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Research Article

On the Generalized Hardy Spaces

M. Fatehi

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran

Correspondence should be addressed to M. Fatehi, fatehi@shirazu.ac.ir

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We introduce new spaces that are extensions of the Hardy spaces and we investigate the continuity of the point evaluations as well as the boundedness and the compactness of the composition operators on these spaces.

1. Introduction

Let U be the open unit disk in the complex plane $\mathbb{C}$, ∂U its boundary, and $H(U)$ the space of all analytic functions on the unit disk.

For an analytic function f on the unit disk and $0 < r < 1$, we define the delay function f_r by $f_r(e^{i\theta}) = f(re^{i\theta})$. It is easy to see that the functions f_r are continuous for each r, hence they are in $L^p(\partial U, d\theta/2\pi)$, where $d\theta/2\pi$ is the normalized arc length on the unit circle.

For $0 < p < \infty$, the Hardy space $H^p(U) = H^p$ is the set of all $f \in H(U)$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} \left| f_r\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < \infty. \tag{1.1}$$

Also we recall that $H^\infty(U) = H^\infty$ is the space of all bounded analytic functions defined on U, with supremum norm $\|f\|_\infty = \sup_{z\in U}|f(z)|$. We know that for $p \geq 1$, H^p is a Banach space (see, e.g., [1, page 37]).

By the Littlewood Subordination Theorem (see [2, Corollary 2.23]), we see that the supremum in the above definition of H^p is actually a limit, that is,

$$\|f\|_p^p = \lim_{r\to 1} \int_0^{2\pi} \left| f_r\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < \infty. \tag{1.2}$$

Another important result is Fatou's Radial Limit Theorem (see [1, Theorems 2.2 and 2.6]), which says, if f is in H^p for some $p > 0$, then the radial limit

$$f^*\left(e^{i\theta}\right) = \lim_{r\to 1} f\left(re^{i\theta}\right) \tag{1.3}$$

exists for almost all θ and the mapping $f \to f^*$ is an isometry of H^p to a closed subspace of $L^p(\partial U, d\theta/2\pi)$. Therefore,

$$\|f\|_p^p = \int_0^{2\pi} \left|f^*\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < \infty. \tag{1.4}$$

We will also write $f(e^{i\theta})$ for $f^*(e^{i\theta})$. If $p = 2$ and $\widehat{f}(n)$ are the nth coefficients of f in its Maclaurin series, then we have another representation for the norm of f on H^2 as follows:

$$\|f\|_2^2 = \sum_{n=0}^{\infty}\left|\widehat{f}(n)\right|^2 < \infty. \tag{1.5}$$

The formula above defines a norm that turns H^2 into a Hilbert space whose inner product is given by

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)} = \int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g(e^{i\theta})}\frac{d\theta}{2\pi} \tag{1.6}$$

for each $f, g \in H^2$ (see, e.g., [2]).

Let e_w be the point evaluation at w, that is, $e_w(f) = f(w)$. It is well known that point evaluations at the points of U are all continuous on H^p (see, e.g., [1, page 36]).

Let $w \in U$ and H be a Hilbert space of analytic functions on U. If e_w is a bounded linear functional on H, then the Riesz Representation Theorem implies that there is a function (which is usually called K_w) in H that induces this linear functional, that is, $e_w(f) = \langle f, K_w\rangle$.

Let φ be an analytic self-map of the unit disk. The linear composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(U)$. It is well known (see, e.g., [1, page 29] or [3, Theorem 1]) that the composition operators are bounded on each of the Hardy spaces H^p $(0 < p < \infty)$. One of the first papers in this research area is [3], while Schwartz in [4] begun the research on compact composition operators on H^p. Shapiro and Taylor in [5] have studied the role of angular derivative for compactness of C_φ in H^p. For some other classical results, see [2, 6].

The boundedness and compactness of composition operators, as well as weighted composition operators and other natural extensions of them, on various spaces of analytic functions have been investigated by many authors; see, books [2, 6], and, for example, the following recent papers [7–28] and the references therein.

Throughout this paper, P denotes the set of all analytic polynomials and for a function F, R_F denotes the range of F.

2. Generalized Hardy Spaces

In this section, we define new spaces and investigate some basic properties of these spaces.

