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Research Article

A New Approximation Method for Common Fixed Points of a Finite Family of Generalized Asymptotically Quasinonexpansive Mappings in Banach Spaces

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We introduce a new iterative scheme to approximate a common fixed point for a finite family of generalized asymptotically quasinonexpansive mappings. Several strong and weak convergence theorems of the proposed iteration in Banach spaces are established. The main results obtianed in this paper generalize and refine many known results in the current literature.

1. Introduction

Let C be a convex subset of a Banach space X, and let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of self-mappings of C. Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned}
x_{n+1} &= (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\
y_{(k-1)n} &= \left(1-\alpha_{(k-1)n}\right)x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\
y_{(k-2)n} &= \left(1-\alpha_{(k-2)n}\right)x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\
&\ \ \vdots \\
y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\
y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n},
\end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all n. The iterative process (1.1) for a finite family of mappings introduced by Khan et al. [1], and the iterative process is the generalized form of the modified Mann (one-step) iterative process by Schu [2], the modified Ishikawa (two-step) iterative process by Tan and Xu [3], and the three-step iterative process by Xu and Noor [4].

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [1, 5–19] are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [12–16, 20, 21].

In 2003, Sun [17] studied an implicit iterative scheme initiated by Xu and Ori [22] for a finite family of asymptotically quasinonexpansive mappings. Shahzand and Udomene [18], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasinonexpence mappings to a common fixed point. Nammanee et al. [23] introduced a three-step iteration scheme for asymptotically nonexpansive mappings and proved weak and strong convergence theorems of that iteration scheme under some control conditions. In 2007, Fukhar-ud-din and Khan [24] studied a new three-step iteration scheme for approximating a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Shahzad and Zegeye [19] introduced a new concept of generalized asymptotially nonexpansive mappings and proved some strong convergence theorems for fixed points of finite family of this class. Recently, Khan et al. [1] introduced the iterative sequence (1.1) for a finite family of asymptotically quasinonexpansive mappings in Banach spaces.

Motivated by Khan et al. [1], we introduce a new iterative scheme for finding a common fixed point of a finite family of generalized asymptotically quasinonexpansive mappings as follows:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
x_{n+1} &= (1-\alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n},\\
y_{(k-1)n} &= \left(1-\alpha_{(k-1)n}\right)y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n},\\
y_{(k-2)n} &= \left(1-\alpha_{(k-2)n}\right)y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n},\\
&\ \ \vdots\\
y_{2n} &= (1-\alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n},\\
y_{1n} &= (1-\alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n},
\end{aligned} \tag{1.2}$$

where $y_{0n} = x_n$ for all n.

The aim of this paper is to obtain strong and weak convergence results for the iterative process (1.2) of a finite family of generalized asymptotically quasinonexpansive mappings in Banach spaces.

2. Preliminaries

In this section, we give some definitions and lemmas used in the main results.

Let C be a nonempty subset of a real Banach space X, and let T be a self-mapping of C. The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Then let T is called

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;

(ii) *quasinonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$;

(iii) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ and $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(iv) *asymptotically quasinonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ and $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(v) *generalized quasinonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \to 0$ as $n \to \infty$ such that $\|T^n x - p\| \leq \|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(vi) *generalized asymptotically quasinonexpansive* [19] if $F(T) \neq \emptyset$ and there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(vii) *uniformly L-Lipschitzian* if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(viii) $(L-\gamma)$ *uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^{\gamma}$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$.

(ix) *semicompact* if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

From the definition of these mappings, it can be seen that

(i) *a quasinonexpansive mapping is generalized quasinonexpansive;*

(ii) *an asymptotically quasinonexpansive mapping is generalized asymptotically quasinonexpansive;*

(iii) *a generalized quasinonexpansive mapping is generalized asymptotically quasinonexpansive;*

(iv) *a uniformly L-Lipschitzian mapping is* $(L - 1)$ *uniform Lipschitz.*

The map $T : C \to X$ is said to be *demiclosed at 0* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and $T(x_n)$ converging strongly to 0, we get $Tx = 0$.

A Banach space X is said to have *Opial's property* if for each sequence $\{x_n\}$ converging weakly to $x \in C$ and $x \neq y$, we have the condition

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|. \tag{2.1}$$

Condition (A″). Let C be a subset of a normed space X. A family of self-mappings $\{T_i : i = 1, 2, \ldots, k\}$ of C is said to have Condition (A″) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T_i x\| \geq f(d(x, F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F = \bigcap_{i=1}^{k} F(T_i)\}$.

