Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 705172, 17 pages
doi:10.1155/2010/705172

Research Article

On Second Order of Accuracy Difference Scheme of the Approximate Solution of Nonlocal Elliptic-Parabolic Problems

Allaberen Ashyralyev[1,2] and Okan Gercek[3,4]

[1] *Department of Mathematics, Fatih University, Buyukcekmece, Istanbul 34500, Turkey*
[2] *Department of Mathematics, ITTU, Ashgabat 744012, Turkmenistan*
[3] *Vocational School, Fatih University, Buyukcekmece, Istanbul 34500, Turkey*
[4] *Department of Mathematics, Yildiz Technical University, 34210 Istanbul, Turkey*

Correspondence should be addressed to Okan Gercek, ogercek@fatih.edu.tr

Received 16 February 2010; Accepted 15 May 2010

Academic Editor: Ağacik Zafer

A second order of accuracy difference scheme for the approximate solution of the abstract nonlocal boundary value problem $-d^2u(t)/dt^2 + Au(t) = g(t)$, $(0 \leq t \leq 1)$, $du(t)/dt - Au(t) = f(t)$, $(-1 \leq t \leq 0)$, $u(1) = u(-1) + \mu$ for differential equations in a Hilbert space H with a self-adjoint positive definite operator A is considered. The well posedness of this difference scheme in Hölder spaces is established. In applications, coercivity inequalities for the solution of a difference scheme for elliptic-parabolic equations are obtained and a numerical example is presented.

1. Introduction

The role played by coercive inequalities in the study of boundary value problems for elliptic and parabolic partial differential equations is well known (see [1–4]).

Nonlocal problems are widely used for mathematical modeling of various processes of physics, biology, chemistry, ecology, engineering, and industry when it is impossible to determine the boundary or initial values of the unknown function. Theory and numerical methods of solutions of the nonlocal boundary value problems for partial differential equations of variable type have been studied extensively by many researchers (see, e.g., [5–34] and the references therein).

In paper [35], the nonlocal boundary value problem

$$-\frac{d^2u(t)}{dt^2}+Au(t)=g(t)\quad (0\le t\le 1),$$
$$\frac{du(t)}{dt}-Au(t)=f(t)\quad (-1\le t\le 0),\tag{1.1}$$
$$u(1)=u(-1)+\mu$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A was considered. The well posedness of problem (1.1) in Hölder spaces was established. The first order of accuracy difference scheme for approximate solutions of nonlocal boundary value problem (1.1) was presented. In applications, the coercivity inequalities for solutions of difference schemes for elliptic-parabolic equations were obtained.

In the present paper, the second order of accuracy difference scheme generated by Crank-Nicholson difference scheme

$$-\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k=g_k,$$
$$g_k=g(t_k),\quad t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,$$
$$\tau^{-1}(u_k-u_{k-1})-\frac{1}{2}(Au_{k-1}+Au_k)=f_k,\quad f_k=f(t_{k-1/2}),\tag{1.2}$$
$$t_{k-1/2}=\left(k-\frac{1}{2}\right)\tau,\quad -(N-1)\le k\le 0,$$
$$u_N=u_{-N}+\mu,\qquad u_2-4u_1+3u_0=-3u_0+4u_{-1}-u_{-2}$$

for the approximate solution of problem (1.1) is presented. The well posedness of difference scheme (1.2) in Hölder spaces is established. As an application, coercivity inequalities for solutions of difference schemes for elliptic-parabolic equations are obtained. A numerical example is given.

2. The Formula for the Solution of Problem (1.2)

The following operators:

$$P=\left(I-\frac{\tau A}{2}\right)G,\qquad G=\left(I+\frac{\tau A}{2}\right)^{-1},\qquad R=(I+\tau B)^{-1},$$
$$T_\tau=\left(I+B^{-1}A\left(I+\tau A+\frac{\tau}{2}G^{-2}\right)K\left(I-R^{2N-1}\right)\right.$$
$$\left.+K(I-\frac{\tau^2A}{2})G^{-2}R^{2N-1}-K\left(I-\frac{\tau^2A}{2}\right)G^{-2}(2I+\tau B)R^NP^N\right)^{-1}\tag{2.1}$$

exist and are bounded for a self-adjoint positive operator A. Here

$$B = \frac{1}{2}\left(\tau A + \sqrt{A(4+\tau^2 A)}\right), \qquad K = \left(I + 2\tau A + \frac{5}{4}(\tau A)^2\right)^{-1}. \tag{2.2}$$

Theorem 2.1. *For any g_k, $1 \le k \le N-1$, and f_k, $-N+1 \le k \le 0$, the solution of problem* (1.2) *exists and the following formula holds:*

$$\begin{aligned} u_k = {}& \left(I - R^{2N}\right)^{-1}\Bigg\{\left[R^k - R^{2N-k}\right]u_0 + \left[R^{N-k} - R^{N+k}\right]\left[P^N u_0 - \tau\sum_{s=-N+1}^{0} P^{s+N-1}Gf_s + \mu\right] \\ & - \left[R^{N-k} - R^{N+k}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s} - R^{N+s}\right]g_s\tau\Bigg\} \\ & + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{|k-s|} - R^{k+s}\right]g_s\tau, \quad 1 \le k \le N, \end{aligned} \tag{2.3}$$

$$u_k = P^{-k}u_0 - \tau\sum_{s=k+1}^{0} P^{s-k-1}Gf_s, \quad -N \le k \le -1, \tag{2.4}$$

