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Research Article

A Refinement of Quasilinearization Method for Caputo's Sense Fractional-Order Differential Equations

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The method of the quasilinearization technique in Caputo's sense fractional-order differential equation is applied to obtain lower and upper sequences in terms of the solutions of linear Caputo's sense fractional-order differential equations. It is also shown that these sequences converge to the unique solution of the nonlinear Caputo's sense fractional-order differential equation uniformly and semiquadratically with less restrictive assumptions.

1. Introduction

The well-known quasilinearization method [1, 2] in differential equation has been employed to obtain a sequence of lower and upper bounds which are the solutions of linear differential equations that converge quadratically. However, the convexity and concavity assumption that is demanded by the method of quasilinearization has been a stumbling block for further development of the theory. Recently, this method has been generalized, refined, and extended in several directions so as to be applicable to a much larger class of nonlinear problems by not demanding convexity and concavity property [1, 3–7]. Moreover, other possibilities that have been explored make the method of generalized quasilinearization universally useful in applications [3, 6, 7].

The theory of nonlinear fractional-order dynamic systems has been investigated depending on the development in the theory of fractional-order differential equations. In this context, generalized quasilinearization method has been reconsidered, and similar results parallel to classical theory of differential equations have been obtained [1, 2, 8].

In this work, the quasilinearization technique coupled with lower and upper solutions is employed to study Caputo's fractional-order differential equation for which particular and general results that include several special cases are obtained. Moreover, one gets monotone sequences whose iterates are the solutions of corresponding linear problems and the sequences converge to the solutions of the original nonlinear problems. Instead of imposing the convexity assumption on the function involved, we assume weaker conditions as well as for the concave functions. This is a definite advantage of this constructive technique. Furthermore, these monotone flows are shown to converge semiquadratically.

Consider the following initial value problem:

$$^{c}D^{q}x(t) = F(t,x), \quad x(t_0) = x_0 \quad \text{for } t \geq t_0, \tag{1.1}$$

where $F \in C([t_0,T] \times \mathbb{R}, \mathbb{R})$ and $^{c}D^{q}$ is Caputo's sense fractional-order derivative. Let $\alpha_0, \beta_0 \in C^{q}([t_0,T], \mathbb{R})$ and be the lower and upper solutions of (1.1) satisfying the following inequalities (1.2) and (1.3), respectively, on J:

$$^{c}D^{q}\alpha_0 \leq F(t,\alpha_0), \quad \alpha_0(t_0) \leq u_0 \quad \text{for } t \geq t_0, \tag{1.2}$$

$$^{c}D^{q}\beta_0 \geq F(t,\beta_0), \quad \beta_0(t_0) \geq u_0 \quad \text{for } t \geq t_0. \tag{1.3}$$

The corresponding Volterra fractional integral equation is

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds. \tag{1.4}$$

Caputo's sense fractional-order differential equation is given by (1.1), and the corresponding Volterra fractional integral equation is given by (1.4). Here, we consider the function $F(t,x)$ on the right-hand side of (1.1) and split it into three parts as $f(t,x)$, $g(t,x)$, and $h(t,x)$, where f satisfies a weaker condition than convexity, g satisfies a weaker condition than concavity, and h is two-sided Lipschitzian.

2. Preliminaries

In this section, we state a comparison result and a corollary. For the proof, please see [2].

Theorem 2.1. *Let $\alpha_0, \beta_0 \in C_p([t_0,T], \mathbb{R})$ be locally Hölder continuous for an exponent $0 < \lambda < 1$ and $\lambda > q$, $p = 1-q$, $F \in C([t_0,T] \times \mathbb{R}, \mathbb{R})$, where $C_p([t_0,T], \mathbb{R}) = [u \in C([t_0,T], \mathbb{R})$ and $u(t) \cdot (t-t_0)^p \in C([t_0,T], \mathbb{R})]$, and*

(i) $D^{q}\alpha_0(t) \leq F(t, \alpha_0(t))$,

(ii) $D^{q}\beta_0(t) \geq F(t, \beta_0(t))$, $t_0 < t \leq T$, *where D^{q} is Riemann-Liouville fractional-order derivative and q is such that $0 < q < 1$.*