The condition (A$''$) defined above by the authors is the generalization of the condition (A) [25] when $k = 1$ and condition (A$'$) [26] for $k = 2$.

The following lemmas are needed for proving our main results.

Lemma 2.1 (cf. [17, Lemma 2.2]). *Let the sequences $\{a_n\}$, $\{\delta_n\}$ and $\{c_n\}$ of real numbers satisfy:*

$$a_{n+1} \leq (1+\delta_n)a_n + c_n, \quad \text{where } a_n \geq 0,\ \delta_n \geq 0,\ c_n \geq 0\ \forall n = 1,2,3,\ldots \tag{2.2}$$

and $\sum_{n=1}^{\infty}, \delta_n < \infty$, $\sum_{n=1}^{\infty}, c_n < \infty$.

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *if* $\lim\inf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [2, Lemma 1.3]). *Let X be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1,2,3,\ldots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$ and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

3. Convergence in Banach Spaces

The aim of this section is to establish the strong convergence of the iterative scheme (1.2) to converge to a common fixed point of a finite family of asymptotically quasinonexpansive mappings in a Banach space under some appropriate conditions.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1,2,\ldots,k\}$ be a finite family of generalized asymtotically quasinonexpansive self-mappings of C, that is, $\|T_i^n x - p_i\| \leq (1+r_{in})\|x - p_i\| + s_{in}$, for all $x \in C$ with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0,\infty)$ and $p_i \in F(T_i)$, $i = 1,2,\ldots,k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$ is defined by (1.2). Let $r_n = \max_{1\leq i\leq k}\{r_{in}\}$ and $s_n = \max_{1\leq i\leq k}\{s_{in}\}$ Then for $p \in F$, we get the following:*

(i) $\|x_n - T_i^n x_n\| \leq (2+r_n)\|x_n - p\| + s_n$, *for all* $i = 1,2,\ldots,k$;

(ii) $\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2+r_n)\|y_{(i-1)n} - p\| + s_n$, *for all* $i = 1,2,\ldots,k$;

(iii) $\|y_{kn} - p\| \leq (1+r_n)^k\|x_n - p\| + s_n\sum_{i=1}^{k}(1+r_n)^{k-i}$;

(iv) $\|x_{n+1} - p\| \leq (1+r_n)^k\|x_n - p\| + s_n\sum_{i=1}^{k}(1+r_n)^{(k-i)}$;

(v) $\|x_{n+1} - p\| \leq (1+\delta_n)\|x_n - p\| + c_n$, *for all* $n \in \mathbb{N}$,
where $c_n = s_n\sum_{i=1}^{k}(1+r_n)^{(k-i)}$ *and* $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$.

(vi) *If* $\sum_{n=1}^{\infty} r_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$, *then* $\lim_{n\to\infty}\|x_n - p\|$ *exists.*

Proof. Let $p \in F$.

(i) For $i = 1,2,3,\ldots,k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - p\| + \|T_i^n x_n - p\| \\ &\leq \|x_n - p\| + (1+r_n)\|x_n - p\| + s_{in}, \\ &\leq (2+r_n)\|x_n - p\| + s_n. \end{aligned} \tag{3.1}$$

(ii) Similarly to part (i), we have

$$\left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\| \leq (2 + r_n)\left\|y_{(i-1)n} - p\right\| + s_n, \quad \forall i = 1, 2, \ldots, k. \tag{3.2}$$

