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Research Article

Weyl-Titchmarsh Theory for Hamiltonian Dynamic Systems

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We establish the Weyl-Titchmarsh theory for singular linear Hamiltonian dynamic systems on a time scale $\mathbb{T}$, which allows one to treat both continuous and discrete linear Hamiltonian systems as special cases for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ within one theory and to explain the discrepancies between these two theories. This paper extends the Weyl-Titchmarsh theory and provides a foundation for studying spectral theory of Hamiltonian dynamic systems. These investigations are part of a larger program which includes the following: (i) $M(\lambda)$ theory for singular Hamiltonian systems, (ii) on the spectrum of Hamiltonian systems, (iii) on boundary value problems for Hamiltonian dynamic systems.

1. Introduction

1.1. Differential Equations

The study of spectral problems for differential operators has played an important role not only in theoretical but also in practical aspects. The study of spectral theory of differential equations has a long history. For this, we refer to [1–12] and references therein along this line.

Spectral problems of differential operators fall into two classifications. First, those defined over finite intervals with well-behaved coefficients are called *regular*. Fine spectral properties can be expected. For example, the spectral set is discrete, infinite, and unbounded, and the eigenfunction basis is complete for a corresponding space.

Spectral problems that are not regular are called *singular*. These are considerably more difficult to discuss because the spectral set can be much more complicated and, as a result, have only been examined closely during the last century. The Weyl-Titchmarsh theory is an important milestone in the study of spectral problems for linear ordinary differential equations [13]. It has started with the celebrated work by H. Weyl in 1910 [14]. He gave a dichotomy of the limit-point and limit-circle cases for singular spectral problems of second-order formally self-adjoint linear differential equations. He was followed by Titchmarsh [12] and many others. From 1910 until 1945, these mathematicians developed and polished the theory of self-adjoint differential operators of the second order to a high degree. Their work was continued by Coddington and Levinson [3], and so forth. in the late 1940s and 1950s. Not only were additional results found for operators of the second order, but operators of higher orders were also examined. At the same time, the Russian school, led by Kreĭn, Naĭmark, Akhiezer, and Glazman, also made major contributions. For a far more comprehensive survey of this work, we recommend the second volume of Dunford and Schwartz [4], where numerous contributions made by many mathematicians are summarized. Further study continued in the 1960s and 1970s with the work of Atkinson [1] on regular Hamiltonian systems

$$\begin{aligned} x'(t) &= A(t)x(t) + (B(t) + \lambda W_2(t))u(t), \\ u'(t) &= (C(t) - \lambda W_1(t))x(t) - A^*(t)u(t) \end{aligned} \tag{1.1}$$

and Everitt and Kumar [15, 16] on higher-order scalar problems. The work for this period is summarized by Atkinson [1], and Everitt and Kumar [15, 16]. Again, there were many other contributors. One contribution, perhaps, deserves special mention. Walker [17] showed that every scalar self-adjoint problem of an arbitrary order can be reformulated as an equivalent self-adjoint Hamiltonian system. This removed the need to discuss scalar problems and systems separately.

In the 1980s and 1990s, Hinton and Shaw [5–9, 11], Krall [18–20], and Remling [21] have made great progress by considering singular spectral problems in the Hamiltonian system format, following the lead of Atkinson [1]. In the 2000s, Brown and Evans [22], Clark and Gesztesy [23], Qi and Chen [24], Qi [25], Remling [21], Shi [26], Sun et al. [27], Zheng and Chen [28] have made progress by considering spectral problems for Hamiltonian differential systems.

1.2. Difference Equations

Spectral problems of discrete linear Hamiltonian systems

$$\begin{aligned} \Delta x(t) &= A(t)x(t+1) + (B(t) + \lambda W_2(t))u(t), \\ \Delta u(t) &= (C(t) - \lambda W_1(t))x(t+1) - A^*(t)u(t) \end{aligned} \tag{1.2}$$

are also divided into two groups: regular and singular problems. Singular spectral problems of second-order self-adjoint scalar difference equations over infinite intervals were first studied by Atkinson [1]. His work was followed by Agarwal et al. [29], Bohner [30], Bohner et al. [31], Clark and Gesztesy [32], Shi [33, 34], and Sun et al. [35]. In [1], Atkinson first studied the Weyl-Titchmarsh theory and the spectral theory for the system (1.2). Following him, Hinton and Shaw have made great progress by considering Weyl-Titchmarsh theory and spectral theory for the system (1.2). Shi studied Weyl-Titchmarsh theory and spectral theory for the system (1.2) in [33, 34]; Clark and Gesztesy established the Weyl-Titchmarsh theory for a class of discrete Hamiltonian systems that include system (1.2) [23]. Sun et al. established the GKN-theory for the system (1.2) [35].

1.3. Dynamic Equations

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis [36]. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [37, 38] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [39], summarizes and organizes much of the time scale calculus. We refer also to the book by Bohner and Peterson [40] for advances in dynamic equations on time scales and to the book by Lakshmikantham et al. [41].

This paper is devoted to the Weyl-Titchmarsh theory for linear Hamiltonian dynamic systems

$$\begin{aligned} x^{\Delta}(t) &= A(t)x^{\sigma}(t) + (B(t) + \lambda W_2(t))u(t), \\ u^{\Delta}(t) &= (C(t) - \lambda W_1(t))x^{\sigma}(t) - A^*(t)u(t), \end{aligned} \tag{1.3}$$

where t takes values in a time scale $\mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$ is the forward jump operator on $\mathbb{T}$, $x^{\sigma} = x \circ \sigma$, and Δ denotes the Hilger derivative. A universal method we provided here allows one to treat both continuous and discrete linear Hamiltonian systems as special cases within one theory and to explain the discrepancies between them. This paper extends the Weyl-Titchmarsh theory and provides a foundation for studying spectral theory of Hamiltonian dynamic systems on time scales. Some ideas in this paper are motivated by some works in [5–9, 11, 18–20, 34, 35, 42].

The paper is organized as follows. Some fundamental theory for Hamiltonian systems is given in Section 2. Some regular spectral problems are considered in Section 3. The Weyl matrix disks are constructed and their properties are studied in Section 4. These matrix disks are nested and converge to a limiting set of the matrix circle. The results are some generalizations of the Weyl-Titchmarsh theory for both Hamiltonian differential systems [6, 9, 18, 20, 26] and discrete Hamiltonian systems [34]. These investigations are part of a larger program which includes the following: (i) $M(\lambda)$ theory for singular Hamiltonian systems, (ii) on the spectrum of Hamiltonian systems, (iii) on boundary value problems for Hamiltonian dynamic systems.

2. Assumptions and Preliminary Results

Throughout we use the following assumption.