Suppose further that the standard Lipschitz condition is satisfied; that is,

$$F(t,x) - F(t \cdot y) \leq L(x-y), \quad x \geq y, \ L > 0. \tag{2.1}$$

Table 1

Quasilinearization method	Integer derivative	Caputo's derivative
Monotone sequences	Yes	Yes
Unique solution exists	Yes	Yes
Uniform convergence	Yes	Yes
Quadratic (semiquadratic) convergence	Yes	Yes

Then, $\alpha_0^0 \leq \beta_0^0$, *where* $\alpha_0^0 = \alpha_0(t)(t-t_0)^{1-q}|_{t=t_0}$ *and* $\beta_0^0 = \beta_0(t)(t-t_0)^{1-q}|_{t=t_0}$ *imply that* $\alpha_0(t) \leq \beta_0(t),\ t_0 \leq t \leq T$.

Corollary 2.2. *The function* $F(t,u) = \sigma(t)u$, *where* $\sigma(t) \leq L$, *is admissible in Theorem 2.1 to yield* $u(t) \leq 0$ *on* $t_0 \leq t \leq T$.

We note that Theorem 2.1 and Corollary 2.2 also hold for Caputo's fractional derivative; see [2].

3. Monotone Technique and Method of Quasilinearization

In monotone iterative technique that we have used an existence result of nonlinear fractional-order differential equations with Caputo's derivative in a sector based on theoretical considerations and described a constructive method which implies monotone sequences of functions that converge to the solution of (1.1). Since each member of these sequences is the solution of a certain linear fractional-order differential equation with Caputo's derivative which can be explicitly computed, the advantage and the importance of the technique need no special emphasis. Moreover, these methods can successfully be employed to generate two-sided pointwise bounds on solutions of initial value problems of fractional-order differential equations with Caputo's derivatives from which qualitative and quantitative behaviors can be investigated.

The idea of relating the study of nonlinear fractional-order differential equations with Caputo's derivative through its related linear fractional-order differential equations with Caputo's derivative finds further extension in the method of quasilinearization. In this case, again, we obtain existence of solutions of (1.1) under certain restrictions after formulating sequences of solutions of related linear fractional-order differential equations with Caputo's derivative. These sequences converge quadratically in the constructive methods. The method involves the formulation of upper and lower solutions.

Due to some advantages of Caputo's derivative, we have applied the quasilinearization technique to the given nonlinear fractional-order differential equations with Caputo's derivative not Riemann-Liouville (R-L) derivative. The main advantage of Caputo's derivative is that the initial conditions for fractional-order differential equations are of the same form as those of ordinary differential equations with integer derivatives. Another difference is that Caputo's derivative for a constant C is zero, while the Riemann-Liouville fractional-order derivative for a constant C is not zero but equals to $D^qC = C(t-t_0)^{-q}/\Gamma(1-q)$, which is not zero. Table 1 depicts the correspondence between the features of quasilinearization in the context of the integer order and fractional-order with Caputo's derivative. Therefore, under the suitable assumptions but different conditions, we have Table 1.

4. Main Result

In this section, we will prove the main theorem that gives several different conditions to apply the method of generalized quasilinearization to the nonlinear fractional-order differential equations with Caputo's derivative and state four remarks for special cases.

Theorem 4.1. *Assume that*

(i) $f, g, h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$, $\alpha_0, \beta_0 \in C^q([t_0, T], \mathbb{R})$, *and*

$$\begin{aligned} {}^cD^q\alpha_0 &\le F(t, \alpha_0), \quad \alpha_0(t_0) \le x_0, \\ {}^cD^q\beta_0 &\ge F(t, \beta_0), \quad \beta_0(t_0) \ge x_0 \end{aligned} \tag{4.1}$$

$\alpha_0(t) \le \beta_0(t)$ *on* $J, \alpha_0(t_0) \le x_0 \le \beta_0(t_0)$, *where* $F(t, x) = f(t, x) + g(t, x) + h(t, x)$ *and* $J = [t_0, T]$.