$$\begin{aligned} u_0 = {}& \frac{1}{2}T_\tau K G^{-2} \\ & \times\Bigg\{\left(2I - \tau^2 A\right)\Bigg\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0} P^{s+N-1}Gf_s + \mu\right] \\ & - R^{N-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s} - R^{N+s}\right]g_s\tau + \left(I - R^{2N}\right)B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau\Bigg\} \\ & + \left(I - R^{2N}\right)(I+\tau B)\left(\tau B^{-1}g_1 - 4GB^{-1}f_0 + PGB^{-1}f_0 + GB^{-1}f_{-1}\right)\Bigg\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} T_\tau = {}& \Bigg(I + B^{-1}A\left(I + \tau A + \frac{\tau}{2}G^{-2}\right)K\left(I - R^{2N-1}\right) \\ & + K\left(I - \frac{\tau^2 A}{2}\right)G^{-2}R^{2N-1} - K\left(I - \frac{\tau^2 A}{2}\right)G^{-2}(2I+\tau B)R^N P^N\Bigg)^{-1}. \end{aligned}$$

Proof. For any $\{f_k\}_{k=-N}^{-1}$ and ξ, the solution of the auxiliary inverse Cauchy difference problem

$$\begin{gathered}\tau^{-1}(u_k - u_{k-1}) - \frac{1}{2}(Au_{k-1} + Au_k) = f_k, \\ -(N-1) \leq k \leq 0, \quad u_0 = \xi\end{gathered} \tag{2.6}$$

exists and the following formula holds [36]

$$u_k = P^{-k}\xi - \tau \sum_{s=k+1}^{0} P^{s-k-1} G f_s, \quad -N \leq k \leq -1. \tag{2.7}$$

Putting $\xi = u_0$, we get (2.4).

Now, we consider the following auxiliary difference problem

$$\begin{gathered}-\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \\ g_k = g(t_k), \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \\ u_0 = \xi, \quad u_N = \varphi.\end{gathered} \tag{2.8}$$

It is well known that for the solution of (2.8) the following formula holds [37, 38]:

$$\begin{aligned}u_k = \left(I - R^{2N}\right)^{-1} \Bigg\{ &\left[R^k - R^{2N-k}\right]\xi + \left[R^{N-k} - R^{N+k}\right]\varphi \\ &- \left[R^{N-k} - R^{N+k}\right](I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \Bigg\} \\ &+ (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{|k-s|} - R^{k+s}\right] g_s \tau, \quad 1 \leq k \leq N.\end{aligned} \tag{2.9}$$

Applying (2.7) and putting $\xi = u_0, \varphi = P^N u_0 - \tau \sum_{s=-N+1}^{0} P^{N+s-1} G f_s + \mu$, in (2.9), we get (2.3).

For u_0, using (2.3), (2.4), and the condition

$$u_2 - 4u_1 + 3u_0 = -3u_0 + 4u_{-1} - u_{-2}, \tag{2.10}$$

we obtain the operator equation

$$\begin{aligned}
&\left(2I-\tau^2A\right)\left\{\left(I-R^{2N}\right)^{-1}\left\{\left[R-R^{2N-1}\right]u_0\right.\right.\\
&\qquad\qquad+\left[R^{N-1}-R^{N+1}\right]\left[P^Nu_0-\tau\sum_{s=-N+1}^{0}P^{s+N-1}Gf_s+\mu\right]\\
&\qquad\qquad\left.-\left[R^{N-1}-R^{N+1}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}\\
&\qquad\left.+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{s-1}-R^{1+s}\right]g_s\tau\right\}\\
&\quad=-\tau^2g_1+G^2\left(2I+4\tau A+\frac{5}{2}(\tau A)^2\right)u_0+4G\tau f_0-PG\tau f_0-G\tau f_{-1}.
\end{aligned}\tag{2.11}$$

The operator

$$\begin{aligned}
&I+B^{-1}A\left(I+\tau A+\frac{\tau}{2}G^{-2}\right)K\left(I-R^{2N-1}\right)\\
&\qquad+K\left(I-\frac{\tau^2A}{2}\right)G^{-2}R^{2N-1}-K\left(I-\frac{\tau^2A}{2}\right)G^{-2}(2I+\tau B)R^NP^N
\end{aligned}\tag{2.12}$$

has an inverse

$$\begin{aligned}
T_\tau=&\left(I+B^{-1}A\left(I+\tau A+\frac{\tau}{2}G^{-2}\right)K\left(I-R^{2N-1}\right)\right.\\
&\left.+K\left(I-\frac{\tau^2A}{2}\right)G^{-2}R^{2N-1}-K\left(I-\frac{\tau^2A}{2}\right)G^{-2}(2I+\tau B)R^NP^N\right)^{-1}.
\end{aligned}\tag{2.13}$$

Hence, we obtain that

$$\begin{aligned}
u_0=&\frac{1}{2}T_\tau KG^{-2}\\
&\times\left\{\left(2I-\tau^2A\right)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}P^{s+N-1}Gf_s+\mu\right]\right.\right.\\
&\qquad\left.-R^{N-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau+\left(I-R^{2N}\right)B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau\right\}\\
&\quad\left.+\left(I-R^{2N}\right)(I+\tau B)\left(\tau B^{-1}g_1-4GB^{-1}f_0+PGB^{-1}f_0+GB^{-1}f_{-1}\right)\right\}.
\end{aligned}\tag{2.14}$$