(iii) By part (i) and $0 \leq \alpha_{in} \leq 1$, we have

$$\begin{aligned}
\left\|y_{1n} - p\right\| &= \left\|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1^n x_n - p)\right\| \\
&\leq (1 - \alpha_{1n})\left\|x_n - p\right\| + \alpha_{1n}\left\|T_1^n x_n - p\right\| \\
&\leq (1 - \alpha_{1n})\left\|x_n - p\right\| + \alpha_{1n}(1 + r_{1n})\left\|x_n - p\right\| + \alpha_{1n}s_{1n} \\
&\leq (1 + r_n)\left\|x_n - p\right\| + s_n, \\
\left\|y_{2n} - p\right\| &= \left\|(1 - \alpha_{2n})(y_{1n} - p) + \alpha_{2n}(T_2^n y_{1n} - p)\right\| \\
&\leq (1 - \alpha_{2n})\left\|y_{1n} - p\right\| + \alpha_{2n}\left\|T_2^n y_{1n} - p\right\| \\
&\leq (1 - \alpha_{2n})\left\|y_{1n} - p\right\| + \alpha_{2n}(1 + r_{2n})\left\|y_{1n} - p\right\| + \alpha_{2n}s_{2n} \\
&\leq (1 + r_n)\left\|y_{1n} - p\right\| + s_n \\
&\leq (1 + r_n)(1 + r_n)\left\|x_n - p\right\| + s_n(1 + r_n) + s_n \\
&\leq (1 + r_n)^2\left\|x_n - p\right\| + s_n((1 + r_n) + 1), \\
\left\|y_{3n} - p\right\| &= \left\|(1 - \alpha_{3n})(y_{2n} - p) + \alpha_{3n}(T_3^n y_{2n} - p)\right\| \\
&\leq (1 - \alpha_{3n})\left\|y_{2n} - p\right\| + \alpha_{3n}\left\|T_3^n y_{2n} - p\right\| \\
&\leq (1 - \alpha_{3n})\left\|y_{2n} - p\right\| + \alpha_{3n}(1 + r_{3n})\left\|y_{2n} - p\right\| + \alpha_{3n}s_{3n} \\
&\leq (1 + r_n)\left\|y_{2n} - p\right\| + s_n \\
&\leq (1 + r_n)^3\left\|x_n - p\right\| + s_n\Big((1 + r_n)^2 + (1 + r_n) + 1\Big), \\
&\ \ \vdots \\
\left\|y_{kn} - p\right\| &\leq (1 + r_n)^k\left\|x_n - p\right\| + s_n\sum_{i=1}^{k}(1 + r_n)^{(k-i)}.
\end{aligned} \tag{3.3}$$

(iv) By part (ii) and part (iii), we get

$$\begin{aligned}
\left\|x_{n+1} - p\right\| &= \left\|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\right\| \\
&\leq (1 - \alpha_{kn})\left\|y_{(k-1)n} - p\right\| + \alpha_{kn}\left\|T_k^n y_{(k-1)n} - p\right\| \\
&\leq (1 - \alpha_{kn})\left\|y_{(k-1)n} - p\right\| + \alpha_{kn}(1 + r_n)\left\|y_{(k-1)n} - p\right\| + \alpha_{kn}s_{kn} \\
&\leq (1 + r_n)\left\|y_{(k-1)n} - p\right\| + s_n \\
&\leq (1 + r_n)(1 + r_n)^{k-1}\left\|x_n - p\right\| + (1 + r_n)s_n\sum_{i=1}^{k-1}(1 + r_n)^{(k-1-i)} + s_n \\
&= (1 + r_n)^k\left\|x_n - p\right\| + s_n\sum_{i=1}^{k}(1 + r_n)^{(k-i)}.
\end{aligned} \tag{3.4}$$

(v) Put $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$ and $c_n = s_n \sum_{i=1}^{k}(1+r_n)^{(k-i)}$. Then (v) is directly obtained by (iv).

(vi) By (v), we have $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n$ for all $n \in N$, where $c_n = s_n \sum_{i=1}^{k}(1 + r_n)^{(k-i)}$ and $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$. From $\Sigma_{n=1}^{\infty} r_n < \infty$ and $\Sigma_{n=1}^{\infty} s_n < \infty$, it follows that $\Sigma_{n=1}^{\infty}\delta_n < \infty$ and $\Sigma_{n=1}^{\infty} c_n < \infty$. By Lemma 2.1, we get $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a finite family of generalized asymtotically quasinonexpansive self-mappings of C, that is, $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is closed, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by* (1.2). *Assume that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, where $r_n = \max_{1\leq i\leq k}\{r_{in}\}$ and $s_n = \max_{1\leq i\leq k}\{s_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F}\|x - p\|$.*

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. Since $1 + t \leq e^t$ for $t \geq 0$, we obtain $(1 + t)^k \leq e^{kt}$, for $k = 1, 2, \ldots$. Thus by Lemma 3.1(iv) and (v), for positive integers m and n, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + r_{(n+m-1)})^k \|x_{(n+m-1)} - p\| + c_{(n+m-1)} \\
&\leq \exp\{k r_{(n+m-1)}\} \|x_{(n+m-1)} - p\| + c_{(n+m-1)} \\
&\leq \cdots \leq \exp\left\{k \sum_{i=n}^{n+m-1} r_i\right\} \|x_n - p\| + \sum_{i=n}^{n+m-1} c_i \\
&\leq \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} \|x_n - p\| + \sum_{i=n}^{\infty} c_i \\
&= M\|x_n - p\| + \sum_{i=n}^{\infty} c_i,
\end{aligned} \tag{3.5}$$

where $M = \exp\{k \sum_{i=1}^{\infty} r_i\}$.