Assumption 1. $\widetilde{\mathbb{T}}$ is a time scale that is unbounded above, that is, $\widetilde{\mathbb{T}}$ is a closed subset of $\mathbb{R}$ such that $\sup \widetilde{\mathbb{T}} = \infty$. We let $a \in \widetilde{\mathbb{T}}$ and define $\mathbb{T} = \widetilde{\mathbb{T}} \cap [a, \infty)$.

In this section, we shall study the fundamental theory and properties of solutions for the Hamiltonian dynamic system (1.3), that is,

$$\mathcal{J}y^{\Delta}(t) := (\lambda\mathcal{W}(t) + \mathcal{P}(t))\widetilde{y}(t) \quad \text{for } t \in \mathbb{T}, \tag{2.1}$$

where

$$y = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \widetilde{y} = \begin{pmatrix} x^{\sigma} \\ u \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} -C & A^* \\ A & B \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2.2}$$

are subject to the following assumptions.

Assumption 2. A, B, C, W_1, W_2 are $n\times n$ complex-valued matrix functions belonging to $C_{\mathrm{rd}}(\mathbb{T})$, $A^*(t)$ is the complex conjugate transpose of $A(t)$, and $B(t)$, $C(t)$, $W_1(t)$, $W_2(t)$ are Hermitian, $W_1(t)$, $W_2(t)$ are nonnegative definite, and

$$\widetilde{A}(t) := I_n - \mu(t)A(t) \quad \text{is nonsingular on } \mathbb{T}, \tag{2.3}$$

I_n is the $n \times n$ identity matrix, and μ is the graininess of $\mathbb{T}$ defined by $\mu(t) := \sigma(t) - t$.

Remark 2.1. If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ and all points in $\mathbb{T}$ satisfy $\sigma(t) = t$, and (1.3) becomes (1.1). If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta} = \Delta x$ and all points in $\mathbb{T}$ satisfy $\sigma(t) = t + 1$, and (1.3) turns into (1.2).

Assumption 3. We always assume that the following definiteness condition holds: for any nontrivial solution y of (2.1), we have

$$\int_a^c \widetilde{y}^*(\tau)\mathcal{W}(\tau)\widetilde{y}(\tau)\Delta\tau > 0, \quad \forall c \in \mathbb{T} \setminus \{a\}. \tag{2.4}$$

Remark 2.2. If $\mathbb{T} = \mathbb{R}$, then the condition (2.4) is just the same as Atkinson's definiteness condition. In the case of $\mathbb{T} = \mathbb{N}$, the condition is the one used in [34, 35].

By a solution of (2.1), we mean an $n \times 1$ vector-valued function y satisfying (2.1) on $\mathbb{T}$. Now we consider the existence of solutions to (2.1).

Theorem 2.3 (Existence and Uniqueness Theorem). *For arbitrary initial data $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{C}^{2n}$, the initial value problem of* (2.1) *with $y(t_0) = y_0$ has a unique solution on $\mathbb{T}$.*

Proof. By [43, Proposition 1.1], we can rewrite (2.1) as

$$y^{\Delta}(t) = \mathcal{A}(t,\lambda)y(t), \quad t \in \mathbb{T}, \tag{2.5}$$

where

$$\mathcal{A}(\cdot,\lambda) = \begin{pmatrix} \widetilde{A}A & \widetilde{A}(B + \lambda W_2) \\ (C - \lambda W_1)\widetilde{A} & \mu(C - \lambda W_1)\widetilde{A}(B + \lambda W_2) - A^* \end{pmatrix} \tag{2.6}$$

and $\mathcal{H}(\cdot,\lambda)$ is symplectic with respect to $\mathbb{T}$, that is,

$$\mathcal{H}^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}+\mathcal{J}\mathcal{H}(\cdot,\lambda)+\mu\mathcal{H}^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}\mathcal{H}(\cdot,\lambda)=0 \quad \text{on } \mathbb{T}, \tag{2.7}$$

and hence $I_{2n}+\mu\mathcal{H}(\cdot,\lambda)$ is symplectic. So $I_{2n}+\mu\mathcal{H}(\cdot,\lambda)$ is invertible and thus (2.1) has a unique solution by [39, Theorem 5.24]. This completes the proof. □

Now we consider the structure of solutions for the system (2.1).

Proposition 2.4. *If y_1, y_2 are solutions of* (2.1)*, then any linear combination of y_1 and y_2 is also a solution of* (2.1)*.*

Proposition 2.5. *There exist $2n$ linearly independent solutions $y_1,\ldots,y_{2n}$ of the system* (2.1)*, and every solution y of the system* (2.1) *can be expressed in the form $y=c_1y_1+\cdots+c_{2n}y_{2n}$, where $c_1,\ldots,c_{2n}\in\mathbb{C}$ are constants.*

Every such set of $2n$ linearly independent solutions $y_1,y_2,\ldots,y_{2n}$ is called a *fundamental solution set*. The matrix-valued function $Y=(y_1,y_2,\ldots,y_{2n})$ is called a *fundamental matrix* for the system (2.1).

Corollary 2.6. *Let Z be a fundamental matrix for* (2.1)*. Then every solution of* (2.1) *can be expressed by $z=Zc$ for some $c\in\mathbb{C}^{2n}$.*

Lemma 2.7. *Let $Y(\cdot,\lambda)$ be a fundamental matrix for the system* (2.1)*. Then*

$$Y^*\left(t,\overline{\lambda}\right)\mathcal{J}Y(t,\lambda)=Y^*\left(a,\overline{\lambda}\right)\mathcal{J}Y(a,\lambda), \quad \forall t\in\mathbb{T}. \tag{2.8}$$

Proof. From (2.5) and (2.7), we have

$$\begin{aligned}\left(Y^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}Y(\cdot,\lambda)\right)^{\Delta}&=(Y^*)^{\Delta}\left(\cdot,\overline{\lambda}\right)\mathcal{J}Y^{\sigma}(\cdot,\lambda)+Y^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}Y^{\Delta}(\cdot,\lambda)\\&=(Y^*)^{\Delta}\left(\cdot,\overline{\lambda}\right)\mathcal{J}\left(Y(\cdot,\lambda)+\mu Y^{\Delta}(\cdot,\lambda)\right)+Y^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}Y^{\Delta}(\cdot,\lambda)\\&=Y^*\left(\cdot,\overline{\lambda}\right)\left[\mathcal{H}^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}+\mathcal{J}\mathcal{H}(\cdot,\lambda)+\mu\mathcal{H}^*\left(\cdot,\overline{\lambda}\right)\mathcal{J}\mathcal{H}(\cdot,\lambda)\right]Y(\cdot,\lambda)\\&=0\end{aligned} \tag{2.9}$$

on $\mathbb{T}$, and so by [39, Corollary 1.68] there exists a constant matrix $\widetilde{C}$ with $Y^*(\cdot,\overline{\lambda})\mathcal{J}Y(\cdot,\lambda)=\widetilde{C}$ on $\mathbb{T}$. This completes the proof. □

