\mathcal{S}_f$ reduces to a singleton, that is, $\mathcal{W}(\mathcal{S}_f) = \{\lambda\}$ for some complex constant λ.*

Proof. Just observe that from Proposition 3.1 one has $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\sigma(\mathcal{S}_f))$. If $\mathcal{S}_f$ is quasinormal, by Theorems 7.1 and 7.2, we infer that $f = \lambda$ for some complex constant λ. The convex hull of a singleton is the set itself. Hence, we obtain $\overline{\mathcal{W}(\mathcal{S}_f)} = \mathcal{H}(\{\lambda\}) = \{\lambda\}$. □

It seems to be of interest to consider also the typical case of bounded harmonic symbols. It is well known that such functions can be decomposed as $f = f_1 + \overline{f_2}$, with Bloch functions f_1 and f_2. Here, we confine ourselves to the related case, where $f = g + \lambda\overline{g}$, for $g \in \mathfrak{H}^\infty(\mathbb{D})$ and λ a complex constant.

Proposition 7.5. *Suppose that $f = g + \lambda\overline{g}$, $0 \neq \lambda \in \mathbb{C}$, and $0 \neq g \in \mathfrak{H}^\infty(\mathbb{D})$. If $\mathcal{S}_f$ is quasinormal, then $\mathcal{S}_f$ is normal and λ must be unimodular.*

Proof. Since $\mathcal{S}_f$ is quasinormal, (7.7) holds. Taking adjoints, we obtain

$$\mathcal{S}_{\overline{f}}\left(\mathcal{S}_{\overline{f}}\mathcal{S}_f - \mathcal{S}_f\mathcal{S}_{\overline{f}}\right) = 0, \qquad \left(\mathcal{S}_{\overline{f}}\mathcal{S}_f - \mathcal{S}_f\mathcal{S}_{\overline{f}}\right)\mathcal{S}_f = 0. \tag{7.16}$$

Therefore, using the Hilbert space orthogonality relations, we see that $\ker([\mathcal{S}_{\overline{f}}, \mathcal{S}_f]) = \overline{\mathcal{R}an(\mathcal{S}_f)}$, which must be nontrivial. Next, a couple of manipulations lead to

$$\left[\mathcal{S}_{\overline{f}}, \mathcal{S}_f\right] = \left(1 - |\lambda|^2\right)\left(\mathcal{S}_{\overline{g}}\mathcal{S}_g - \mathcal{S}_g\mathcal{S}_{\overline{g}}\right) = \left(1 - |\lambda|^2\right)\left[\mathcal{S}_{\overline{g}}, \mathcal{S}_g\right]. \tag{7.17}$$

Now, (2.3) as well as the fact that g is analytic yields $[\mathcal{S}_{\overline{g}}, \mathcal{S}_g] = -H_{\overline{g}}H_{\overline{g}}^*$, whence

$$\left[\mathcal{S}_{\overline{f}}, \mathcal{S}_f\right] = \left(|\lambda|^2 - 1\right) H_{\overline{g}} H_{\overline{g}}^*. \tag{7.18}$$

Since the Hankel operator $H_{\overline{g}}$ is one-to-one (as $g \in \mathfrak{H}^\infty$), we infer that $H_{\overline{g}}H_{\overline{g}}^*$ has a trivial kernel. This contradicts the fact that $[\mathcal{S}_{\overline{f}}, \mathcal{S}_f]$ has a nontrivial kernel, unless $[\mathcal{S}_{\overline{f}}, \mathcal{S}_f] = 0$, which happens only if $|\lambda|^2 = 1$. Thus $\mathcal{S}_f$ is normal and λ must be unimodular. □

Corollary 7.6. *Let f be as in Proposition 7.5 and suppose that $\mathcal{S}_f$ is quasinormal. Then, $\mathcal{W}(\mathcal{S}_f) = (a, b)$ for some complex constants a and b.*

Proof. Combining Proposition 7.5 and Theorem 3.6, the result follows. □

At this stage, we would like to conclude with a crucial point, which probably sheds some light on the fifth Halmos' problem [32]. This problem asks whether every subnormal Hardy space Toeplitz operator is either normal or analytic. In the Hardy space setting, the original general problem was weakened to whether every quasinormal Toeplitz operator is either normal or analytic, and it was completely solved positively by Amemiya et al. [17], whereas Cowen and Long [19] answered the original problem in the negative. For further results in this direction, see [16, 18, 20–23]. The Bergman space analog seems to be still pending [24]. However, for dual Toeplitz operators, a similar conjecture can be stated with slight modifications, namely: `every quasinormal dual Toeplitz operator must be normal`. Theorems 7.1 and 7.2, Proposition 7.5 as well as Remark 7.3 support this conjecture.

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References

[1] K. Stroethoff and D. Zheng, "Algebraic and spectral properties of dual Toeplitz operators," *Transactions of the American Mathematical Society*, vol. 354, no. 6, pp. 2495–2520, 2002.

[2] K. Stroethoff and D. Zheng, "Products of Hankel and Toeplitz operators on the Bergman space," *Journal of Functional Analysis*, vol. 169, no. 1, pp. 289–313, 1999.