(ii) *Assume also that* $f_x(t, x)$ *exists and* $f_x(t, x)$ *is nondecreasing in* x *for each* t *as*

$$\begin{aligned} f(t, x) &\ge f(t, y) + f_x(t, y)(x - y), \quad x \ge y, \\ \left|f_x(t, x) - f_x(t, y)\right| &\le L_1|x - y| \quad \text{with } L_1 \ge 0. \end{aligned} \tag{4.2}$$

Furthermore, $g_x(t, x)$ *exists and* $g_x(t, x)$ *is nonincreasing in* x *for each* t *as*

$$\begin{aligned} g(t, x) &\ge g(t, y) + g_x(t, x)(x - y), \quad x \ge y, \\ \left|g_x(t, x) - g_x(t, y)\right| &\le L_2|x - y| \quad \text{with } L_2 \ge 0. \end{aligned} \tag{4.3}$$

(iii) *Moreover assume that* $h(t, x)$ *is two-sided Lipschitzian in* x *such that* $|h(t, x) - h(t, y)| \le K|x - y|$, *where* $K > 0$ *is the Lipschitz constant.*

Then, there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *which converge uniformly and monotonically to the unique solution* $x(t)$ *of* (1.1) *and the convergence is semiquadratic.*

Proof. Consider the following linear fractional-order initial value problems with Caputo's derivatives order q:

$$\begin{aligned} {}^cD^q\alpha_{k+1} &= F(t, \alpha_k) + \left[f_x(t, \alpha_k) + g_x(t, \beta_k) - k\right](\alpha_{k+1} - \alpha_k), \\ \alpha_{k+1}(t_0) &= x_0, \\ {}^cD^q\beta_{k+1} &= F(t, \beta_k) + \left[f_x(t, \alpha_k) + g_x(t, \beta_k) - k\right](\beta_{k+1} - \beta_k), \\ \beta_{k+1}(t_0) &= x_0. \end{aligned} \tag{4.4}$$

Since the right-hand sides of the equations satisfy a Lipschitz condition, it is obvious that unique solutions exist. We will show that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le \beta_k \le \cdots \le \beta_1 \le \beta_0 \quad \text{on } J. \tag{4.5}$$

First, we will prove that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 \quad \text{on } J. \tag{4.6}$$

Put $p(t) = \alpha_0(t) - \alpha_1(t)$ on J. Then,

$$\begin{aligned}
{}^cD^q p &= {}^cD^q \alpha_0 - {}^cD^q \alpha_1 \\
&\leq F(t, \alpha_0) - F(t, \alpha_0) - \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\alpha_1 - \alpha_0) \\
&= \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right]p,
\end{aligned} \tag{4.7}$$

and $p(t_0) \leq 0$. Hence, applying Corollary 2.2, we get

$$\alpha_0(t) \leq \alpha_1(t) \quad \text{on } J. \tag{4.8}$$

Let us set $p(t) = \alpha_1(t) - \beta_0(t)$; then, using (ii) and (iii) and the fact that $\beta_0 \geq \alpha_0$, we have

$$\begin{aligned}
{}^cD^q p &= {}^cD^q \alpha_1 - {}^cD^q \beta_0 \\
&\leq F(t, \alpha_0) + \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\alpha_1 - \alpha_0) - F(t, \beta_0) \\
&\leq F(t, \alpha_0) + \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\alpha_1 - \alpha_0) \\
&\quad - f(t, \alpha_0) - f_x(t, \alpha_0)(\beta_0 - \alpha_0) - g(t, \alpha_0) - g_x(t, \beta_0)(\beta_0 - \alpha_0) \\
&\quad - h(t, \alpha_0) + K(\beta_0 - \alpha_0) \\
&= \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\alpha_1 - \beta_0).
\end{aligned} \tag{4.9}$$

This implies that

$${}^cD^q p \leq \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right]p, \qquad p(t_0) \leq 0, \tag{4.10}$$

which because of Corollary 2.2 yields $p(t) \leq 0$ on J. Thus, we have $\alpha_1 \leq \beta_0$ on J. Similarly, one can prove that $\alpha_0 \leq \beta_1 \leq \beta_0$ on J. We now prove that $\alpha_1(t) \leq \beta_1(t)$ on J. For this purpose we set $p(t) = \alpha_1 - \beta_1$ and note that $p(t_0) = 0$.