In the rest of the paper, we use the following notation for the imaginary part of a complex number or matrix:

$$\Im\lambda=\frac{\lambda-\overline{\lambda}}{2i}, \qquad \Im M=\frac{M-M^*}{2i}. \tag{2.10}$$

3. Eigenvalue Problems

Let $y \in \mathbb{C}^{2n}$ be defined on $[a,b] \subset \mathbb{T}$ with $b \in \mathbb{T}$ and let $\mathcal{M}, \mathcal{N} \in \mathbb{C}^{2n\times 2n}$. We consider the boundary condition

$$\mathcal{M}y(a) + \mathcal{N}y(b) = 0. \tag{3.1}$$

Definition 3.1. The boundary condition (3.1) is called *formally self adjoint* if

$$y_1^* \mathcal{J} y_2 \big|_a^b = 0, \quad \forall y_1, y_2 \in \mathbb{C}^{2n} \quad \text{satisfying (3.1).} \tag{3.2}$$

Lemma 3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be $2n \times 2n$ matrices such that* $\operatorname{rank}(\mathcal{M}, \mathcal{N}) = 2n$. *Then the boundary condition* (3.1) *is formally self adjoint if and only if $\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{N}\mathcal{J}\mathcal{N}^*$.*

Proof. Let $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ be any matrix with $\operatorname{Im} P = \operatorname{Ker}(\mathcal{M}, \mathcal{N})$. Then $\mathcal{M}P_1 + \mathcal{N}P_2 = 0$, $\operatorname{rank} P = 2n$, and so $\mathcal{M}y(a) + \mathcal{N}y(b) = 0$ if and only if $y(a) = P_1 c$ and $y(b) = P_2 c$ for some $c \in \mathbb{C}^{2n}$. This yields that the boundary condition (3.1) is formally self adjoint if and only if

$$y_1^* \mathcal{J} y_2 \big|_a^b = c_1^* \left(P_2^* \mathcal{J} P_2 - P_1^* \mathcal{J} P_1 \right) c_2 = 0, \quad \forall c_1, c_2 \in \mathbb{C}^{2n}, \tag{3.3}$$

that is,

$$P_2^* \mathcal{J} P_2 = P_1^* \mathcal{J} P_1. \tag{3.4}$$

First, assume that $\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{N}\mathcal{J}\mathcal{N}^*$. From $\operatorname{rank}(\mathcal{M}, \mathcal{N}) = 2n$, we can conclude that

$$\operatorname{Im} \begin{pmatrix} \mathcal{J}\mathcal{M}^* \\ -\mathcal{J}\mathcal{N}^* \end{pmatrix} = \operatorname{Ker}(\mathcal{M}, \mathcal{N}). \tag{3.5}$$

Hence, the matrix P above can be taken to be $\begin{pmatrix} \mathcal{J}\mathcal{M}^* \\ -\mathcal{J}\mathcal{N}^* \end{pmatrix}$ and then $\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{N}\mathcal{J}\mathcal{N}^*$ yields $P_2^* \mathcal{J} P_2 = P_1^* \mathcal{J} P_1$, which means that the boundary condition (3.1) is formally self adjoint.

Next, assume that the boundary condition is self adjoint, that is, $P_2^* \mathcal{J} P_2 = P_1^* \mathcal{J} P_1$. Then $P_1^* \mathcal{J} P_1 - P_2^* \mathcal{J} P_2 = 0$, $P_1^* \mathcal{M}^* + P_2^* \mathcal{N}^* = 0$ and $\operatorname{rank} P = \operatorname{rank}(\mathcal{M}, \mathcal{N}) = 2n$ imply that

$$\operatorname{Ker}\left(P_1^*, P_2^*\right) = \operatorname{Im} \begin{pmatrix} \mathcal{M}^* \\ \mathcal{N}^* \end{pmatrix} = \operatorname{Im} \begin{pmatrix} \mathcal{J} P_1 \\ -\mathcal{J} P_2 \end{pmatrix}. \tag{3.6}$$

Hence $\mathcal{M}^* = \mathcal{J} P_1 S$ and $\mathcal{N}^* = -\mathcal{J} P_2 S$ for some invertible matrix S, and it follows that

$$\mathcal{M}\mathcal{J}\mathcal{M}^* = S^* P_1^* \mathcal{J}^* \mathcal{J}\mathcal{J} P_1 S = S^* P_2^* \mathcal{J}^* \mathcal{J}\mathcal{J} P_2 S = N\mathcal{J}N^*, \tag{3.7}$$

which completes the proof. $\square$

Now we consider the system (2.1) with the formally self-adjoint boundary conditions

$$\alpha y(a) = 0, \qquad \beta y(b) = 0, \tag{3.8}$$

where α and β are $n \times 2n$ matrices satisfying the self-adjoint conditions

$$\begin{aligned} &\operatorname{rank} \alpha = n, \qquad \alpha\alpha^* = I_n, \qquad \alpha\mathcal{J}\alpha^* = 0, \\ &\operatorname{rank} \beta = n, \qquad \beta\beta^* = I_n, \qquad \beta\mathcal{J}\beta^* = 0. \end{aligned} \tag{3.9}$$

Since (3.9) can be written as $\mathcal{M}y(a) = 0$, $\mathcal{N}y(b) = 0$, where

$$\mathcal{M} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \tag{3.10}$$

we have $\mathcal{M}\mathcal{J}\mathcal{M}^* = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}\mathcal{J}(\alpha^* \;\; 0) = 0$ and $\mathcal{N}\mathcal{J}\mathcal{N}^* = \begin{pmatrix} 0 \\ \beta \end{pmatrix}\mathcal{J}(0 \;\; \beta^*) = 0$. Hence, by Lemma 3.2, the boundary condition (3.8) is self adjoint.

Let $\theta(\cdot, \lambda)$ and $\phi(\cdot, \lambda)$ be the $2n \times n$ matrix-valued solutions of (2.1) satisfying

$$\theta(a, \lambda) = \alpha^*, \qquad \phi(a, \lambda) = \mathcal{J}\alpha^*. \tag{3.11}$$

It is clear that $\alpha\theta(a, \lambda) = I_n$ and $\alpha\phi(a, \lambda) = 0$. Set $Y = (\theta \;\; \phi)$. Then $Y(\cdot, \lambda)$ is the fundamental matrix for (2.1) satisfying $Y(a, \lambda) = (\alpha^* \;\; \mathcal{J}\alpha^*)$.