Definition 2.1. Let $F : H(U) \to H(U)$ be a linear operator such that $F(f) = 0$ if and only if $f = 0$, that is, F is 1-1. For $1 \le p < \infty$, the generalized Hardy space $H_{F,p}(U) = H_{F,p}$ is defined to be the collection of all analytic functions f on U for which

$$\sup_{0<r<1} \int_0^{2\pi} \left| (F(f))_r \left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi} < \infty. \tag{2.1}$$

Denote the pth root of this supremum by $\|f\|_{H_{F,p}}$. Since $|F(f)|^p$ is a subharmonic function, so by [2, Corollary 2.23], we have

$$\|f\|_{H_{F,p}}^p = \lim_{r \to 1} \int_0^{2\pi} \left| (F(f))_r \left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi}. \tag{2.2}$$

Therefore, $f \in H_{F,p}$ if and only if $F(f) \in H^p$ and

$$\|f\|_{H_{F,p}}^p = \|F(f)\|_p^p = \int_0^{2\pi} \left| F(f) \left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi}. \tag{2.3}$$

It is easy to see that $H_{F,p}$ is a normed space with the norm $\|\cdot\|_{H_{F,p}}$.

From now on, unless otherwise stated, we assume that F satisfies the conditions of Definition 2.1.

In this section, we first set some conditions on F such that $H_{F,p}$ becomes a Banach space. In the following theorem, we obtain a necessary and sufficient condition for $H_{F,p}$ to be a Banach space.

Theorem 2.2. *Let $1 \le p < \infty$ and $P \subseteq R_F$. Then H^p is a subspace of R_F if and only if $H_{F,p}$ is a Banach space.*

Proof. Suppose that $H^p \subseteq R_F$. Since $H_{F,p}$ is a normed space, it suffices to show that it is complete. Let $\{f_n\}$ be a Cauchy sequence in $H_{F,p}$ and set $F(f_n) = g_n$. Then $\{g_n\}$ is a Cauchy sequence in H^p. Since H^p is complete, there is a $g \in H^p$ such that

$$\|g_n - g\|_p \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

Since $H^p \subseteq R_F$, there is an $f \in H(U)$ such that $F(f) = g$. Now we show that this f is the $H_{F,p}$-limit of $\{f_n\}$. We have

$$\|f_n - f\|_{H_{F,p}} = \|g_n - g\|_p \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

Hence $f_n - f \in H_{F,p}$ for sufficiently large positive integer n, which implies that $f \in H_{F,p}$. So $f_n \to f$ in $H_{F,p}$ as $n \to \infty$.

Conversely, suppose that $H_{F,p}$ is a Banach space. If $H^p \nsubseteq R_F$, then there is a $g \in H^p$ such that g is not in R_F. Since the polynomials are dense in H^p, there is a sequence $\{p_n\}$ in P such that $\|p_n - g\|_p \to 0$ as $n \to \infty$. Let $q_n = F^{-1}(p_n)$. Then $\{q_n\}$ is a Cauchy sequence in $H_{F,p}$ and so there is a $q \in H_{F,p}$ such that $\|q_n - q\|_{H_{F,p}} \to 0$ as $n \to \infty$. Hence $\|F(q_n) - F(q)\|_p \to 0$ as $n \to \infty$. On the other hand, $\|F(q_n) - g\|_p \to 0$ as $n \to \infty$. This shows that $g = F(q)$ which is a contradiction. □

Example 2.3. Let $F(\sum_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty}(a_j/2^j)z^j$, where $\sum_{j=0}^{\infty} a_j z^j \in H(U)$. It is not hard to see that $g(z) = \sum_{n=0}^{\infty} 2^{-n/2} z^n \in H^2$ and g is not in R_F. So by Theorem 2.2, $H_{F,2}$ is not a Banach space.

Proposition 2.4. *If $H^2 \subseteq R_F$, then $H_{F,2}$ is a Hilbert space.*

Proof. We define a scalar product on $H_{F,2}$ by

$$\langle f, g \rangle_{H_{F,2}} = \langle F(f), F(g) \rangle_{H^2}. \tag{2.6}$$

It is easy to show that this scalar product defines an inner product on $H_{F,2}$. □

There is a Banach space $H_{F,p}$ such that it does not satisfy the condition of Theorem 2.2. For example, let $1 \le p < \infty$, $F(f) = zf$ for each $f \in H(U)$. Then $1 \notin R_F$. By the following proposition, we see that although $H^p \not\subseteq R_F$, $H_{F,p}$ is a Banach space.