This concludes the proof of Theorem 2.1. □

3. Main Theorems

Here, we study well posedness of problem (1.2). First, we give some necessary estimates for P^k, R^k, and T_τ. For a self-adjoint positive operator A, the following estimates are satisfied [36, 38, 39]:

$$\left\|P^k\right\|_{H\to H} \leq 1, \qquad \|G\|_{H\to H} \leq 1, \qquad k\tau\left\|AP^kG^2\right\|_{H\to H} \leq M, \quad k\geq 1,\ \delta>0, \tag{3.1}$$

$$\left\|R^k\right\|_{H\to H} \leq M(1+\delta\tau)^{-k}, \qquad k\tau\left\|BR^k\right\|_{H\to H} \leq M, \quad k\geq 1,\ \delta>0, \tag{3.2}$$

where M is independent of τ. From these estimates, it follows that

$$\begin{aligned}
&\left\|\left(I+B^{-1}A\left(I+\tau A+\frac{\tau}{2}G^{-2}\right)K\left(I-R^{2N-1}\right)+K\left(I-\frac{\tau^2A}{2}\right)G^{-2}R^{2N-1}\right.\right. \\
&\qquad \left.\left.-K\left(I-\frac{\tau^2 A}{2}\right)G^{-2}(2I+\tau B)R^NP^N\right)^{-1}\right)^{-1}\right\|_{H\to H} \leq M.
\end{aligned} \tag{3.3}$$

Let $F_\tau(H) = F([a,b]_\tau, H)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_{N_a}^{N_b}$ defined on $[a,b]_\tau = \{t_k = kh, N_a \leq k \leq N_b, N_a\tau = a, N_b\tau = b\}$ with values in the Hilbert space H. Next on $F_\tau(H)$ we denote $C([a,b]_\tau, H)$ and $C_{0,1}^\alpha([-1,1]_\tau, H), C_{0,1}^\alpha([-1,0]_\tau, H), C_0^\alpha([0,1]_\tau, H) (0 < \alpha < 1)$ Banach spaces with the norms

$$\begin{aligned}
\|\varphi^\tau\|_{C([a,b]_\tau,H)} &= \max_{N_a\leq k\leq N_b}\|\varphi_k\|_H, \\
\|\varphi^\tau\|_{C_{0,1}^\alpha([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)} + \sup_{-N\leq k<k+r\leq 0}\|\varphi_{k+r}-\varphi_k\|_E\frac{(-k)^\alpha}{r^\alpha} \\
&\quad + \sup_{1\leq k<k+r\leq N-1}\|\varphi_{k+r}-\varphi_k\|_E\frac{((k+r)\tau)^\alpha(N-k)^\alpha}{r^\alpha}, \\
\|\varphi^\tau\|_{C_0^\alpha([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)} + \sup_{-N\leq k<k+r\leq 0}\|\varphi_{k+r}-\varphi_k\|_E\frac{(-k)^\alpha}{r^\alpha}, \\
\|\varphi^\tau\|_{C_{0,1}^\alpha([0,1]_\tau,H)} &= \|\varphi^\tau\|_{C([0,1]_\tau,H)} \\
&\quad + \sup_{1\leq k<k+r\leq N-1}\|\varphi_{k+r}-\varphi_k\|_E\frac{((k+r)\tau)^\alpha(N-k)^\alpha}{r^\alpha}.
\end{aligned} \tag{3.4}$$

Nonlocal boundary value problem (1.2) is said to be stable in $F([-1,1]_\tau, H)$ if we have the inequality

$$\|u^\tau\|_{F([-1,1]_\tau,H)} \leq M\left[\|f^\tau\|_{F([-1,0]_\tau,H)} + \|g^\tau\|_{F([0,1]_\tau,H)} + \|\mu\|_H\right], \tag{3.5}$$

where M is independent of not only $f^\tau, g^\tau,$ and μ but also τ.

Theorem 3.1. *Nonlocal boundary value problem* (1.2) *is stable in* $C([-1,1]_\tau, H)$ *norm.*

Proof. By [38], we have

$$\left\|\{u_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} \leq M\left[\|g^\tau\|_{C([0,1]_\tau,H)} + \|u_0\|_H + \|u_N\|_H\right] \tag{3.6}$$

for the solution of boundary value problem (2.8).

By [36], we get

$$\left\|\{u_k\}_{-N}^{0}\right\|_{C([-1,0]_\tau,H)} \leq M\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|u_0\|_H\right] \tag{3.7}$$

for the solution of an inverse Cauchy difference problem (2.6). Then, the proof of Theorem 3.1 is based on the stability inequalities (3.6), (3.7), and on the estimates

$$\|u_0\|_H \leq M\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([1,0]_\tau,H)} + \|\mu\|_H\right], \tag{3.8}$$

$$\|u_N\|_H \leq M\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([1,0]_\tau,H)} + \|\mu\|_H\right], \tag{3.9}$$

for the solution of the boundary value problem (1.2). Estimates (3.8) and (3.9) follow from formula (2.5) and estimates (3.1), (3.2), and (3.3) which conclude the proof of Theorem 3.1. □

Theorem 3.2. *Assume that* $\mu \in D(A)$ *and* $f_0, f_{-1}, g_1 \in D(I + \tau B)$. *Then, for the solution of difference problem* (1.2), *we have the following almost coercivity inequality:*

$$\begin{aligned}
&\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} \\
&\quad + \left\|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \\
&\quad + \left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k + Au_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \\
&\leq M\left[\min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}\right]\right. \\
&\quad \left. + \|A\mu\|_H + \|(I+\tau B)f_0\|_H + \|(I+\tau B)g_1\|_H + \|(I+\tau B)f_{-1}\|_H\right],
\end{aligned} \tag{3.10}$$

where M *does not dependent on not only* $f^\tau, g^\tau,$ *and* μ *but also* τ.