Lemma 3.3. *Let $Y(\cdot, \lambda)$ be the fundamental matrix for* (2.1) *satisfying $Y(a, \lambda) = (\alpha^* \;\; \mathcal{J}\alpha^*)$. Then*

$$Y^*\left(\cdot, \overline{\lambda}\right)\mathcal{J}Y(\cdot, \lambda) = Y(\cdot, \lambda)\mathcal{J}Y^*\left(\cdot, \overline{\lambda}\right) = \mathcal{J} \quad \text{on } \mathbb{T}. \tag{3.12}$$

Proof. From Lemma 2.7,

$$Y^*\left(t, \overline{\lambda}\right)\mathcal{J}Y(t, \lambda) = Y^*\left(a, \overline{\lambda}\right)\mathcal{J}Y(a, \lambda) = \begin{pmatrix} \alpha \\ \alpha\mathcal{J}^* \end{pmatrix}\mathcal{J}(\alpha^* \;\; \mathcal{J}\alpha^*) = \mathcal{J}, \quad \forall t \in \mathbb{T}. \tag{3.13}$$

Furthermore, $-\mathcal{J}Y^*(\cdot, \overline{\lambda})\mathcal{J}Y(\cdot, \lambda) = I_{2n}$ on $\mathbb{T}$ implies $\mathcal{J}Y(\cdot, \lambda)(-\mathcal{J}Y^*(\cdot, \overline{\lambda})) = I_{2n}$ on $\mathbb{T}$. It follows that $Y(\cdot, \lambda)\mathcal{J}Y^*(\cdot, \overline{\lambda}) = \mathcal{J}$ on $\mathbb{T}$. This completes the proof. □

Theorem 3.4. *Assume* (3.9). *Then λ is an eigenvalue of the problem* (2.1), (3.8) *if and only if* $\det(\beta\phi(b, \lambda)) = 0$, *and $y(\cdot, \lambda)$ is a corresponding eigenfunction if and only if there exists a vector $\xi \in \mathbb{C}^n$ such that $y(\cdot, \lambda) = \phi(\cdot, \lambda)\xi$ on $\mathbb{T}$, where ξ is a nonzero solution of the equation $\beta\phi(b, \lambda)\xi = 0$.*

Proof. Assume (3.9). Let λ be an eigenvalue of the eigenvalue problem (2.1), (3.8) with corresponding eigenfunction $y(\cdot, \lambda)$. Then there exists a unique constant vector $\eta \in \mathbb{C}^{2n} \setminus \{0\}$ such that

$$y(t, \lambda) = Y(t, \lambda)\eta, \quad \forall t \in \mathbb{T}. \tag{3.14}$$

Then, using (3.8) and (3.9),

$$0 = \alpha y(a,\lambda) = \alpha Y(a,\lambda)\eta = \alpha(\alpha^* \quad \mathcal{J}\alpha^*)\eta = (I_n \quad 0)\eta = \zeta, \quad \text{where } \eta = \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \tag{3.15}$$

with $\zeta, \xi \in \mathbb{C}^n$. Thus $y(t,\lambda) = \phi(t,\lambda)\xi$, and (3.8) implies that $\beta\phi(b,\lambda)\xi = 0$. Clearly, $\xi \neq 0$, since $y(\cdot,\lambda) \neq 0$. Hence ξ is a nonzero solution of $\beta\phi(b,\lambda)\xi = 0$. Thus $\det(\beta\phi(b,\lambda)) = 0$.

Conversely, if λ satisfies $\det(\beta\phi(b,\lambda)) = 0$, then $\beta\phi(b,\lambda)\xi = 0$ has a nonzero solution ξ. Let $y(\cdot,\lambda) = \phi(\cdot,\lambda)\xi$. Then $\beta y(b,\lambda) = 0$. Moreover, $\alpha y(a,\lambda) = \alpha\phi(a,\lambda)\xi = \alpha\mathcal{J}\alpha^* = 0$ by (3.9). Taking into account $\operatorname{rank}\phi(a,\lambda) = \operatorname{rank}(\mathcal{J}\alpha^*) = n$, we get that $y(\cdot,\lambda)$ is a nontrivial solution of (2.1). This completes the proof. □

Lemma 3.5. *Let $y(\cdot,\lambda)$ and $y(\cdot,\nu)$ be any solutions of* (2.1) *corresponding to the parameters λ, $\nu \in \mathbb{C}$. Then*

$$y^*(t,\nu)\mathcal{J}y(t,\lambda)\big|_a^b = (\lambda - \overline{\nu})\int_a^b \tilde{y}^*(t,\nu)\mathcal{W}(t)\tilde{y}(t,\lambda)\Delta t. \tag{3.16}$$

In particular,

$$y^*(t,\lambda)\mathcal{J}y(t,\lambda)\big|_a^b = 2i\Im\lambda\int_a^b \tilde{y}^*(t,\lambda)\mathcal{W}(t)\tilde{y}(t,\lambda)\Delta t. \tag{3.17}$$

Proof. Set

$$(ly)(t,\lambda) = \mathcal{J}y^{\Delta}(t,\lambda) - \mathcal{P}(t)\tilde{y}(t,\lambda). \tag{3.18}$$

Then from [44, Lemma 2] we have

$$\begin{aligned} y^*(t,\nu)\mathcal{J}y(t,\lambda)\big|_a^b &= \int_a^b \left[\tilde{y}^*(t,\nu)(ly)(t,\lambda) - (ly)^*(t,\nu)\tilde{y}(t,\lambda)\right]\Delta t \\ &= \int_a^b \left[\tilde{y}^*(t,\nu)\lambda\mathcal{W}(t)\tilde{y}(t,\lambda) - (\nu\mathcal{W}(t)\tilde{y}(t,\nu))^*\tilde{y}(t,\lambda)\right]\Delta t \\ &= (\lambda - \overline{\nu})\int_a^b \tilde{y}^*(t,\nu)\mathcal{W}(t)\tilde{y}(t,\lambda)\Delta t. \end{aligned} \tag{3.19}$$

This completes the proof. □

Theorem 3.6. *Assume* (3.9)*. Then all eigenvalues of* (2.1), (3.8) *are real, and eigenvectors corresponding to different eigenvalues are orthogonal.*

Proof. Assume (3.9) and let λ be an eigenvalue of (2.1), (3.8) with corresponding eigenfunction $y(\cdot,\lambda)$. Hence $y(\cdot,\lambda)$ satisfies (2.1) and

$$y(a,\lambda) \in \operatorname{Ker}\alpha = \operatorname{Im}\mathcal{J}\alpha^*, \qquad y(b,\lambda) \in \operatorname{Ker}\beta = \operatorname{Im}\mathcal{J}\beta^*, \tag{3.20}$$

which follows from (3.9) and [10, Corollary 3.1.3]. Thus there exist $c_1, c_2 \in \mathbb{C}^n$ such that

$$y(a,\lambda) = \mathcal{J}\alpha^* c_1, \qquad y(b,\lambda) = \mathcal{J}\beta^* c_2. \tag{3.21}$$