By Lemma 3.1(v), we have

$$\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n, \quad \forall p \in F, \tag{3.6}$$

where $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$ and $c_n = s_n \sum_{i=1}^{k}(1 + r_n)^{(k-i)}$.

It follows that

$$d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F). \tag{3.7}$$

From the given condition $\liminf_{n\to\infty} d(x_n, F) = 0$ and Lemma 2.1, we get

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.8}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. By (3.8) and $\sum_{n=1}^{\infty} c_n < \infty$, we get that for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{3M}, \qquad \sum_{n=n_0}^{\infty} c_n < \frac{\epsilon}{3}. \tag{3.9}$$

From $d(x_n, F) < \epsilon/3M$, there exists $p_0 \in F$ such that

$$\left\| x_{n_0} - p_0 \right\| < \frac{\epsilon}{3M}. \tag{3.10}$$

For any positive integer m, by (3.5), (3.9), and (3.10), we have

$$\begin{aligned} \| x_{n_0+m} - x_{n_0} \| &\leq \left\| x_{n_0+m} - p_0 \right\| + \left\| x_{n_0} - p_0 \right\| \\ &\leq M \left\| x_{n_0} - p_0 \right\| + \sum_{i=n_0}^{\infty} c_i + \left\| x_{n_0} - p_0 \right\| \\ &< M \frac{\epsilon}{3M} + \frac{\epsilon}{3} + M \frac{\epsilon}{3M} = \epsilon. \end{aligned} \tag{3.11}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $x_n \to q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and C is a closed subset of X. Next we show that $q \in F$. Since $F = \bigcap_{i=1}^{k} F(T_i)$ is closed, by the continuity of $d(x, F)$ with $d(x_n, F) \to 0$ and $x_n \to q$ as $n \to \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete. □

Since any asymptotically quasinonexpansive mapping is generalized asymptotically quasinonexpansive, the next corollary is obtained immediately from Theorem 3.2.

Corollary 3.3 (see [5, Theorem 3.2]). *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a finite family of asymptotically quasinonexpansive self-mappings of C, that is, $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$. $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by* (1.2). *Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf_{p\in F} \|x - p\|$.

4. Convergence in Uniformly Convex Banach Spaces

In this section, strong and weak convergence results for the iterative process (1.2) on uniformly convex Banach spaces are proved without using the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ appearing in Section 3.

Theorem 4.1. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X. Let $\{T_i : i = 1, 2, \ldots, k\}$ be a finite family of uniformly $(L - \gamma_i)$ Lipschitzian and generalized asymptotically quasinonexpansive self-mappings of C, that is, $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$. Suppose that $\{T_i : i = 1, 2, \ldots, k\}$ satisfies condition (A'') and $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by* (1.2) *with $\{\alpha_{in}\}_{i=1}^{n} \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$,*

$\sum_{n=1}^{\infty} s_n < \infty$ *where* $r_n = \max_{1\leq i\leq k}\{r_{in}\}$ *and* $s_n = \max_{1\leq i\leq k}\{s_{in}\}$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of the family of mappings.*

Proof. Let $p \in F$. By Lemma 3.1(vi), we get that $\lim_{n\to\infty}\|x_n - p\|$ exists. Then there is a real number $c \geq 0$ such that

$$\lim_{n\to\infty}\|x_n - p\| = c. \tag{4.1}$$

By Lemma 3.1(iii), we have

$$\|y_{kn} - p\| \leq (1+r_n)^k\|x_n - p\| + s_n\sum_{i=1}^{k}(1+r_n)^{(k-i)}, \quad \forall p \in F. \tag{4.2}$$