Proof. By [40], we have

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)}+\left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|A\|_{H\to H}|\right\}\|f^\tau\|_{C([-1,0]_\tau)}+\|Au_0\|_H\right] \tag{3.11}$$

for the solution of an inverse Cauchy difference problem (2.6).

By [38], we get

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_{1}^{N-1}\right\|_{C([0,1]_\tau,H)}+\left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} \leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|A\|_{H\to H}|\right\}\|g^\tau\|_{C([0,1]_\tau,H)}+\|Au_0\|_H+\|Au_N\|_H\right] \tag{3.12}$$

for the solution of boundary value problem (2.8).

Then, the proof of Theorem 3.2 is based on almost coercivity inequalities (3.11), (3.12), and on the estimates

$$\begin{aligned}\|Au_0\|_H &\leq M\Bigg[\|A\mu\|_H+\|(I+\tau B)f_0\|_H\\ &\qquad+\min\left\{\ln\frac{1}{\tau},1+|\ln\|A\|_{H\to H}|\right\}\left[\|f^\tau\|_{C([-1,0]_\tau,H)}+\|g^\tau\|_{C([0,1]_\tau,H)}\right]\Bigg],\\ \|Au_N\|_H &\leq M\Bigg[[\|A\mu\|_H+\|(I+\tau B)f_0\|_H]\\ &\qquad+\min\left\{\ln\frac{1}{\tau},1+|\ln\|A\|_{H\to H}|\right\}\left[\|f^\tau\|_{C([-1,0]_\tau,H)}+\|g^\tau\|_{C([0,1]_\tau,H)}\right]\Bigg]\end{aligned} \tag{3.13}$$

for the solution of boundary value problem (1.2). The proof of these estimates follows the scheme of the papers [38, 40] and relies on both the formula (2.5) and the estimates (3.1), (3.2), and (3.3).

This concludes the proof of Theorem 3.2. □

Let $\widetilde{C}^{\alpha}_{0,1}([-1,1]_\tau,H),\widetilde{C}^{\alpha}_{0}([-1,0]_\tau,H), 0<\alpha<1$ be the Banach spaces with the norms

$$\begin{aligned}\|\varphi^\tau\|_{\widetilde{C}^{\alpha}_{0,1}([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)}+\sup_{-N\leq k<k+2r\leq 0}\|\varphi_{k+2r}-\varphi_k\|_E\frac{(-k)^\alpha}{(2r)^\alpha}\\ &\quad+\sup_{1\leq k<k+r\leq N-1}\|\varphi_{k+2r}-\varphi_k\|_E\frac{((k+r)\tau)^\alpha(N-k)^\alpha}{r^\alpha},\\ \|\varphi^\tau\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)}+\sup_{-N\leq k<k+2r\leq 0}\|\varphi_{k+2r}-\varphi_k\|_E\frac{(-k)^\alpha}{(2r)^\alpha}.\end{aligned} \tag{3.14}$$

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied. Then, boundary value problem* (1.2) *is well posed in Hölder spaces* $C^{\alpha}_{0,1}([-1,1]_{\tau},H)$ *and* $\widetilde{C}^{\alpha}_{0,1}([-1,1]_{\tau},H)$, *and the following coercivity inequalities hold:*

$$\begin{aligned}
&\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)} \\
&\quad+\left\|\{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\quad+\left\|\{Au_k\}_1^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\leq M\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^{\tau}\|_{C^{\alpha}_{0}([-1,0]_{\tau},H)}+\|g^{\tau}\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}\right]+\|A\mu\|_H\right. \\
&\quad\left.+\|(I+\tau B)f_0\|_H+\|(I+\tau B)g_1\|_H+\|(I+\tau B)f_{-1}\|_H\right], \\
&\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)} \\
&\quad+\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\quad+\left\|\{Au_k\}_1^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\leq M\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^{\tau}\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}+\|g^{\tau}\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}\right]+\|A\mu\|_H\right. \\
&\qquad\left.+\|(I+\tau B)f_0\|_H+\|(I+\tau B)g_1\|_H+\|(I+\tau B)f_{-1}\|_H\right],
\end{aligned} \tag{3.15}$$

where M is independent of not only f^{τ}, g^{τ}, and μ but also τ and α.

Proof. By [39, 40],

$$\begin{aligned}
&\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}+\left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\quad\leq M\left[\frac{1}{\alpha(1-\alpha)}\|f^{\tau}\|_{C^{\alpha}_{0}([-1,0]_{\tau},H)}+\|Au_0\|_H\right],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}+\left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)} \\
&\quad\leq M\left[\frac{1}{\alpha(1-\alpha)}\|f^{\tau}\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}+\|Au_0\|_H\right]
\end{aligned} \tag{3.17}$$

for the solution of an inverse Cauchy difference problem (2.6) can be written.

By [37, 38], we get

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}+\left\|\{Au_k\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}$$
$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}+\|Au_0\|_H+\|Au_N\|_H\right] \tag{3.18}$$

for the solution of boundary value problem (2.8).

Then, the proof of Theorem 3.3 is based on coercivity inequalities (3.16)–(3.18), and the estimates

$$\|Au_0\|_H\leq M\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau,H)}+\|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}\right]\right.$$
$$\left.+\|Au\|_H+\left\|(I+\tau B)f_0\right\|_H+\left\|(I+\tau B)g_1\right\|_H+\left\|(I+\tau B)f_{-1}\right\|_H\right], \tag{3.19}$$

$$\|Au_N\|_H\leq M\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau,H)}+\|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}\right]\right.$$
$$\left.+\left\|A\mu\right\|_H+\left\|(I+\tau B)f_0\right\|_H+\left\|(I+\tau B)g_1\right\|_H+\left\|(I+\tau B)f_{-1}\right\|_H\right] \tag{3.20}$$

for the solution of boundary value problem (1.2).