Proposition 2.5. *Suppose $1 \le p < \infty$, $h \in H(U), h \not\equiv 0$, and $F(f) = fh$ for every $f \in H(U)$. Then $H_{F,p}$ is a Banach space.*

Proof. If $H^p \subseteq R_F$, then by Theorem 2.2, the proposition holds. Otherwise, let $\{f_n\}$ be a Cauchy sequence in $H_{F,p}$. Seting $F(f_n) = g_n$, so $\{g_n\}$ is a Cauchy sequence in H^p. Therefore, there is a $g \in H^p$ such that $\|g_n - g\|_p \to 0$ as $n \to \infty$. If $g \in R_F$, then the proof is similar to the proof of Theorem 2.2.

Now suppose that g is not in R_F. Then there are $z_0 \in U$, $m_1 \ge 0$, and $m_2 > m_1$ such that

$$\begin{aligned} g(z) &= (z - z_0)^{m_1} g_0(z), \\ h(z) &= (z - z_0)^{m_2} h_0(z), \end{aligned} \tag{2.7}$$

where $h_0, g_0 \in H(U)$, $g_0(z_0) \ne 0$, and $h_0(z_0) \ne 0$. Therefore, we have

$$\begin{aligned} \|g_n - g\|_p &= \|h f_n - g\|_p \\ &= \int_0^{2\pi} \left| \left(e^{i\theta} - z_0\right)^{m_2} h_0\left(e^{i\theta}\right) f_n\left(e^{i\theta}\right) - \left(e^{i\theta} - z_0\right)^{m_1} g_0\left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \left| \left(e^{i\theta} - z_0\right) \right|^{m_1} \left| \left(e^{i\theta} - z_0\right)^{m_2 - m_1} h_0\left(e^{i\theta}\right) f_n\left(e^{i\theta}\right) - g_0\left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi} \\ &\ge (1 - |z_0|)^{m_1} \int_0^{2\pi} \left| \left(e^{i\theta} - z_0\right)^{m_2 - m_1} h_0\left(e^{i\theta}\right) f_n\left(e^{i\theta}\right) - g_0\left(e^{i\theta}\right) \right|^p \frac{d\theta}{2\pi}. \end{aligned} \tag{2.8}$$

Since $\|g_n - g\|_p \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\int_0^{2\pi}\left|\left(e^{i\theta}-z_0\right)^{m_2-m_1}h_0\left(e^{i\theta}\right)f_n\left(e^{i\theta}\right)-g_0\left(e^{i\theta}\right)\right|^p\frac{d\theta}{2\pi}=0. \tag{2.9}$$

Hence $\|(z-z_0)^{m_2-m_1}h_0f_n - g_0\|_p \to 0$ as $n \to \infty$. Since the point evaluation at z_0 is a bounded linear functional on H^p, we have

$$(z_0-z_0)^{m_2-m_1}h_0(z_0)f_n(z_0)-g_0(z_0)\longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty. \tag{2.10}$$

So $g_0(z_0) = 0$, which is a contradiction. □

The set of all analytic polynomials is dense in H^p, but this is not the case for each space $H_{F,p}$. Also it is possible that $P \not\subseteq H_{F,p}$. For example, let $p = 2$, $g(z) = 1/(1-z)$, and $F(f) = fg$. Then 1 is not in $H_{F,2}$, for if $1 \in H_{F,2}$, then $F(1) = g \in H^2$, which is a contradiction.

In the following proposition, we will find a dense subset in $H_{F,p}$, whenever $P \subseteq R_F$.

Proposition 2.6. *Suppose* $1 \le p < \infty$ *and* $P \subseteq R_F$. *Then* $\overline{\{F^{-1}(p) : p \in P\}} = H_{F,p}$.

Proof. It is clear that $\{F^{-1}(p) : p \in P\} \subseteq H_{F,p}$. Suppose that $f \in H_{F,p}$. Then there is a sequence $\{h_n\}$ in P such that $\|h_n - F(f)\|_p \to 0$ as $n \to \infty$. Setting $f_n = F^{-1}(h_n)$, we have

$$\left\|f_n-f\right\|_{H_{F,p}}=\left\|h_n-F(f)\right\|_p, \tag{2.11}$$

so the result follows. □

Corollary 2.7. *Suppose* $1 \le p < \infty$, $P \subseteq R_F$, *and* $F^{-1}(p) \in P$ *for each* $p \in P$. *Then* $\overline{P \cap H_{F,p}} = H_{F,p}$.

3. Point Evaluations

In this section, we investigate the continuity of the point evaluations on $H_{F,p}$. The idea behind Theorem 3.1 is similar to the one found in [29, page 51].