Using Lemma 3.5 and (3.9), we have

$$2i\Im\lambda \int_a^b \tilde{y}^*(t,\lambda)\mathcal{W}(t)\tilde{y}(t,\lambda)\Delta t = y^*(t,\lambda)\mathcal{J}y(t,\lambda)|_a^b = c_2^*\beta\mathcal{J}^*\mathcal{J}\mathcal{J}\beta^* c_2 - c_1^*\alpha\mathcal{J}^*\mathcal{J}\mathcal{J}\alpha^* c_1 = 0 \tag{3.22}$$

so that $\Im\lambda = 0$ and $\lambda \in \mathbb{R}$. Now let $y(\cdot,\lambda)$ and $y(\cdot,\nu)$ be eigenfunctions corresponding to the eigenvalues $\lambda \neq \nu$. Then, using Lemma 3.5 and proceeding as above, we have

$$(\lambda - \overline{\nu}) \int_a^b \tilde{y}^*(t,\nu)\mathcal{W}(t)\tilde{y}(t,\lambda)\Delta t = y^*(t,\nu)\mathcal{J}y(t,\lambda)|_a^b = 0 \tag{3.23}$$

so that $y(\cdot,\lambda)$ and $y(\cdot,\nu)$ are orthogonal. □

4. Weyl-Titchmarsh Circles and Disks

In this section, we consider the construction of Weyl-Titchmarsh disks and circles for Hamiltonian dynamic systems (2.1). Assume (3.9) and let α, β and $Y(\cdot,\lambda)$ be defined as in Section 3. Suppose $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and set

$$\chi_b(\cdot,\lambda) = Y(\cdot,\lambda)\begin{pmatrix} I_n \\ M(b,\lambda) \end{pmatrix}, \quad \text{where } M(b,\lambda) = M_\beta(b,\lambda) = -(\beta\phi(b,\lambda))^{-1}\beta\theta(b,\lambda) \tag{4.1}$$

(observe Theorems 3.4 and 3.6). For any $n \times n$ matrix M, define

$$\begin{gathered} \mathcal{E}(M,b,\lambda) := -i\,\mathrm{sgn}(\Im\lambda)(I_n \;\; M^*)Y^*(b,\lambda)\mathcal{J}Y(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix}, \\ \chi(\cdot,\lambda) = Y(\cdot,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix}. \end{gathered} \tag{4.2}$$

It is clear that

$$\mathcal{E}(M(b,\lambda),b,\lambda) = -i\,\mathrm{sgn}(\Im\lambda)\chi_b^*(b,\lambda)\mathcal{J}\chi_b(b,\lambda). \tag{4.3}$$

Definition 4.1. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The sets

$$\mathfrak{D}(b,\lambda) = \{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}(M,b,\lambda) \leq 0\} \quad \text{and} \quad \mathcal{K}(b,\lambda) = \{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}(M,b,\lambda) = 0\} \tag{4.4}$$

are called a *Weyl disk* and a *Weyl circle*, respectively.

Theorem 4.2. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathcal{K}(b,\lambda) = \{M_\beta(b,\lambda) \mid \beta \text{ satisfies } (3.9)\}. \tag{4.5}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Assume that β satisfies (3.9). Let $\eta \in \mathbb{C}^{2n}$. Then $\beta\chi_b(b,\lambda)\eta = 0$ so that (use again (3.9) and [10, Corollary 3.1.3])

$$\chi_b(b,\lambda)\eta \in \operatorname{Ker}\beta = \operatorname{Im}\mathcal{J}^*\beta, \tag{4.6}$$

and thus there exists $c \in \mathbb{C}^n$ such that $\chi_b(b,\lambda)\eta = \mathcal{J}^*\beta c$. Hence

$$\eta^*\chi_b^*(b,\lambda)\mathcal{J}\chi_b(b,\lambda)\eta = c^*\beta^*\mathcal{J}\mathcal{J}\mathcal{J}^*\beta c = c^*\beta^*\mathcal{J}\beta c = 0 \tag{4.7}$$

by (3.9). So $\chi_b^*(b,\lambda)\mathcal{J}\chi_b(b,\lambda) = 0$, that is, $\mathcal{E}(M(b,\lambda),b,\lambda) = 0$.

Conversely, if $\mathcal{E}(M,b,\lambda) = 0$, then

$$0 = \begin{pmatrix} I_n & M^* \end{pmatrix} Y^*(b,\lambda)\mathcal{J}Y(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix} = \gamma\mathcal{J}\gamma^*, \quad \text{where } \gamma = \begin{pmatrix} I_n & M^* \end{pmatrix} Y^*(b,\lambda)\mathcal{J}. \tag{4.8}$$

Then $\operatorname{rank}\gamma = n$ and $\gamma\chi(b,\lambda) = 0$. Since

$$\gamma\gamma^* = \begin{pmatrix} I_n & M^* \end{pmatrix} Y^*(b,\lambda)Y(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix} > 0, \tag{4.9}$$

we can define $\beta = (\gamma\gamma^*)^{-1/2}\gamma$. Then β satisfies (3.9) and $\beta Y(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix} = 0$. It follows that $M = -(\beta\phi(b,\lambda))^{-1}\beta\theta(b,\lambda) = M(b,\lambda)$. □

Let

$$\mathcal{F}(b,\lambda) := -i\operatorname{sgn}(\Im\lambda)Y^*(b,\lambda)\mathcal{J}Y(b,\lambda). \tag{4.10}$$

Then $\mathcal{F}(b,\lambda)$ is a $2n \times 2n$ Hermitian matrix and

$$\mathcal{E}(M,b,\lambda) = \begin{pmatrix} I_n & M^* \end{pmatrix} \mathcal{F}(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix}. \tag{4.11}$$

Lemma 4.3. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $b \geq a$, we have*

$$\mathcal{F}(b,\lambda) = \operatorname{sgn}(\Im\lambda)\left(-i\mathcal{J} + 2\Im\lambda\int_a^b \widetilde{Y}^*(t,\lambda)\mathcal{W}(t)\widetilde{Y}(t,\lambda)\Delta t\right), \tag{4.12}$$

$$\int_a^b \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t = \frac{1}{2|\Im\lambda|}\mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im\lambda}. \tag{4.13}$$