Estimates (3.19) and (3.20) follow from the formulas

$$\begin{aligned}
Au_0 &= \frac{1}{2}T_\tau KG^{-2}\\
&\times\left\{\left(2I-\tau^2A\right)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G\left(f_s-f_{-N+1}\right)+A\mu\right]\right.\right.\\
&\qquad -R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}(g_s-g_{N-1})\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}(g_s-g_1)\tau\\
&\qquad \left.+\left(I-R^{2N}\right)AB^{-1}\sum_{s=1}^{N-1}R^{s-1}(g_s-g_1)\tau\right\}\\
&\quad+\left(I-R^{2N}\right)(I+\tau B)\left(\tau B^{-1}Ag_1-4GB^{-1}Af_0+PGB^{-1}Af_0+GB^{-1}Af_{-1}\right)\\
&\quad+\left(2I-\tau^2A\right)(2+\tau B)R^N\left(P^N-I\right)f_{-N+1}\\
&\quad\left.+AB^{-2}\left(R^{N-1}-I\right)\left\{R^{N-1}g_{N-1}+\left(R^{2N}-R^{2N-1}-I\right)g_1\right\}\right\},
\end{aligned}$$

$$
\begin{aligned}
Au_N ={}& \frac{1}{2}P^N T_\tau K G^{-2} \\
&\times \Bigg\{ \left(2I-\tau^2 A\right)\Bigg\{ (2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0} AP^{s+N-1}G\left(f_s - f_{-N+1}\right) + A\mu\right] \\
&\qquad - R^{N-1}AB^{-1}\sum_{s=1}^{N-1} R^{N-s}\left(g_s - g_{N-1}\right)\tau + R^{N-1}AB^{-1}\sum_{s=1}^{N-1} R^{N+s}\left(g_s - g_1\right)\tau \\
&\qquad + \left(I - R^{2N}\right)AB^{-1}\sum_{s=1}^{N-1} BR^{s-1}\left(g_s - g_1\right)\tau \Bigg\} \\
&\quad + \left(I - R^{2N}\right)(I+\tau B)\left(\tau B^{-1}Ag_1 - 4GB^{-1}Af_0 + PGB^{-1}Af_0 + GB^{-1}Af_{-1}\right) \\
&\quad + \left(2I-\tau^2 A\right)(2+\tau B)R^N\left(P^N - I\right)f_{-N+1} \\
&\quad + AB^{-2}\left(R^{N-1} - I\right)\left\{R^{N-1}g_{N-1} + \left(R^{2N} - R^{2N-1} - I\right)g_1\right\}\Bigg\} \\
&- \tau\sum_{s=-N+1}^{0} AP^{s+N-1}G\left(f_s - f_{-N+1}\right) + A\mu + \left(P^N - I\right)f_{-N+1}
\end{aligned}
\tag{3.21}
$$

for the solution of problem (1.2) and estimates (3.1), (3.2), and (3.3).

This finalizes the proof of Theorem 3.3. □

4. Applications

In this section, we indicate applications of Theorems 3.1, 3.2, and 3.3 to obtain the stability, the almost coercive stability, and the coercive stability estimates for the solutions of these difference schemes for the approximate solution of nonlocal mixed problems. First, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with boundary $S, \overline{\Omega} = \Omega \cup S$. In $[-1,1] \times \Omega$, the boundary value problem for the multidimensional elliptic-parabolic equation

$$
\begin{gathered}
-u_{tt} - \sum_{r=1}^{n}\left(a_r(x)u_{x_r}\right)_{x_r} = g(t,x), \quad 0 < t < 1,\ x \in \Omega, \\
u_t + \sum_{r=1}^{n}\left(a_r(x)u_{x_r}\right)_{x_r} = f(t,x), \quad -1 < t < 0,\ x \in \Omega, \\
u(t,x) = 0, \quad x \in S,\ -1 \le t \le 1, \qquad u(1,x) = u(-1,x) + \mu(x), \quad x \in \overline{\Omega}, \\
u(0+,x) = u(0-,x), \qquad u_t(0+,x) = u_t(0-,x), \quad x \in \overline{\Omega}
\end{gathered}
\tag{4.1}
$$

is considered. Problem (4.1) has a unique smooth solution $u(t,x)$ for $f(t,x)$ $(t \in (-1,0),\ x \in \overline{\Omega})$, $g(t,x)$ $(t \in (0,1),\ x \in \overline{\Omega})$ the smooth functions, and $a_r(x) \geqslant a > 0 (x \in \Omega)$.

The discretization of problem (4.1) is carried out in two steps. In the first step, the grid sets

$$\begin{gathered}\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \dots, h_n m_n), m = (m_1, \dots, m_n),\\ 0 \le m_r \le N_r, h_r N_r = 1, r = 1, \dots, n\},\\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S\end{gathered} \tag{4.2}$$

are defined. To the differential operator A generated by problem (4.1), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} \left(a_r(x) u_{\overline{x}_r}^h \right)_{x_r, m_r} \tag{4.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{gathered}-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), \quad 0 < t < 1,\ x \in \Omega_h,\\ \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad -1 < t < 0,\ x \in \Omega_h,\\ u^h(1,x) = u^h(-1,x) + \mu^h(x), \quad x \in \widetilde{\Omega}_h,\\ u^h(0+,x) = u^h(0-,x), \qquad \frac{du^h(0+,x)}{dt} = \frac{du^h(0-,x)}{dt}, \quad x \in \widetilde{\Omega}_h\end{gathered} \tag{4.4}$$

for an infinite system of ordinary differential equations.