Theorem 3.1. *Suppose that* $w \in U$ *and* $H^2 \subseteq R_F$. *Then we have the following.*

(a) *If* $p \ge 2$ *and* $\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(w)}z^j \in H^2$, *then* e_w *is continuous on* $H_{F,p}$.

(b) *Let* $1 \le p < 2$ *and* $\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(w)}z^j \in H^\infty$. *If for each* $0 < r < 1$ *and* $f \in H_{F,1}$, $(F(f))_r = F(f_r)$, *then* e_w *is continuous on* $H_{F,p}$.

Proof. (a) By Proposition 2.4, $H_{F,2}$ is a Hilbert space. Since

$$\begin{gathered}\left\langle F^{-1}\left(z^j\right),F^{-1}\left(z^k\right)\right\rangle_{H_{F,2}}=\left\langle z^j,z^k\right\rangle_{H^2}=0,\\ \left\|F^{-1}(z^j)\right\|_{H_{F,2}}=\left\|z^j\right\|_{H^2}=1\end{gathered} \tag{3.1}$$

for each $j,k\in\mathbb{N}\cup\{0\}$, where $j\neq k$, $\{F^{-1}(z^j):j\in\mathbb{N}\cup\{0\}\}$ is an orthonormal set in $H_{F,2}$. Also if $g\in H_{F,2}$ and for each $j\in\mathbb{N}\cup\{0\}$, $\langle g,F^{-1}(z^j)\rangle_{H_{F,2}}=0$, then for each $j\in\mathbb{N}\cup\{0\}$, $\langle F(g),z^j\rangle_{H^2}=0$. Since $\{z^j:j\in\mathbb{N}\cup\{0\}\}$ is a basis for H^2, $g\equiv 0$. Therefore, $\{F^{-1}(z^j):j\in\mathbb{N}\cup\{0\}\}$ is a basis for $H_{F,2}$. Since $\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(w)}z^j\in H^2$, there is an $h\in H_{F,2}$ such that $F(h)=\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(w)}z^j$. For each $j\in\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}\left\langle F^{-1}\left(z^j\right),h\right\rangle_{H_{F,2}}&=\left\langle z^j,F(h)\right\rangle_{H^2}\\&=\left\langle z^j,\sum_{k=0}^{\infty}\overline{F^{-1}\left(z^k\right)(w)}z^k\right\rangle_{H^2}\\&=F^{-1}\left(z^j\right)(w).\end{aligned}\tag{3.2}$$

Hence $h=K_w\in H_{F,2}$ and e_w is continuous on $H_{F,2}$.

Let $p\geq 2$. If $f\in H_{F,p}$, then

$$|f(w)|\leq\|K_w\|_{H_{F,2}}\|f\|_{H_{F,2}}\leq\|K_w\|_{H_{F,2}}\|f\|_{H_{F,p}},\tag{3.3}$$

so e_w is continuous on $H_{F,p}$.

(b) Let $f\in H_{F,1}$. Then for each $0<r<1$, $f_r\in H_{F,2}$ and so

$$\begin{aligned}f_r(w)&=\langle f_r,K_w\rangle_{H_{F,2}}\\&=\langle F(f_r),F(K_w)\rangle_{H^2}\\&=\int_0^{2\pi}F(f_r)\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi}.\end{aligned}\tag{3.4}$$

Also by [1, Theorem 2.6], $\|(F(f))_r-F(f)\|_1\to 0$ as $r\to 1$. Therefore, by Holder's inequality and the fact that $F(K_w)=\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(w)}z^j$, we have

$$\begin{aligned}&\left|\int_0^{2\pi}(F(f))_r\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi}-\int_0^{2\pi}F(f)\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi}\right|\\&\quad\leq\|F(K_w)\|_\infty\int_0^{2\pi}\left|\left(F(f_r)-F(f)\right)\left(e^{i\theta}\right)\right|\frac{d\theta}{2\pi}\\&\quad\leq\|F(K_w)\|_\infty\left\|(F(f))_r-F(f)\right\|_1\longrightarrow 0,\end{aligned}\tag{3.5}$$

as $r \to 1$. So we obtain

$$
\begin{aligned}
f(w) &= \lim_{r\to 1} f_r(w) \\
&= \lim_{r\to 1}\int_0^{2\pi} F(f_r)\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} F(f)\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi}.
\end{aligned}
\tag{3.6}
$$