By taking lim sup on both sides of the above inequality, we get

$$\limsup_{n\to\infty}\|y_{in} - p\| \leq c, \quad \text{for } i = 1,2,\ldots,k-1. \tag{4.3}$$

Since $\|T_i^n y_{(i-1)n} - p\| \leq (1+r_n)\|y_{(i-1)n} - p\| + s_n$ and (4.3), we obtain

$$\limsup_{n\to\infty}\|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1,2,\ldots,k. \tag{4.4}$$

Since $\lim_{n\to\infty}\|x_{n+1} - p\| = c$, we have

$$\lim_{n\to\infty}\|(1-\alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = c. \tag{4.5}$$

Using (4.1), (4.4), and Lemma 2.2, we conclude that

$$\lim_{n\to\infty}\|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \tag{4.6}$$

We assume that

$$\lim_{n\to\infty}\left\|y_{(j-1)n} - T_j^n y_{(j-1)n}\right\| = 0, \quad \text{for some } 2 \leq j \leq k. \tag{4.7}$$

It follows from (4.5) and (4.7) that

$$c \leq \liminf_{n\to\infty}\|y_{(j-1)n} - p\|, \quad \text{for } 2 \leq j \leq k. \tag{4.8}$$

By Lemma 3.1(iv), (1.2), and (4.8), we get

$$\lim_{n\to\infty}\left\|(1-\alpha_{(j-1)n})(y_{(j-2)n} - p) + \alpha_{(j-1)n}\left(T_{j-1}^n y_{(j-2)n} - p\right)\right\| = \lim_{n\to\infty}\|y_{(j-1)n} - p\| = c. \tag{4.9}$$

Using (4.3), (4.4) and Lemma 2.2, we conclude that

$$\lim_{n\to\infty}\left\|y_{(j-2)n}-T_{j-1}^{n}y_{(j-2)n}\right\|=0. \tag{4.10}$$

Therefore, by mathematical induction, we obtain

$$\lim_{n\to\infty}\left\|y_{(i-1)n}-T_{i}^{n}y_{(i-1)n}\right\|=0,\quad \text{for } i=1,2,\ldots,k. \tag{4.11}$$

From (1.2), we have

$$\left\|y_{in}-y_{(i-1)n}\right\|=\alpha_{in}\left\|T_{i}^{n}y_{(i-1)n}-y_{(j-1)n}\right\|,\quad \text{for } i=1,2,\ldots,k-1. \tag{4.12}$$

By (4.11), we obtain that

$$\left\|y_{in}-y_{(i-1)n}\right\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty,\ \text{for } i=1,2,\ldots,k-1. \tag{4.13}$$

From

$$\left\|x_{n}-y_{in}\right\|\leq\left\|x_{n}-y_{1n}\right\|+\left\|y_{1n}-y_{2n}\right\|+\cdots+\left\|y_{(i-1)n}-y_{in}\right\|, \tag{4.14}$$

for $i=1,2,\ldots,k-1$, it follows by (4.13) that

$$\left\|x_{n}-y_{in}\right\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty,\ \text{for } i=1,2,\ldots,k-1. \tag{4.15}$$

From (4.11), when $i=1$, we get $\lim_{n\to\infty}\|x_n-T_1^n x_n\|=0$. For $2\leq i\leq k$, we have

$$\begin{aligned}\left\|x_{n}-T_{i}^{n}x_{n}\right\|&\leq\left\|x_{n}-y_{(i-1)n}\right\|+\left\|y_{(i-1)n}-T_{i}^{n}y_{(i-1)n}\right\|+\left\|T_{i}^{n}y_{(i-1)n}-T_{i}^{n}x_{n}\right\|\\&\leq\left\|x_{n}-y_{(i-1)n}\right\|+\left\|y_{(i-1)n}-T_{i}^{n}y_{(i-1)n}\right\|+L\left\|y_{(i-1)n}-x_{n}\right\|^{\gamma_i}.\end{aligned} \tag{4.16}$$