Replacing problem (4.4) by the difference scheme (1.2), one can obtain the second order of accuracy difference scheme

$$\begin{gathered}-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = g_k^h(x),\\ g_k^h(x) = g(t_k, x_n), \quad t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = 1,\ x \in \Omega_h,\\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} - \frac{A_h^x}{2}\left(u_k^h(x) + u_{k-1}^h(x) \right) = f_k^h(x),\\ f_k^h(x) = f(t_{k-1/2}, x_n), \quad t_{k-1/2} = \left(k - \frac{1}{2}\right)\tau,\ -N+1 \le k \le 0,\ x \in \Omega_h,\\ u_N^h(x) = u_{-N}^h(x) + \mu^h(x), \quad x \in \widetilde{\Omega}_h,\\ -u_2^h(x) + 4u_1^h(x) - 3u_0^h(x) = 3u_0^h(x) - 4u_{-1}^h(x) + u_{-2}^h(x), \quad x \in \widetilde{\Omega}_h.\end{gathered} \tag{4.5}$$

Let us give a corollary of Theorems 3.1 and 3.2.

Theorem 4.1. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, solutions of difference scheme* (4.5) *satisfy the following stability and almost coercivity estimates:*

$$\begin{aligned}
&\left\| \left\{ u_k^h \right\}_{-N}^{N-1} \right\|_{C([-1,1]_\tau, L_{2h})} \le M \left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{C([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_{1}^{N-1} \right\|_{C([0,1]_\tau, L_{2h})} + \left\| \mu^h \right\|_{L_{2h}} \right], \\
&\left\| \left\{ \tau^{-2} \left(u_{k+1}^h - 2u_k^h + u_{k-1}^h \right) \right\}_1^{N-1} \right\|_{C([0,1]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C([0,1]_\tau, W_{2h}^2)} + \left\| \left\{ \tau^{-1} \left(u_k^h - u_{k-1}^h \right) \right\}_{-N+1}^{0} \right\|_{C([-1,0]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ \frac{u_k^h + u_{k-1}^h}{2} \right\}_{-N+1}^{0} \right\|_{C([-1,0]_\tau, W_{2h}^2)} \\
&\le M \Bigg[\left\| f_0^h \right\|_{L_{2h}} + \left\| f_{-1}^h \right\|_{L_{2h}} + \left\| g_1^h \right\|_{L_{2h}} + \left\| \mu^h \right\|_{W_{2h}^2} \\
&\qquad + \tau \left\| f_0^h \right\|_{W_{2h}^1} + \tau \left\| f_{-1}^h \right\|_{W_{2h}^1} + \tau \left\| g_1^h \right\|_{W_{2h}^1} \\
&\qquad + \ln \frac{1}{\tau + |h|} \left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{C([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C([0,1]_\tau, L_{2h})} \right] \Bigg].
\end{aligned} \tag{4.6}$$

Here, M is independent of not only $\tau, h, \mu^h(x)$ but also $g_k^h(x), 1 \le k \le N-1$, and $f_k^h, -N+1 \le k \le 0$.

The proof of Theorem 4.1 is based on Theorems 3.1 and 3.2, the estimate

$$\min\left\{ \ln \frac{1}{\tau}, 1 + \left| \ln \left\| A_h^x \right\|_{L_{2h} \to L_{2h}} \right| \right\} \le M \ln \frac{1}{\tau + |h|}, \tag{4.7}$$

the symmetry properties of the difference operator A_h^x defined by formula (4.3) in L_{2h}, and the following theorem.

Theorem 4.2. *For the solution of the elliptic difference problem*

$$\begin{gathered}
A_h^x u^h(x) = \omega^h(x), \quad x \in \Omega_h, \\
u^h(x) = 0, \quad x \in S_h
\end{gathered} \tag{4.8}$$

the following coercivity inequality holds [41]:

$$\sum_{r=1}^{n} \left\| \left(u^h \right)_{\overline{x}_r x_r, m_r} \right\|_{L_{2h}} \le M \left\| \omega^h \right\|_{L_{2h}}. \tag{4.9}$$

Let us give a corollary of Theorem 3.3.

Theorem 4.3. *Let τ and $|h|$ be sufficiently small positive numbers. Then, solutions of difference scheme* (4.5) *satisfy the following coercivity stability estimates:*

$$\begin{aligned}
&\left\| \left\{ \tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, W_{2h}^2)} \\
&\quad + \left\| \left\{ \frac{u_k^h + u_{k-1}^h}{2} \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, W_{2h}^2)} \\
&\le M\Bigg[\left\|\mu^h\right\|_{W_{2h}^2} + \tau\left\|f_0^h\right\|_{W_{2h}^1} + \tau\left\|f_{-1}^h\right\|_{W_{2h}^1} + \tau\left\|g_1^h\right\|_{W_{2h}^1} \\
&\quad + \frac{1}{\alpha(1-\alpha)}\left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{C_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \right]\Bigg], \\
&\left\| \left\{ \tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ \frac{u_k^h + u_{k-1}^h}{2} \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, W_{2h}^2)} \\
&\quad + \left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, W_{2h}^2)} \\
&\le M\Bigg[\left\|\mu^h\right\|_{W_{2h}^2} + \tau\left\|f_0^h\right\|_{W_{2h}^1} + \tau\left\|f_{-1}^h\right\|_{W_{2h}^1} + \tau\left\|g_1^h\right\|_{W_{2h}^1} \\
&\quad + \frac{1}{\alpha(1-\alpha)}\left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \right]\Bigg],
\end{aligned} \tag{4.10}$$

where M is independent of not only $\tau, h,$ and $\mu^h(x)$ but also $g_k^h(x), 1 \le k \le N-1$ and $f_k^h, -N+1 \le k \le 0$.