Hence

$$
\begin{aligned}
|f(w)| &= \left|\int_0^{2\pi} F(f)\left(e^{i\theta}\right)\overline{F(K_w)}\left(e^{i\theta}\right)\frac{d\theta}{2\pi}\right| \\
&\le \|F(K_w)\|_\infty \int_0^{2\pi}\left|F(f)\left(e^{i\theta}\right)\right|\frac{d\theta}{2\pi} \\
&= \|F(K_w)\|_\infty \|f\|_{H_{F,1}}
\end{aligned}
\tag{3.7}
$$

for each $f \in H_{F,1}$. Now let $1 \le p < 2$. If $f \in H_{F,p}$, then

$$
|f(w)| \le \|F(K_w)\|_\infty \|f\|_{H_{F,1}} \le \|F(K_w)\|_\infty \|f\|_{H_{F,p}},
\tag{3.8}
$$

so the result follows. □

Suppose that w and F satisfy the hypotheses in the first line of Theorem 3.1. It is easy to see that if e_w is continuous on $H_{F,2}$, then according to the proof of the previous theorem, $F(K_w) = \sum_{j=0}^{\infty} \overline{F^{-1}(z^j)(w)} z^j$.

Now we give two examples for the preceding theorem.

Example 3.2. (a) Let $w \in U$ and $F(\sum_{j=0}^{\infty} a_j z^j) = a_0 + \sum_{j=1}^{\infty} j a_j z^j$, where $\sum_{j=0}^{\infty} a_j z^j \in H(U)$. It is easy to see that $\sum_{j=0}^{\infty} \overline{F^{-1}(z^j)(w)} z^j = 1 + \sum_{j=1}^{\infty}((\overline{w})^j/j)z^j \in H^\infty$ and for each $f \in H_{F,1}$, $F(f_r) = (F(f))_r$. Therefore, by Theorem 3.1, e_w is continuous on $H_{F,p}$ for each $1 \le p < \infty$.

(b) Let $p \ge 2$ and $F(\sum_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty} b_j z^j$ such that $(b_0, b_1, \ldots, b_n)$ is a permutation of $(a_0, a_1, \ldots, a_n)$ for some $n \in \mathbb{N}$ and $b_k = a_k$ for $k > n$. Then by Theorem 3.1, for each point w in the open unit disk, e_w is continuous on $H_{F,p}$. Also if $p = 2$ and S is the above permutation, then

$$
F(K_w) = \sum_{j=0}^{\infty} \overline{F^{-1}(z^j)(w)} z^j = \sum_{j=0}^{\infty} (\overline{w})^{S^{-1}(j)} z^j.
\tag{3.9}
$$

Therefore, $K_w(z) = \sum_{j=0}^{\infty} (\overline{w})^{S^{-1}(j)} z^{S^{-1}(j)}$.

There is a Banach space $H_{F,p}$ such that it does not satisfy the conditions of Theorem 3.1, but for each $w \in U$, e_w is continuous on $H_{F,p}$. For example, let $1 \le p < \infty$, $g(z) = 1/2 - z$, and $F(f) = fg$ for each $f \in H(U)$. Then 1 is not in R_F and $F((1)_r) \not\equiv (F(1))_r$. By the following theorem, we see that although the hypotheses of Theorem 3.1 do not hold, e_w is continuous on $H_{F,p}$ for each $w \in U$.

Theorem 3.3. *Let $1 \le p < \infty$, $w \in U$, $h \in H(U)$, $h \not\equiv 0$, and for each $f \in H(U)$, $F(f) = fh$. Then e_w is continuous on $H_{F,p}$.*

Proof. We break the proof into two parts.

(1) Let $h(w) \neq 0$. If $|w| < r < 1$ and Γ_r is the circle of radius r with center at the origin, then the Cauchy formula shows that for any f in $H_{F,p}$,

$$\begin{aligned} f(w)h(w) &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(\zeta)h(\zeta)}{\zeta - w} d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})h(re^{i\theta})}{re^{i\theta} - w} rie^{i\theta} d\theta \\ &= \int_0^{2\pi} f\left(re^{i\theta}\right)h\left(re^{i\theta}\right)\frac{r}{r - we^{-i\theta}}\frac{d\theta}{2\pi}. \end{aligned} \tag{3.10}$$

It follows that

$$\left|f(w)\right||h(w)| \le \|(fh)_r\|_p \left\|\frac{r}{r - we^{-i\theta}}\right\|_q, \tag{3.11}$$

where $1/p + 1/q = 1$. Now if r tends to 1, $|r/(r - we^{-i\theta})|$ converges uniformly to the bounded function $|1 - we^{i\theta}|^{-1}$ and $\|(fh)_r\|_p \le \|fh\|_p$. Hence there is an $M < \infty$ such that

$$\left|f(w)\right| \le \frac{M}{|h(w)|}\|f\|_{H_{F,p}}, \tag{3.12}$$

and the result follows.