Proof. From Lemma 3.5, we obtain

$$\begin{aligned}
Y^*(b,\lambda)\mathcal{J}Y(b,\lambda) &= Y^*(a,\lambda)\mathcal{J}Y(a,\lambda) + 2i\Im\lambda \int_a^b \widetilde{Y}^*(t,\lambda)\mathcal{W}(t)\widetilde{Y}(t,\lambda)\Delta t \\
&= \mathcal{J} + 2i\Im\lambda \int_a^b \widetilde{Y}^*(t,\lambda)\mathcal{W}(t)\widetilde{Y}(t,\lambda)\Delta t,
\end{aligned} \tag{4.14}$$

and so (4.12) follows from (4.10). From (4.12), we obtain

$$\begin{aligned}
\int_a^b \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t &= \begin{pmatrix} I_n & M^* \end{pmatrix} \int_a^b \widetilde{Y}^*(t,\lambda)\mathcal{W}(t)\widetilde{Y}(t,\lambda)\Delta t \begin{pmatrix} I_n \\ M \end{pmatrix} \\
&= \frac{1}{2|\Im\lambda|}\begin{pmatrix} I_n & M^* \end{pmatrix}\left[\mathcal{F}(b,\lambda) + i\,\mathrm{sgn}(\Im\lambda)\mathcal{J}\right]\begin{pmatrix} I_n \\ M \end{pmatrix} \\
&= \frac{1}{2|\Im\lambda|}\begin{pmatrix} I_n & M^* \end{pmatrix}\mathcal{F}(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix} + \frac{\Im M}{\Im\lambda} \\
&= \frac{1}{2|\Im\lambda|}\mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im\lambda}.
\end{aligned} \tag{4.15}$$

This completes the proof. □

Theorem 4.4. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathfrak{D}(b_2,\lambda) \subset \mathfrak{D}(b_1,\lambda) \quad \text{for any } b_1, b_2 \in \mathbb{T} \quad \text{with } b_1 < b_2. \tag{4.16}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $b_1 < b_2$. Assume $M \in \mathfrak{D}(b_2,\lambda)$. Then $\mathcal{E}(M,b_2,\lambda) \le 0$. By Lemma 3.5 and Assumption 2,

$$\mathcal{F}(b_2,\lambda) - \mathcal{F}(b_1,\lambda) = 2|\Im\lambda| \int_{b_1}^{b_2} \widetilde{Y}^*(t,\lambda)\mathcal{W}(t)\widetilde{Y}(t,\lambda)\Delta t > 0, \tag{4.17}$$

which implies that $\mathcal{E}(M,b_2,\lambda) \ge \mathcal{E}(M,b_1,\lambda)$. From this, we have $\mathcal{E}(M,b_1,\lambda) \le 0$. Thus $M \in \mathfrak{D}(b_1,\lambda)$. □

Now we study convergence of the disks. For this purpose, we denote

$$\mathcal{F}(b,\lambda) = \begin{pmatrix} F_{11}(b,\lambda) & F_{12}(b,\lambda) \\ F_{12}^*(b,\lambda) & F_{22}(b,\lambda) \end{pmatrix}, \tag{4.18}$$

where $F_{11}(b,\lambda), F_{12}(b,\lambda)$, and $F_{22}(b,\lambda)$ are $n \times n$ matrices.

Lemma 4.5. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $F_{11}(b,\lambda)$ and $F_{22}(b,\lambda)$ are positive definite and nondecreasing in b.*

Proof. From (4.10), (4.12), and (4.13), we have

$$F_{11}(b,\lambda) = -i\,\mathrm{sgn}(\Im\lambda)\theta^*(b,\lambda)\mathcal{J}\theta(b,\lambda) = 2|\Im\lambda|\int_a^b \tilde{\theta}^*(t,\lambda)\mathcal{W}(t)\tilde{\theta}(t,\lambda)\Delta t,$$
$$F_{22}(b,\lambda) = -i\,\mathrm{sgn}(\Im\lambda)\phi^*(b,\lambda)\mathcal{J}\phi(b,\lambda) = 2|\Im\lambda|\int_a^b \tilde{\phi}^*(t,\lambda)\mathcal{W}(t)\tilde{\phi}(t,\lambda)\Delta t. \tag{4.19}$$

Employing Assumption 2 completes the proof. □

Using the notation of (4.13), we find that (4.11) can be rewritten as

$$\begin{aligned}\mathcal{E}(M,b,\lambda) &= M^*F_{22}(b,\lambda)M + F_{12}(b,\lambda)M + M^*F_{12}^*(b,\lambda) + F_{11}(b,\lambda)\\ &= \left[M + F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)\right]^* F_{22}(b,\lambda)\left[M + F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)\right]\\ &\quad + F_{11}(b,\lambda) - F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda).\end{aligned} \tag{4.20}$$

Lemma 4.6. *For $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda) - F_{11}(b,\lambda) = F_{22}^{-1}(b,\overline{\lambda})$.*

Proof. By applying Lemma 3.3 twice, we find

$$\begin{aligned}\mathcal{F}^*(b,\lambda)\mathcal{J}\mathcal{F}\left(b,\overline{\lambda}\right) &= (-i^*\,\mathrm{sgn}(\Im\lambda))\left(-i\,\mathrm{sgn}\left(\Im\overline{\lambda}\right)\right)\Upsilon^*(b,\lambda)\mathcal{J}^*\Upsilon(b,\lambda)\mathcal{J}\Upsilon^*\left(b,\overline{\lambda}\right)\mathcal{J}\Upsilon\left(b,\overline{\lambda}\right)\\ &= -\Upsilon^*(b,\lambda)\mathcal{J}^*\mathcal{J}\mathcal{J}\Upsilon\left(b,\overline{\lambda}\right) = -\Upsilon^*(b,\lambda)\mathcal{J}\Upsilon\left(b,\overline{\lambda}\right) = -\mathcal{J}.\end{aligned} \tag{4.21}$$

Hence

$$F_{12}(b,\lambda)F_{12}\left(b,\overline{\lambda}\right) - F_{11}(b,\lambda)F_{22}\left(b,\overline{\lambda}\right) = I_n, \qquad F_{22}(b,\lambda)F_{12}\left(b,\overline{\lambda}\right) - F_{12}^*(b,\lambda)F_{22}\left(b,\overline{\lambda}\right) = 0. \tag{4.22}$$

From the second relation in (4.22), we have (observe Lemma 4.3)

$$F_{12}\left(b,\overline{\lambda}\right)F_{22}^{-1}\left(b,\overline{\lambda}\right) = F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda), \tag{4.23}$$

and hence, using also the first relation in (4.22), we obtain

$$\begin{aligned}F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda) - F_{11}(b,\lambda) &= F_{12}(b,\lambda)F_{12}\left(b,\overline{\lambda}\right)F_{22}^{-1}\left(b,\overline{\lambda}\right) - F_{11}(b,\lambda)\\ &= \left(I_n + F_{11}(b,\lambda)F_{22}\left(b,\overline{\lambda}\right)\right)F_{22}^{-1}\left(b,\overline{\lambda}\right) - F_{11}(b,\lambda)\\ &= F_{22}^{-1}\left(b,\overline{\lambda}\right),\end{aligned} \tag{4.24}$$

which completes the proof. □

From Lemma 4.6, (4.11), and hence (4.20), can be rewritten in the form

$$\mathcal{E}(M,b,\lambda) = (M - \mathfrak{C}(b,\lambda))^* \mathcal{R}^{-2}(b,\lambda)(M - \mathfrak{C}(b,\lambda)) - \mathcal{R}^2\left(b,\overline{\lambda}\right), \tag{4.25}$$

where

$$\mathfrak{C}(b,\lambda) = -F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda), \qquad \mathcal{R}(b,\lambda) = F_{22}^{-1/2}(b,\lambda). \tag{4.26}$$

Definition 4.7. $\mathfrak{C}(b,\lambda)$ is called the *center* of the Weyl disk $\mathfrak{D}(b,\lambda)$ or the Weyl circle $\mathcal{K}(b,\lambda)$, while $\mathcal{R}(b,\lambda)$ and $\mathcal{R}(b,\overline{\lambda})$ are called the matrix *radii* of $\mathfrak{D}(b,\lambda)$ or $\mathcal{K}(b,\lambda)$.