Then,

$$\begin{aligned}
{}^cD^q p &= {}^cD^q \alpha_1 - {}^cD^q \beta_1 \\
&= f(t, \alpha_0) + g(t, \alpha_0) + h(t, \alpha_0) + \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\alpha_1 - \alpha_0) \\
&\quad - f(t, \beta_0) - g(t, \beta_0) - h(t, \beta_0) - \left[f_x(t, \alpha_0) + g_x(t, \beta_0) - K\right](\beta_1 - \beta_0).
\end{aligned} \tag{4.11}$$

Since $\beta_0 \geq \alpha_0$, by using nondecreasing property of f_x and nonincreasing property of g_x, we obtain

$$\begin{aligned} {}^cD^q p &\leq f_x(t,\alpha_0)(\alpha_0-\beta_0) + g_x(t,\beta_0)(\alpha_0-\beta_0) - K(\alpha_0-\beta_0) \\ &\quad + \left[f_x(t,\alpha_0) + g_x(t,\beta_0) - K\right](\alpha_1-\alpha_0-\beta_1+\beta_0) \\ &= \left[f_x(t,\alpha_0) + g_x(t,\beta_0) - K\right](\alpha_1-\beta_1), \end{aligned} \tag{4.12}$$

which shows that ${}^cD^q p \leq [f_x(t,\alpha_0) + g_x(t,\beta_0) - K]p$. This proves that $p(t) \leq 0$. Therefore, $\alpha_1(t) \leq \beta_1(t)$ on J. Hence, (4.6) is proved.

Using mathematical induction with $k > 1$, we obtain

$$\alpha_0 \leq \alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \leq \beta_0 \quad \text{on } J. \tag{4.13}$$

We must prove that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad \text{on } J. \tag{4.14}$$

To do so, we set $p(t) = \alpha_k - \alpha_{k+1}$. Then,

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_k - {}^cD^q\alpha_{k+1} \\ &= F(t,\alpha_{k-1}) + \left[f_x(t,\alpha_{k-1}) + g_x(t,\beta_{k-1}) - K\right](\alpha_k-\alpha_{k-1}) \\ &\quad - F(t,\alpha_k) - \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_{k+1}-\alpha_k) \\ &= \left[f(t,\alpha_{k-1}) - f(t,\alpha_k)\right] + \left[g(t,\alpha_{k-1}) - g(t,\alpha_k)\right] + \left[h(t,\alpha_{k-1}) - h(t,\alpha_k)\right] \\ &\quad + \left[f_x(t,\alpha_{k-1}) + g_x(t,\beta_{k-1}) - K\right](\alpha_k-\alpha_{k-1}) - \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_{k+1}-\alpha_k) \\ &\leq f_x(t,\alpha_{k-1})(\alpha_{k-1}-\alpha_k) + g_x(t,\beta_{k-1})(\alpha_{k-1}-\alpha_k) + K(\alpha_k-\alpha_{k-1}) + f_x(t,\alpha_{k-1})(\alpha_k-\alpha_{k-1}) \\ &\quad + g_x(t,\beta_{k-1})(\alpha_k-\alpha_{k-1}) + \left[f_x(t,\alpha_k) + g_x(t,\beta_k)\right](\alpha_k-\alpha_{k+1}) + K(\alpha_{k-1}-\alpha_k+\alpha_{k+1}-\alpha_k) \\ &= \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_k-\alpha_{k+1}), \end{aligned} \tag{4.15}$$

where we have used the inequalities in (ii), (iii) and the fact that f_x is nondecreasing in x and g_x is nonincreasing in x. Thus, we have

$${}^cD^q p \leq \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right]p, \quad p(t_0) = 0. \tag{4.16}$$

Again, from Corollary 2.2, we get $\alpha_k \leq \alpha_{k+1}$ on J. Similarly, it can be shown that $\beta_{k+1} \leq \beta_k$ on J.