From (4.11) and (4.15), we conclude that

$$\lim_{n\to\infty}\gamma_{in}=0,\quad \text{for } i=1,2,\ldots,k, \tag{4.17}$$

where $\gamma_{in}=\|x_n-T_i^n x_n\|$. From (1.2), we have

$$\begin{aligned}\|x_{n+1}-x_{n}\|&\leq(1-\alpha_{kn})\left\|y_{(k-1)n}-x_{n}\right\|+\alpha_{kn}\left\|T_{k}^{n}y_{(k-1)n}-x_{n}\right\|\\&\leq(1-\alpha_{kn})\left\|y_{(k-1)n}-x_{n}\right\|+\alpha_{kn}\left(\left\|T_{k}^{n}y_{(k-1)n}-y_{(k-1)n}\right\|+\left\|y_{(k-1)n}-x_{n}\right\|\right)\\&=\left\|y_{(k-1)n}-x_{n}\right\|+\alpha_{kn}\left\|T_{k}^{n}y_{(k-1)n}-y_{(k-1)n}\right\|.\end{aligned} \tag{4.18}$$

From (4.11) and (4.15),

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.19}$$

For $i=1,2,\ldots,k$, we have

$$\begin{aligned}
\|x_{n+1}-T_ix_{n+1}\| &\le \left\|x_{n+1}-T_i^{n+1}x_{n+1}\right\|+\left\|T_ix_{n+1}-T_i^{n+1}x_{n+1}\right\|\\
&\le \gamma_{i(n+1)}+L\left\|x_{n+1}-T_i^nx_{n+1}\right\|^{\gamma_i}\\
&\le \gamma_{i(n+1)}+L\left(\|x_{n+1}-x_n\|+\left\|x_n-T_i^nx_n\right\|+\left\|T_i^nx_n-T_i^nx_{n+1}\right\|\right)^{\gamma_i}\\
&\le \gamma_{i(n+1)}+L\left(\|x_{n+1}-x_n\|+\gamma_{in}+L\|x_n-x_{n+1}\|^{\gamma_i}\right)^{\gamma_i}.
\end{aligned} \tag{4.20}$$

Using (4.17) and (4.19), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-T_ix_{n+1}\|=0,\quad \text{for } i=1,2,\ldots,k. \tag{4.21}$$

Therefore, by using condition (A″), there exists a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$\lim_{n\to\infty}f(d(x_n,F))\le\lim_{n\to\infty}\left\|x_n-T_jx_n\right\|=0, \tag{4.22}$$

for some $1\le j\le k$, that is

$$\lim_{n\to\infty}d(x_n,F)=0. \tag{4.23}$$

By Theorem 3.2, we conclude that $\{x_n\}$ converges strongly to a point $p\in F$. □

Lemma 4.2. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X, and $\{T_i : i=1,2,\ldots,k\}$ be a family of $(L-\gamma_i)$ uniform Lipschitz and generalized asymtotically quasinonexpansive self-mappings of C, that is, $\|T_i^nx-T_i^ny\|\le L\|x-y\|^{\gamma_i}$ and $\|T_i^nx-p_i\|\le(1+r_{in})\|x-p_i\|+s_{in}$, for all $x,y\in C$ and $p_i\in F(T_i)$, $i=1,2,\ldots,k$. Suppose that $F=\bigcap_{i=1}^kF(T_i)\neq\emptyset$. Let $x_1\in C$ and the iterative sequence $\{x_n\}$ be defined by (1.2) with $\{\alpha_{in}\}_{i=1}^n\subset[a,b]$, where $0<a<b<1$. Assume that $\sum_{n=1}^\infty r_n<\infty$, $\sum_{n=1}^\infty s_n<\infty$ where $r_n=\max_{1\le i\le k}\{r_{in}\}$ and $s_n=\max_{1\le i\le k}\{s_{in}\}$. Then,*

(i) $\lim_{n\to\infty}\|x_n-T_i^ny_{(i-1)n}\|=0$, *for all* $i=1,2,,\ldots,k$;

(ii) $\lim_{n\to\infty}\|x_n-T_ix_n\|=0$, *for all* $i=1,2,,\ldots,k$.