Theorem 4.8. *Define the unit matrix circle and the unit matrix disk by*

$$\partial D = \left\{U \in \mathbb{C}^{n\times n} \mid U^*U = I_n\right\} \qquad \text{and} \qquad D = \left\{V \in \mathbb{C}^{n\times n} \mid V^*V \le I_n\right\}, \tag{4.27}$$

respectively. Then

$$\begin{gathered}\mathcal{K}(b,\lambda) = \left\{\mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}\left(b,\overline{\lambda}\right) \mid U \in \partial D\right\},\\ \mathfrak{D}(b,\lambda) = \left\{\mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)V\mathcal{R}\left(b,\overline{\lambda}\right) \mid V \in D\right\}.\end{gathered} \tag{4.28}$$

Proof. We only prove the first statement as the second one can be shown similarly. From (4.25),

$$\begin{gathered}\mathcal{E}(M,b,\lambda) = 0 \quad \text{if and only if}\\ \left[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}\left(b,\overline{\lambda}\right)\right]^*\left[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}\left(b,\overline{\lambda}\right)\right] = I_n.\end{gathered} \tag{4.29}$$

First, let $M \in \mathcal{K}(b,\lambda)$ and put $U = \mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\overline{\lambda})$. Then $M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}(b,\overline{\lambda})$ and (4.29) yields $U^*U = I_n$. Conversely, let U be unitary and define $M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}(b,\overline{\lambda})$. Then $U = \mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\overline{\lambda})$, so that

$$\left[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}\left(b,\overline{\lambda}\right)\right]^*\left[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}\left(b,\overline{\lambda}\right)\right] = I_n, \tag{4.30}$$

and hence (4.29) yields $M \in \mathcal{K}(b,\lambda)$. □

Theorem 4.9. *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lim_{b\to\infty}\mathcal{R}(b,\lambda)$ *exists and* $\lim_{b\to\infty}\mathcal{R}(b,\lambda) \ge 0$.

Proof. From Lemma 4.5, $F_{22}(b,\lambda) > 0$ is Hermitian and nondecreasing in b. Thus $\mathcal{R}(b,\lambda) = F_{22}^{-1/2}(b,\lambda) > 0$ is Hermitian and nonincreasing in b. Hence $\lim_{b\to\infty}\mathcal{R}(b,\lambda)$ exists and is nonnegative definite. □

Theorem 4.10. *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lim_{b\to\infty}\mathfrak{C}(b,\lambda)$ *exists.*

Proof. Let $b_1, b_2 \in \mathbb{T}$ with $b_1 < b_2$. Let $V \in D$ and define

$$M = \mathfrak{C}(b_2, \lambda) + \mathcal{R}(b_2, \lambda) V \mathcal{R}\left(b_2, \overline{\lambda}\right). \tag{4.31}$$

By Theorem 4.8, $M \in \mathfrak{D}(b_2, \lambda)$. Hence, by Theorem 4.4, $M \in \mathfrak{D}(b_1, \lambda)$. Again by Theorem 4.8, there exists $\Phi(V) \in D$ with

$$M = \mathfrak{C}(b_1, \lambda) + \mathcal{R}(b_1, \lambda) \Phi(V) \mathcal{R}\left(b_1, \overline{\lambda}\right). \tag{4.32}$$

Thus $\Phi : D \to D$ satisfies

$$\Phi(V) = \mathcal{R}^{-1}(b_1, \lambda)\left[\mathfrak{C}(b_2, \lambda) - \mathfrak{C}(b_1, \lambda) + \mathcal{R}(b_2, \lambda) V \mathcal{R}\left(b_2, \overline{\lambda}\right)\right]\mathcal{R}^{-1}\left(b_1, \overline{\lambda}\right) \tag{4.33}$$

for all $V \in D$. This implies

$$\Phi(V_I) - \Phi(V_{II}) = \mathcal{R}^{-1}(b_1, \lambda)\mathcal{R}(b_1, \lambda)[V_I - V_{II}]\mathcal{R}\left(b_2, \overline{\lambda}\right)\mathcal{R}^{-1}\left(b_2, \overline{\lambda}\right) \tag{4.34}$$

for all $V_I, V_{II} \in D$. Thus $\Phi : D \to D$ is continuous and hence has a fixed point $\widetilde{V} \in D$ by Brouwer's fixed point theorem. Letting $\Phi(\widetilde{V}) = \widetilde{V}$ in (4.33), we have

$$\begin{aligned}
\|\mathfrak{C}(b_2, \lambda) - \mathfrak{C}(b_1, \lambda)\| &= \left\|\mathcal{R}(b_1, \lambda)\widetilde{V}\mathcal{R}\left(b_1, \overline{\lambda}\right) - \mathcal{R}(b_2, \lambda)\widetilde{V}\mathcal{R}\left(b_2, \overline{\lambda}\right)\right\| \\
&\leq \left\|\mathcal{R}(b_1, \lambda)\widetilde{V}\mathcal{R}\left(b_1, \overline{\lambda}\right) - \mathcal{R}(b_1, \lambda)\widetilde{V}\mathcal{R}\left(b_2, \overline{\lambda}\right)\right\| \\
&\quad + \left\|\mathcal{R}(b_1, \lambda)\widetilde{V}\mathcal{R}\left(b_2, \overline{\lambda}\right) - \mathcal{R}(b_2, \lambda)\widetilde{V}\mathcal{R}\left(b_2, \overline{\lambda}\right)\right\| \\
&\leq \|\mathcal{R}(b_1, \lambda)\|\left\|\mathcal{R}\left(b_2, \overline{\lambda}\right) - \mathcal{R}\left(b_1, \overline{\lambda}\right)\right\| \\
&\quad + \|\mathcal{R}(b_2, \lambda) - \mathcal{R}(b_1, \lambda)\|\left\|\mathcal{R}\left(b_2, \overline{\lambda}\right)\right\|,
\end{aligned} \tag{4.35}$$

where $\|\cdot\|$ is a matrix norm. Using Theorem 4.9 completes the proof. □

Definition 4.11. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and define

$$\mathfrak{C}_0(\lambda) := \lim_{b \to \infty} \mathfrak{C}(b, \lambda), \qquad \mathcal{R}_0(\lambda) := \lim_{b \to \infty} \mathcal{R}(b, \lambda). \tag{4.36}$$

Then $\mathfrak{C}_0(\lambda)$ is called the *center*, and $\mathcal{R}_0(\lambda)$ and $\mathcal{R}_0(\overline{\lambda})$ are called the matrix *radii* of the limiting set

$$\mathfrak{D}_0(\lambda) := \left\{\mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda) V \mathcal{R}_0\left(\overline{\lambda}\right) \mid V \in D\right\}. \tag{4.37}$$

The following result gives another expression for $\mathfrak{D}_0(\lambda)$.