Next we need to show that $\alpha_{k+1} \le \beta_{k+1}$ on J.

Set $p(t) = \alpha_{k+1} - \beta_{k+1}$; then,

$$\begin{aligned}
{}^{c}D^{q}p &= {}^{c}D^{q}\alpha_{k+1} - {}^{c}D^{q}\beta_{k+1} \\
&= F(t,\alpha_k) + \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_{k+1} - \alpha_k) \\
&\quad - F(t,\beta_k) + \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\beta_{k+1} - \beta_k) \\
&\le f_x(t,\alpha_k)(\alpha_k - \beta_k) + g_x(t,\beta_k)(\alpha_k - \beta_k) - K(\alpha_k - \beta_k) \\
&\quad + \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_{k+1} - \alpha_k - \beta_{k+1} + \beta_k) \\
&\le \left[f_x(t,\alpha_k) + g_x(t,\beta_k) - K\right](\alpha_{k+1} - \beta_{k+1}).
\end{aligned} \tag{4.17}$$

Thus, we have ${}^{c}D^{q}p \le [f_x(t,\alpha_k) + g_x(t,\beta_k) - K]p$ and $p(t_0) = 0$. Consequently, as before, it follows from Corollary 2.2 we get that $\alpha_{k+1} \le \beta_{k+1}$ on J.

Employing the standard arguments [2], one can easily show that $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly and monotonically to the unique solution of (1.1).

To prove the semiquadratic convergence, we set $p_{n+1} = x - \alpha_{n+1}$ and $r_{n+1} = \beta_{n+1} - x$. Note that $p_{n+1}(t_0) = r_{n+1}(t_0) = 0$ and

$$\begin{aligned}
{}^{c}D^{q}p_{n+1} &= {}^{c}D^{q}x - {}^{c}D^{q}\alpha_{n+1} \\
&= F(t,x) - F(t,\alpha_n) - \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right](\alpha_{n+1} - \alpha_n) \\
&= f_x(t,\xi)(x - \alpha_n) + g_x(t,\eta)(x - \alpha_n) + K(x - \alpha_n) \\
&\quad + \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right](\alpha_n - \alpha_{n+1}),
\end{aligned} \tag{4.18}$$

where $\alpha_n \le \xi, \eta \le x$. Now using the nondecreasing property of f_x and nonincreasing property of g_x, we get

$$\begin{aligned}
{}^{c}D^{q}p_{n+1} &\le \left[f_x(t,x) + g_x(t,\alpha_n) + K\right]p_n + \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right](p_{n+1} - p_n) \\
&= \left[f_x(t,x) - f_x(t,\alpha_n)\right]p_n + \left[g_x(t,\alpha_n) - g_x(t,\beta_n)\right]p_n + 2Kp_n \\
&\quad + \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right]p_{n+1} \\
&\le L_1 p_n^2 + L_2(p_n + r_n)p_n + 2Kp_n + \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right]p_{n+1} \\
&\le \left(L_1 + \frac{3}{2}L_2\right)p_n^2 + \frac{1}{2}L_2 r_n^2 + 2Kp_n + \left[f_x(t,\alpha_n) + g_x(t,\beta_n) - K\right]p_{n+1}.
\end{aligned} \tag{4.19}$$

Thus, we have

$$ {}^{c}D^{q}p_{n+1} \leq \left(L_1 + \frac{3}{2}L_2\right)\left|p_n\right|_0^2 + \frac{1}{2}L_2|r_n|_0^2 + 2K\left|p_n\right|_0 + K^{*}p_{n+1}, \tag{4.20} $$

where $|p_n|_0 = \max_{t\in J}|p_n(t)|$, $|r_n|_0 = \max_{t\in J}|r_n(t)|$, $|f_x(t,\alpha_n)| \leq K_1$, $|g_x(t,\beta_n)| \leq K_2$, and $K^{*} = K_1 + K_2 - K$.