Proof. (i) Let $p\in F$. By Lemma 3.1(vi), we obtain that $\lim_{n\to\infty}\|x_n-p\|$ exists and we then suppose that

$$\lim_{n\to\infty}\left\|x_n-p\right\|=c. \tag{4.24}$$

By (4.24) and Lemma 3.1(iii), we have

$$\limsup_{n\to\infty}\|y_{in}-p\| \le c, \quad \text{for } i=1,2,\ldots,k-1. \tag{4.25}$$

By (1.2), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le (1-\alpha_{kn})\|y_{(k-1)n}-p\| + \alpha_{kn}\|T_k^n y_{(k-1)n}-p\| \\
&\le (1-\alpha_{kn})\|y_{(k-1)n}-p\| + \alpha_{kn}(1+r_n)\|y_{(k-1)n}-p\| + \alpha_{kn}s_{kn} \\
&\le (1+r_n)\|y_{(k-1)n}-p\| + s_n \\
&= (1+r_n)\left\|(1-\alpha_{(k-1)n})(y_{(k-2)n}-p) + \alpha_{(k-1)n}(T_{k-1}^n y_{(k-2)n}-p)\right\| + s_n \\
&\le (1+r_n)\left[(1-\alpha_{(k-1)n})\|y_{(k-2)n}-p\| + \alpha_{(k-1)n}\|T_{k-1}^n y_{(k-2)n}-p\|\right] + s_n \\
&\le (1+r_n)\left[(1-\alpha_{(k-1)n})\|y_{(k-2)n}-p\| + \alpha_{(k-1)n}(1+r_n)\|y_{(k-2)n}-p\| + \alpha_{(k-1)n}s_n\right] + s_n \\
&\le (1+r_n)^2\|y_{(k-2)n}-p\| + s_n(1+r_n) + s_n \\
&\ \ \vdots \\
&\le (1+r_n)^{k-i}\|y_{in}-p\| + s_n\sum_{i=1}^{k-1}(1+r_n)^{k-i},
\end{aligned} \tag{4.26}$$

for $i=1,2,\ldots,k-1$. It follows that

$$c \le \liminf_{n\to\infty}\|y_{in}-p\|, \quad \text{for } i=1,2,\ldots,k-1. \tag{4.27}$$

From (4.25) and (4.27), we obtain

$$\lim_{n\to\infty}\|y_{in}-p\| = c, \quad \text{for } i=1,2,\ldots,k-1, \tag{4.28}$$

and then

$$\lim_{n\to\infty}\left\|(1-\alpha_{in})(y_{(i-1)n}-p) + \alpha_{in}(T_i^n y_{(i-1)n}-p)\right\| = c, \tag{4.29}$$

for $i=1,2,\ldots,k-1$.

Since $\|T_i^n y_{(i-1)n}-p\| \le (1+r_n)\|y_{(i-1)n}-p\| + s_n$, for $i=1,2,\ldots,k-1$, we have

$$\limsup_{n\to\infty}\|T_i^n y_{(i-1)n}-p\| \le c, \quad \text{for } i=1,2,\ldots,k-1. \tag{4.30}$$

From (4.25), (4.29), (4.30) and Lemma 2.2, we obtain

$$\lim_{n\to\infty}\left\|T_i^n y_{(i-1)n} - y_{(i-1)n}\right\| = 0, \quad \text{for } i = 1,2,\ldots,k-1. \tag{4.31}$$

Now we want to show that (4.31) is also true for $i = k$.

Since $\|T_k^n y_{(k-1)n} - p\| \le (1+r_n)\|y_{(k-1)n} - p\| + s_n$, $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} s_n = 0$, it follows by (4.28),

$$\limsup_{n\to\infty}\left\|T_k^n y_{(k-1)n} - p\right\| \le c. \tag{4.32}$$

We also have

$$\lim_{n\to\infty}\left\|(1-\alpha_{kn})\left(y_{(k-1)n} - p\right) + \alpha_{kn}\left(T_k^n y_{(k-1)n} - p\right)\right\| = \lim_{n\to\infty}\|x_{n+1} - p\| = c. \tag{4.33}$$

Hence, by (4.25), (4.32), and Lemma 2.2, we obtain

$$\lim_{n\to\infty}\left\|y_{(k-1)n} - T_k^n y_{(k-1)n}\right\| = 0. \tag{4.34}$$

Then, (4.31) and (4.34) give us

$$\lim_{n\to\infty}\left\|T_i^n y_{(i-1)n} - y_{(i-1)n}\right\| = 0, \quad \text{for } i = 1,2,\ldots,k. \tag{4.35}$$

From

$$\left\|x_n - T_i^n y_{(i-1)n}\right\| \le \left\|x_n - y_{(i-1)n}\right\| + \left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\|, \tag{4.36}$$

it implies by (4.15) and (4.35) that

$$\lim_{n\to\infty}\left\|x_n - T_i^n y_{(i-1)n}\right\| = 0, \tag{4.37}$$

for $i = 1,2,3,\ldots,k$.