The proof of Theorem 4.3 is based on the abstract Theorems 3.3 and 4.2, and the symmetry properties of the difference operator A_h^x defined by the formula (4.3).

Table 1: Comparison of the errors.

Method	$N = M = 20$	$N = M = 30$	$N = M = 60$
1st order of accuracy d. s.	0.043541	0.030515	0.015973
2nd order of accuracy d. s	0.000627	0.000283	0.000071

Second, the mixed boundary value problem for the elliptic-parabolic equation

$$\begin{gathered} -u_{tt} - (a(x)u_x)_x + \delta u = g(t,x), \quad 0 < t < 1,\ 0 < x < 1, \\ u_t + (a(x)u_x)_x - \delta u = f(t,x), \quad -1 < t < 0,\ 0 < x < 1, \\ u(t,0) = u(t,1), \qquad u_x(t,0) = u_x(t,1), \quad -1 \le t \le 1, \\ u(1,x) = u(-1,x) + \mu(x), \quad 0 \le x \le 1, \\ u(0+,x) = u(0-,x), \qquad u_t(0+,x) = u_t(0-,x), \quad 0 \le x \le 1 \end{gathered} \tag{4.11}$$

is considered. Problem (4.11) has a unique smooth solution $u(t,x)$ for $f(t,x)(t \in [-1,0], x \in [0,1]), g(t,x)(t \in [0,1], x \in [0,1])$, the smooth functions, and $a(x) \geqslant a > 0(x \in (0,1)), \delta = \text{const} > 0$.

Note that in a similar manner one can construct the difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the boundary value problem (4.11). Abstract theorems given above permit us to obtain the stability, the almost stability and the coercive stability estimates for the solutions of these difference schemes.

5. Numerical Results

We consider the nonlocal boundary value problem

$$\begin{gathered} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = \sin x, \quad -1 < t \le 0,\ 0 < x < \pi, \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \sin x, \quad 0 < t < 1,\ 0 < x < \pi, \\ u(1,x) = u(-1,x) + 2\sinh 1 \sin x, \quad 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad -1 \le t \le 1 \end{gathered} \tag{5.1}$$

for the elliptic-parabolic equation.

The exact solution of this problem is $u(t,x) = (e^t - 1)\sin x$.

Now, we give the results of the numerical analysis. The errors computed by

$$E_M^N = \max_{-N \le k \le N, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right| \tag{5.2}$$

of the numerical solutions are given in Table 1.

Thus, the second order of accuracy difference scheme is more accurate than the first order of accuracy difference scheme.

Acknowledgments

The authors would like to thank the referees and Professor P. E. Sobolevskii (Jerusalem, Israel) for helpful suggestions to the improvement of this paper.

References

[1] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, NY, USA, 1968.
[2] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, vol. 23 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1968.
[3] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II," *Communications on Pure and Applied Mathematics*, vol. 17, pp. 35–92, 1964.
[4] M. I. Vislik, A. D. Myshkis, and O. A. Oleinik, "Partial differential equations," in *Mathematics in USSR in the Last 40 Years*, vol. 1, pp. 563–599, Fizmatgiz, Moscow, Russia, 1959.
[5] D. Bazarov and H. Soltanov, *Some Local and Nonlocal Boundary Value Problems for Equations of Mixed and Mixed-Composite Types*, Ylim, Ashgabat, Turkmenistan, 1995.
[6] A. Ashyralye, "A note on the nonlocal boundary value problem for elliptic-parabolic equations," *Nonlinear Studies*, vol. 13, no. 4, pp. 327–333, 2006.
[7] M. Sapagovas, "On the stability of a finite-difference scheme for nonlocal parabolic boundary-value problems," *Lithuanian Mathematical Journal*, vol. 48, no. 3, pp. 339–356, 2008.
[8] M. S. Salakhitdinov, *Uravneniya Smeshanno-Sostavnogo Tipa*, Fan, Tashkent, Uzbekistan, 1974.
[9] R. Ewing, R. Lazarov, and Y. Lin, "Finite volume element approximations of nonlocal reactive flows in porous media," *Numerical Methods for Partial Differential Equations*, vol. 16, no. 3, pp. 285–311, 2000.
[10] J. R. Cannon, S. Pérez Esteva, and J. van der Hoek, "A Galerkin procedure for the diffusion equation subject to the specification of mass," *SIAM Journal on Numerical Analysis*, vol. 24, no. 3, pp. 499–515, 1987.
[11] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, pp. 155–160, 1963.
[12] N. Gordeziani, P. Natalini, and P. E. Ricci, "Finite-difference methods for solution of nonlocal boundary value problems," *Computers & Mathematics with Applications*, vol. 50, no. 8-9, pp. 1333–1344, 2005.
[13] M. Dehghan, "On the solution of the diffusion equation with a nonlocal boundary condition," *Numerical Methods for Partial Differential Equations*, vol. 21, no. 1, pp. 24–40, 2005.
[14] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, vol. 1–11, Masson, Paris, France, 1988.
[15] A. Ashyralyev, "On well-posedness of the nonlocal boundary value problems for elliptic equations," *Numerical Functional Analysis and Optimization*, vol. 24, no. 1-2, pp. 1–15, 2003.
[16] A. Ashyralyev, "High-accuracy stable difference schemes for well-posed NBVP," in *Modern Analysis and Applications*, vol. 191 of *Oper. Theory Adv. Appl.*, pp. 229–252, Birkhäuser, Basel, Switzerland, 2009.
[17] D. Guidetti, B. Karasözen, and S. Piskarev, "Approximation of abstract differential equations," *Journal of Mathematical Sciences*, vol. 122, no. 2, pp. 3013–3054, 2004.
[18] A. Ashyralyev, S. Piskarev, and L. Weis, "On well-posedness of difference schemes for abstract parabolic equations in Lp([0,T];E) spaces," *Numerical Functional Analysis and Optimization*, vol. 23, no. 7-8, pp. 669–693, 2002.
[19] V. L. Makarov and D. T. Kulyev, "Solution of a boundary value problem for a quasilinear equation of parabolic type with nonclassical boundary condition," *Differential Equations*, vol. 21, no. 2, pp. 296–305, 1985.