Then, we obtain

$$ {}^{c}D^{q}p_{n+1} \leq \left(\left(L_1 + \frac{3}{2}L_2\right)\left|p_n\right|_0^2 + \frac{1}{2}L_2|r_n|_0^2 + 2K\left|p_n\right|_0\right)\int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(K^{*}(t-s))ds, \tag{4.21} $$

where $E_{q,q}$ is the Mittag-Leffler function.

Let $W = (1/q)(T-t_0)^q E_{q,q}(K^{*}(T-t_0)^q)$; then,

$$ {}^{c}D^{q}p_{n+1} \leq U_1\left|p_n\right|_0^2 + U_2|r_n|_0^2 + U_3\left|p_n\right|_0, \tag{4.22} $$

where $U_1 = (L_1 + (3/2)L_2)W$, $U_2 = (1/2)L_2W$, and $U_3 = 2kW$.

Thus, we reach the desired result

$$ \max_{[t_0,T]}|x-\alpha_{n+1}| \leq U_1\max_{[t_0,T]}|x-\alpha_n|^2 + U_2\max_{[t_0,T]}\left|\beta_n - x\right|^2 + U_3\max_{[t_0,T]}|x-\alpha_n| \tag{4.23} $$

which shows the semiquadratic convergence.

Similarly, using suitable computation, we arrive at

$$ \max_{[t_0,T]}\left|\beta_{n+1} - x\right| \leq V_1\max_{[t_0,T]}|x-\alpha_n|^2 + V_2\max_{[t_0,T]}\left|\beta_n - x\right|^2 + V_3\max_{[t_0,T]}\left|\beta_n - x\right|, \tag{4.24} $$

where $V_1 = (1/2)L_1W$, and $V_2 = ((3/2)L_1 + L_2)W$, and $V_3 = 2kW$. □

Remark 4.2. Let $f(t,x)+g(t,x) = 0$; then, we have the monotone method, and the convergence is linear.

Remark 4.3. Let $g(t,x) + h(t,x) = 0$; then, Theorem 4.1 reduces to Theorem 3.1 of [4], and the convergence is quadratic.

Remark 4.4. Let $f(t,x) + h(t,x) = 0$ and $g(t,x)$ be concave; then, Theorem 4.1 reduces to Theorem 4.1 of [4], and the convergence is quadratic.

Remark 4.5. Let $h(t,x) = 0$; then, Theorem 4.1 reduces to the theorem in [5].

5. An Example

The following example illustrates how the main result of the theorem may be applied for the nonlinear fractional differential equation order $q = 1/2$ and $t_0 = 0$.

Example 5.1. Let us consider the following nonlinear fractional-order initial value problem with Caputo's derivative order $q = 1/2$:

$$ {}^{c}D^{1/2}x(t) = \sqrt{\frac{t}{\pi}}x^2 - 3\sqrt{\frac{t}{\pi}}x + \sqrt{\frac{t}{\pi}}(1+x), \quad x(0) = 0, \ 0 \le t \le 10, \tag{5.1} $$

where $f(t,x) = \sqrt{t/\pi}x^2$, $g(t,x) = -3\sqrt{t/\pi}x$, $h(t,x) = \sqrt{t/\pi}(1+x)$ and $f, g, h \in C([0,10] \times \mathbb{R}, \mathbb{R})$.