Theorem 4.12. *The set $\mathfrak{D}_0(\lambda)$ is given by $\mathfrak{D}_0(\lambda) = \bigcap_{b\ge a} \mathfrak{D}(b,\lambda)$.*

Proof. If $M \in \mathfrak{D}_0(\lambda)$, then there exists $V \in D$ such that $M = \mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda)V\mathcal{R}_0(\overline{\lambda})$. Hence $M = \lim_{b\to\infty} M(b)$, where $M(b) = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)V\mathcal{R}(b,\overline{\lambda})$. Let $\widetilde{b} \ge a$. Then $M(b) \in \mathfrak{D}(b,\lambda) \subset \mathfrak{D}(\widetilde{b},\lambda)$ for all $b \ge \widetilde{b}$ by Theorem 4.4 and thus $M = \lim_{b\to\infty} M(b) \in \mathfrak{D}(\widetilde{b},\lambda)$. Therefore $M \in \bigcap_{b\ge a} \mathfrak{D}(b,\lambda)$.

Conversely, if $M \in \bigcap_{b\ge a} \mathfrak{D}(b,\lambda)$, then for all $b \ge a$, there exists $V_b \in D$ such that $M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)V_b R(b,\overline{\lambda})$. Since D is compact, there exist a sequence $\{b_k\}$ and $V \in D$ such that $V_{b_k} \to V$ as $k \to \infty$. Thus $M = \mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda)V\mathcal{R}_0(\overline{\lambda}) \in \mathfrak{D}_0(\lambda)$. □

Theorem 4.13. *For all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and for $M \in \mathfrak{D}_0(\lambda)$, we have $\Im\lambda \cdot \Im M > 0$.*

Proof. Assume that $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and let $M \in \mathfrak{D}_0(\lambda)$. Fix an arbitrary $b > a$. From Theorem 4.12, $\mathfrak{D}_0(\lambda) \subset \mathfrak{D}(b,\lambda)$. Hence $M \in \mathfrak{D}(b,\lambda)$, and thus $\mathcal{E}(M,b,\lambda) \le 0$. Therefore, (4.13) and Assumption 2 yield

$$\frac{\Im M}{\Im\lambda} \ge \int_a^b \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t > 0. \tag{4.38}$$

The proof is complete. □

Definition 4.14. Let M be an $n\times n$ matrix and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. We say that

(1) M lies in the limit circle if $M \in \mathfrak{D}_0(\lambda)$;

(2) M lies on the boundary of the limit circle if $M \in \mathfrak{D}_0(\lambda)$ and there exists a sequence $b_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} \mathcal{E}(M,b_k,\lambda) = 0$.

Theorem 4.15. *Let $M \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

(1) *M lies in the limit circle if and only if*

$$\int_a^\infty \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t \le \frac{\Im M}{\Im\lambda}; \tag{4.39}$$

(2) *M lies on the boundary of the limit circle if and only if*

$$\int_a^\infty \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t = \frac{\Im M}{\Im\lambda}. \tag{4.40}$$

Proof. Assume $M \in \mathfrak{D}_0(\lambda)$. Let $b > a$. Then $M \in \mathfrak{D}(b,\lambda)$ by Theorem 4.12. Hence $\mathcal{E}(M,b,\lambda) \le 0$. From (4.13), we have that

$$\int_a^b \widetilde{\chi}^*(t,\lambda)\mathcal{W}(t)\widetilde{\chi}(t,\lambda)\Delta t = \frac{1}{2|\Im\lambda|}\mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im\lambda} \le \frac{\Im M}{\Im\lambda}. \tag{4.41}$$

Letting $b \to \infty$, we arrive at (4.39). Conversely, assume that (4.39) holds. Let $b \geq a$. By Assumption 2,

$$\int_a^b \tilde{\chi}^*(t,\lambda)\mathcal{W}(t)\tilde{\chi}(t,\lambda)\Delta t \leq \int_a^\infty \tilde{\chi}^*(t,\lambda)\mathcal{W}(t)\tilde{\chi}(t,\lambda)\Delta t \leq \frac{\Im M}{\Im\lambda}. \tag{4.42}$$

So $\mathcal{E}(M,b,\lambda) \leq 0$ by (4.13). This shows that $M \in \mathfrak{D}(b,\lambda)$. Using Theorem 4.12 yields $M \in \mathfrak{D}_0(\lambda)$. This proves (1), and (2) can be concluded immediately by result (1) and (4.13). □

Theorem 4.16. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then M lies on the boundary of the limit circle if and only if $\lim_{t\to\infty}\chi^*(t,\lambda)\mathcal{J}\chi(t,\lambda) = 0$.*

Proof. From Lemma 3.5, for any $t > a$, we have that

$$\chi^*(\cdot,\lambda)\mathcal{J}\chi(\cdot,\lambda)\big|_a^t = 2i\Im\lambda\int_a^t \tilde{\chi}^*(s,\lambda)\mathcal{W}(s)\tilde{\chi}(s,\lambda)\Delta s. \tag{4.43}$$

Since

$$\chi^*(a,\lambda)\mathcal{J}\chi(a,\lambda) = M^* - M = -2i\Im M, \tag{4.44}$$

we get

$$\chi^*(t,\lambda)\mathcal{J}\chi(t,\lambda) = 2i\Im\lambda\int_a^t \tilde{\chi}^*(s,\lambda)\mathcal{W}(s)\tilde{\chi}(s,\lambda)\Delta s - 2i\Im M. \tag{4.45}$$

From Theorem 4.15, M is on the boundary of the limit circle if and only if

$$\Im\lambda\cdot\int_a^\infty \tilde{\chi}^*(t,\lambda)\mathcal{W}(t)\tilde{\chi}(t,\lambda)\Delta t - \Im M = 0. \tag{4.46}$$

So by (4.45), we have that M is on the boundary of the limit circle if and only if

$$\lim_{t\to\infty}\chi^*(t,\lambda)\mathcal{J}\chi(t,\lambda) = 0. \tag{4.47}$$

This completes the proof. □

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