[20] I. P. Gavrilyuk and V. L. Makarov, “Exponentially convergent parallel discretization methods for the first order evolution equations,” *Computational Methods in Applied Mathematics*, vol. 1, no. 4, pp. 333–355, 2001.

[21] I. P. Gavrilyuk and V. L. Makarov, “Algorithms without accuracy saturation for evolution equations in Hilbert and Banach spaces,” *Mathematics of Computation*, vol. 74, no. 250, pp. 555–583, 2005.

[22] I. P. Gavrilyuk and V. L. Makarov, “Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces,” *SIAM Journal on Numerical Analysis*, vol. 43, no. 5, pp. 2144–2171, 2005.

[23] D. Gordeziani, H. Meladze, and G. Avalishvili, “On one class of nonlocal in time problems for first-order evolution equations,” *Zhurnal Obchyslyuval’nö ta Prykladnö Matematyky*, vol. 88, no. 1, pp. 66–78, 2003.

[24] D. G. Gordeziani and G. A. Avalishvili, “Time-nonlocal problems for Schrödinger-type equations. I. Problems in abstract spaces,” *Differential Equations*, vol. 41, no. 5, pp. 703–711, 2005.

[25] D. Gordeziani, G. Avalishvili, and M. Avalishvili, “Hierarchical models of elastic shells in curvilinear coordinates,” *Computers & Mathematics with Applications*, vol. 51, no. 12, pp. 1789–1808, 2006.

[26] R. P. Agarwal, M. Bohner, and V. B. Shakhmurov, “Maximal regular boundary value problems in Banach-valued weighted space,” *Boundary Value Problems*, vol. 2005, no. 1, pp. 9–42, 2005.

[27] V. B. Shakhmurov, “Coercive boundary value problems for regular degenerate differential-operator equations,” *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 605–620, 2004.

[28] A. Favini, V. Shakhmurov, and Y. Yakubov, “Regular boundary value problems for complete second order elliptic differential-operator equations in UMD Banach spaces,” *Semigroup Forum*, vol. 79, no. 1, pp. 22–54, 2009.

[29] A. V. Gulin and V. A. Morozova, “On the stability of a nonlocal difference boundary value problem,” *Differential Equations*, vol. 39, no. 7, pp. 962–967, 2003 (Russian).

[30] A. V. Gulin, N. I. Ionkin, and V. A. Morozova, “On the stability of a nonlocal two-dimensional difference problem,” *Differential Equations*, vol. 37, no. 7, pp. 970–978, 2001 (Russian).

[31] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, Mass, USA, 2000.

[32] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and Their Applications, 16, Birkhäuser, Basel, Switzerland, 1995.

[33] A. K. Ratyni, “On the solvability of the first nonlocal boundary value problem for an elliptic equation,” *Differential Equations*, vol. 45, no. 6, pp. 862–872, 2009.

[34] T. A. Jangveladze and G. B. Lobjanidze, “On a variational statement of a nonlocal boundary value problem for a fourth-order ordinary differential equation,” *Differential Equations*, vol. 45, no. 3, pp. 335–343, 2009.

[35] A. Ashyralyev and O. Gercek, “Nonlocal boundary value problems for elliptic-parabolic differential and difference equations,” *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 904824, 16 pages, 2008.

[36] P. E. Sobolevskii, “On the stability and convergence of the Crank-Nicolson scheme,” in *Variational-Difference Methods in Mathematical Physics*, pp. 146–151, Vychisl.Tsentr Sibirsk. Otdel. Akademii Nauk SSSR, Novosibirsk, Russia, 1974.

[37] P. E. Sobolevskiĭ, “The theory of semigroups and the stability of difference schemes,” in *Operator Theory in Function Spaces*, pp. 304–337, Nauka, Novosibirsk, Russia, 1977.

[38] P. E. Sobolevskiĭ, “The coercive solvability of difference equations,” *Doklady Akademii Nauk SSSR*, vol. 201, pp. 1063–1066, 1971.

[39] A. O. Ashyralyev and P. E. Sobolevskiĭ, “Coercive stability of a Crank-Nicolson difference scheme in spaces $\tilde{C}_0^\alpha$,” in *Approximate Methods for Investigating Differential Equations and Their Applications*, pp. 16–24, Kuĭbyshev. Gos. Univ., Kuybyshev, Russia, 1982.

[40] A. O. Ashyralyev and P. E. Sobolevskiĭ, “Correct solvability of the Crank-Nicholson scheme for parabolic equations,” *Izvestiya Akademii Nauk Turkmenskoĭ SSR. Seriya Fiziko-Tekhnicheskikh, Khimicheskikh i Geologicheskikh Nauk*, no. 6, pp. 10–16, 1981.

[41] P. E. Sobolevskii, *Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, Russia, 1975.