(ii) From part (i), for $i = 1$, we have

$$\lim_{n\to\infty}\left\|T_1^n x_n - x_n\right\| = 0. \tag{4.38}$$

For $i = 2,3,4,\ldots,k$, we get

$$\begin{aligned}\left\|T_i^n x_n - x_n\right\| &\le \left\|T_i^n x_n - T_i^n y_{(i-1)n}\right\| + \left\|T_i^n y_{(i-1)n} - x_n\right\| \\ &\le L\left\|x_n - y_{(i-1)n}\right\|^{\gamma^i} + \left\|T_i^n y_{(i-1)n} - x_n\right\|.\end{aligned} \tag{4.39}$$

By part (i) and (4.15),

$$\lim_{n\to\infty}\left\|T_i^n x_n - x_n\right\| = 0, \quad \text{for } i = 1,2,\ldots,k. \tag{4.40}$$

For $1 \le i \le k$, we obtain

$$\begin{aligned}
\|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\
&\quad + \left\|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\right\| + \left\|T_i^{n+1} x_n - T_i x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\
&\quad + L\|x_{n+1} - x_n\|^{\gamma_i} + L\left\|T_i^n x_n - x_n\right\|^{\gamma_i}.
\end{aligned} \tag{4.41}$$

From (4.19) and (4.40), we then have

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0, \quad \text{for } i = 1,2,\ldots,k. \tag{4.42}$$

□

Theorem 4.3. *Under the hypotheses of Lemma 4.2, assume that T_j^m is semicompact for some positive integers m and $1 \le j \le k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1,2,\ldots,k\}$.*

Proof. Suppose that T_j^m is semicompact for some positive integers $m \ge 1$ and $1 \le j \le k$. We have

$$\begin{aligned}
\left\|T_j^m x_n - x_n\right\| &\le \left\|T_j^m x_n - T_j^{m-1} x_n\right\| + \left\|T_j^{m-1} x_n - T_j^{m-2} x_n\right\| \\
&\quad + \cdots + \left\|T_j^2 x_n - T_j x_n\right\| + \left\|T_j x_n - x_n\right\| \\
&\le (m-1)L\left\|T_j x_n - x_n\right\|^{\gamma_j} + \left\|T_j x_n - x_n\right\|.
\end{aligned} \tag{4.43}$$

Then, by Lemma 4.2(ii), we get $\|T_j^m x_n - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and T_j^m is semicompact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \to q \in C$ as $l \to \infty$.

By continuity of T_i and Lemma 4.2(ii), we obtain

$$\left\|q - T_j q\right\| = \lim_{l\to\infty}\left\|x_{n_l} - T_j\, x_{n_l}\right\| = 0, \quad \forall j = 1,2,\ldots,k. \tag{4.44}$$

Therefore, $q \in F$ and then Theorem 3.2 implies that $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1,2,\ldots,k\}$. □

We note that in practical Theorem 4.3 is very useful in the case that one of T_i, $i = 1,2,3,\ldots,k$, is semicompact.

Theorem 4.4. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1,2,\ldots,k\}$ be a family of $(L-\gamma_i)$ uniform Lipschitz and* *generalized asymtotically quasinonexpansive self-mappings of* C*, that is,* $\|T_i^n x - T_i^n y\| \leq L\|x-y\|^{\gamma_i}$ *and* $\|T_i^n x - p_i\| \leq (1+r_{in})\|x-p_i\| + s_{in}$*, for all* $x, y \in C$ *and* $p_i \in F(T_i)$*,* $i = 1, 2, \ldots, k$*. Suppose that* $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$*. Let* $x_1 \in C$ *and the iterative sequence* $\{x_n\}$ *be defined by* (1.2) *with* $\{\alpha_{in}\}_{i=1}^{n} \subset [a, b]$*, where* $0 < a < b < 1$*. Assume that* $\sum_{n=1}^{\infty} r_n < \infty$*, where* $r_n = \max_{1 \leq i \leq k}\{r_{in}\}$*. If* $I - T_i$*,* $i = 1, 2, \ldots, k$*, is demiclosed at 0, then* $\{x_n\}$ *converges weakly to a common fixed point of the family of mappings.*

Proof. Let $p \in F$. By Lemma 3.1(v), we get $\lim_{n\to\infty}\|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [1] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$. □

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