[3] Y. Lu, "Commuting dual Toeplitz operators with pluriharmonic symbols," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 1, pp. 149–156, 2005.

[4] G. Z. Cheng and T. Yu, "Dual Toeplitz algebra on the polydisk," *Journal of Mathematical Research and Exposition*, vol. 28, no. 2, pp. 366–370, 2008.

[5] T. Yu and S. Y. Wu, "Commuting dual Toeplitz operators on the orthogonal complement of the Dirichlet space," *Acta Mathematica Sinica (English Series)*, vol. 25, no. 2, pp. 245–252, 2009.

[6] A. Brown and P. R. Halmos, "Algebraic properties of Toeplitz operators," *Journal für die Reine und Angewandte Mathematik*, vol. 213, pp. 89–102, 1964.

[7] P. R. Halmos, *A Hilbert Space Problem Book*, vol. 19 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1982.

[8] E. M. Klein, "The numerical range of a Toeplitz operator," *Proceedings of the American Mathematical Society*, vol. 35, pp. 101–103, 1972.

[9] J. K. Thukral, "The numerical range of a Toeplitz operator with harmonic symbol," *Journal of Operator Theory*, vol. 34, no. 2, pp. 213–216, 1995.

[10] B. R. Choe and Y. J. Lee, "The numerical range and normality of Toeplitz operators," *Far East Journal of Mathematical Sciences (FJMS)*, no. Special Volume, Part I, pp. 71–80, 2001.

[11] D. G. Gu, "The numerical range of Toeplitz operator on the polydisk," *Abstract and Applied Analysis*, vol. 2009, Article ID 757964, 8 pages, 2009.

[12] K. Z. Wang and P. Y. Wu, "Numerical ranges of radial Toeplitz operators on Bergman space," *Integral Equations and Operator Theory*, vol. 65, no. 4, pp. 581–591, 2009.

[13] M. Schreiber, "Numerical range and spectral sets," *The Michigan Mathematical Journal*, vol. 10, pp. 283–288, 1963.

[14] S. Hildebrandt, "The closure of the numerical range of an operator as spectral set," *Communications on Pure and Applied Mathematics*, vol. 17, pp. 415–421, 1964.

[15] D. N. Clark, "Toeplitz operators and k-spectral sets," *Indiana University Mathematics Journal*, vol. 33, no. 1, pp. 127–141, 1984.

[16] T. Itô and T. K. Wong, "Subnormality and quasinormality of Toeplitz operators," *Proceedings of the American Mathematical Society*, vol. 34, pp. 157–164, 1972.

[17] I. Amemiya, T. Ito, and T. K. Wong, "On quasinormal Toeplitz operators," *Proceedings of the American Mathematical Society*, vol. 50, pp. 254–258, 1975.

[18] M. B. Abrahamse, "Subnormal Toeplitz operators and functions of bounded type," *Duke Mathematical Journal*, vol. 43, no. 3, pp. 597–604, 1976.

[19] C. C. Cowen and J. J. Long, "Some subnormal Toeplitz operators," *Journal für die Reine und Angewandte Mathematik*, vol. 351, pp. 216–220, 1984.

[20] C. C. Cowen, "Hyponormal and subnormal Toeplitz operators," in *Surveys of Some Recent Results in Operator Theory, Vol. I*, vol. 171, pp. 155–167, Longman Sci. Tech., Harlow, UK, 1988.

[21] C. C. Cowen, "More subnormal Toeplitz operators," *Journal für die Reine und Angewandte Mathematik*, vol. 1986, no. 367, pp. 215–219, 2009.

[22] W. Y. Lee, "Open problems in Toeplitz operator theory," *Trends in Mathematics. Information Center for Mathematical Sciences*, vol. 4, no. 2, pp. 133–150, 2001.

[23] T. Yoshino, "The conditions that the Toeplitz operator is normal or analytic," *Nihonkai Mathematical Journal*, vol. 13, no. 2, pp. 167–177, 2002.

[24] N. Faour, "On quasinormal, subnormal, and hyponormal Toeplitz operators," *Rendiconti del Circolo Matematico di Palermo. Serie II*, vol. 38, no. 1, pp. 121–129, 1989.

[25] W. F. Donoghue, Jr., "On the numerical range of a bounded operator," *The Michigan Mathematical Journal*, vol. 4, pp. 261–263, 1957.

[26] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, NY, USA, 1972.

[27] B. Istratescu, *Introduction to Linear Operator Thoery*, Marcel Dekker, New York, NY, USA, 1981.

[28] F. A. Valentine, *Convex Sets*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, NY, USA, 1964.

[29] G. McDonald and C. Sundberg, "Toeplitz operators on the disc," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 595–611, 1979.

[30] S. Axler and Ž. Čučković, "Commuting Toeplitz operators with harmonic symbols," *Integral Equations and Operator Theory*, vol. 14, no. 1, pp. 1–12, 1991.

[31] Y. M. Han and A.-H. Kim, "Weyl spectra of the χ-class operators," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 1, pp. 163–174, 2001.

[32] P. R. Halmos, "Ten problems in Hilbert space," *Bulletin of the American Mathematical Society*, vol. 76, pp. 887–933, 1970.