Let $\alpha_0(t) = (1/2)(1 - 1/\sqrt{1+t}), \alpha_0(0) = 0$ and $\beta_0(t) = 2(1 - 1/\sqrt{1+t}), \beta_0(0) = 0$ for $0 \le t \le 10$ be lower and upper solutions of the fractional-order differential equation with Caputo's derivative order $q = 1/2$, respectively. Then, $\alpha_0(t)$ and $\beta_0(t)$ satisfy the inequalities in assumption (i) as

$$ \begin{aligned} {}^{c}D^{1/2}\alpha_0 &\le \sqrt{\frac{t}{\pi}}\alpha_0^2 - 3\sqrt{\frac{t}{\pi}}\alpha_0 + \sqrt{\frac{t}{\pi}}(1+\alpha_0), \quad \alpha_0(0) \le 0 \quad \text{for } 0 \le t \le 10, \\ {}^{c}D^{1/2}\beta_0 &\ge \sqrt{\frac{t}{\pi}}\beta_0^2 - 3\sqrt{\frac{t}{\pi}}\beta_0 + \sqrt{\frac{t}{\pi}}(1+\beta_0), \quad \beta_0(0) \ge 0 \quad \text{for } 0 \le t \le 10. \end{aligned} \tag{5.2} $$

On the other hand, $f_x(t,x) = 2\sqrt{t/\pi}x$, $g_x(t,x) = -3\sqrt{t/\pi}$ exist, and $f_x(t,x)$ is nondecreasing and $g_x(t,x)$ is nonincreasing in x for each t in assumption (ii). Also, it can be shown that these three functions f, g, and h hold in the correspondence inequality in assumptions (ii) and (iii) with the nonnegative constants $L_1 \ge 2\sqrt{10/\pi}$, $L_2 \ge 0$, and $K \ge \sqrt{10/\pi}$.

Therefore, we can construct the monotone sequences $\{\alpha_{k+1}\}$ and $\{\beta_{k+1}\}$ whose elements are solutions of linear fractional-order differential equations with Caputo's derivatives order $q = 1/2$ of (5.3) and (5.4), respectively, as

$$ \begin{gathered} {}^{c}D^{1/2}\alpha_{k+1} = \sqrt{\frac{t}{\pi}}\alpha_k^2 - 3\sqrt{\frac{t}{\pi}}\alpha_k + \sqrt{\frac{t}{\pi}}(1+\alpha_k) + \left[2\sqrt{\frac{t}{\pi}}\alpha_k - 3\sqrt{\frac{t}{\pi}} - \sqrt{\frac{10}{\pi}}\right](\alpha_{k+1} - \alpha_k), \\ \alpha_{k+1}(0) = 0 \quad \text{for } K = \sqrt{\frac{10}{\pi}}, \ 0 \le t \le 10, \end{gathered} \tag{5.3} $$

$$ \begin{gathered} {}^{c}D^{1/2}\beta_{k+1} = \sqrt{\frac{t}{\pi}}\beta_k^2 - 3\sqrt{\frac{t}{\pi}}\beta_k + \sqrt{\frac{t}{\pi}}(1+\beta_k) + \left[2\sqrt{\frac{t}{\pi}}\alpha_k - 3\sqrt{\frac{t}{\pi}} - \sqrt{\frac{10}{\pi}}\right](\beta_{k+1} - \beta_k), \\ \beta_{k+1}(0) = 0 \quad \text{for } K = \sqrt{\frac{10}{\pi}}, \ 0 \le t \le 10. \end{gathered} \tag{5.4} $$

Since the right-hand sides of the equations satisfy a Lipschitz condition, it is obvious that unique solutions exist such that, for all $k = n$,

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k \le \cdots \le \beta_1 \le \beta_0 \quad \text{on } J = [0, 10]. \tag{5.5}$$

Employing the standard techniques [2], sequences $\{\alpha_k\}$ and $\{\beta_k\}$ converge uniformly and monotonically to the unique solution $x(t) = 1 - 1/\sqrt{1+t}$ of ${}^cD^{1/2}x(t) = \sqrt{t/\pi}x^2 - 3\sqrt{t/\pi}x + \sqrt{t/\pi}(1+x)$, $x(0) = 0$, since using the fact that F satisfies a Lipschitz condition that is F_x is bounded on the sector

$$[\alpha_0, \beta_0] = \left[x : \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+t}}\right) \le x \le 2\left(1 - \frac{1}{\sqrt{1+t}}\right)\right] \quad \text{for } 0 \le t \le 10. \tag{5.6}$$

Moreover, the convergence is semiquadratic.

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