(2) Let $h(w) = 0$. Then $h(z) = (z - w)^m h_0(z)$, where $m \in \mathbb{N}$, $h_0 \in H(U)$, and $h_0(w) \neq 0$. Let $F_1(f) = fh_0$ for each $f \in H(U)$, it is easy to see that $H_{F,p} \subseteq H_{F_1,p}$. Then by the preceding part, there is a constant $0 < C < \infty$ such that

$$\begin{aligned} \left|f(w)\right|^p \le C\|fh_0\|_p^p &= C\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^p \left|h_0\left(e^{i\theta}\right)\right|^p \frac{\left|e^{i\theta} - w\right|^{mp}}{\left|e^{i\theta} - w\right|^{mp}}\frac{d\theta}{2\pi} \\ &\le \frac{C}{(1 - |w|)^{mp}}\int_0^{2\pi} \left|f\left(e^{i\theta}\right)h\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} \\ &= \frac{C}{(1 - |w|)^{mp}}\|f\|_{H_{F,p}}^p \end{aligned} \tag{3.13}$$

for each $f \in H_{F,p}$. So e_w is continuous on $H_{F,p}$. □

There is an example of $H_{F,p}$ such that $H_{F,2}$ is not a Hilbert space and e_w is continuous for some $w \in U$.

Example 3.4. Let $p = 2$ and $F(\sum_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty} (a_j/2^j) z^j$, where $\sum_{j=0}^{\infty} a_j z^j \in H(U)$. We can show that for each $w \in (1/2)U$, e_w is continuous on $H_{F,2}$. Suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j \in H_{F,2}$, we have

$$\begin{aligned} |f(w)| &= \left| \sum_{j=0}^{\infty} a_j w^j \right| = \left| \sum_{j=0}^{\infty} \frac{a_j}{2^j} (2w)^j \right| \\ &\le \left(\sum_{j=0}^{\infty} \left| \frac{a_j}{2^j} \right|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \left| (2w)^j \right|^2 \right)^{1/2} \\ &= \|f\|_{H_{F,2}} \left(\sum_{j=0}^{\infty} |2w|^{2j} \right)^{1/2}, \end{aligned} \tag{3.14}$$

as desired. Also it is easy to see that e_w is continuous on $H_{F,p}$ for each $p > 2$ and $w \in (1/2)U$.

4. Continuity of the Composition Operators on $H_{F,p}$

In the most important classical spaces, all analytic maps of the unit disk into itself induce bounded composition operators, but there are analytic self-maps of the unit disk which do not give bounded composition operators on some generalized Hardy spaces.

Example 4.1. (a) Suppose that $\varphi(z) = z^2$ and for each $f \in H(U)$, $F(f) = fg$, where $g(z) = 1 - z$. Let $f_j(z) = 1 + z + z^2 + \cdots + z^j$ for each $j \in \mathbb{N} \cup \{0\}$. Then $f_j \in H_{F,2}$. We see that

$$\begin{gathered} \|C_{z^2}(f_j)\|_{H_{F,2}}^2 = \left\| f_j(z^2) g(z) \right\|_2^2 = \left\| (1 + z^2 + \cdots + z^{2j})(1 - z) \right\|_2^2 = 2j + 2, \\ \|f_j\|_{H_{F,2}}^2 = \|f_j g\|_2^2 = \left\| 1 - z^{j+1} \right\|_2^2 = 2. \end{gathered} \tag{4.1}$$

So $\|C_{z^2}\|^2 \ge j + 1$ and C_{z^2} is not bounded on $H_{F,2}$.

(b) Let $F(\sum_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty} a_j \beta(j) z^j$, where $\sum_{j=0}^{\infty} a_j z^j \in H(U)$ and $\{\beta(j)\}$ is given in [2, Example 3.4]. If $\varphi(z) = z^2$, then in [2, Example 3.4] shows that C_φ is not bounded on $H_{F,2}$.

In this section, we investigate the continuity of the composition operator on some spaces $H_{F,p}$ in terms of a Carleson measure criterion. This criterion has been used to characterize the boundedness and the compactness of the composition operators in different papers (see, e.g., [16, 30]).

Definition 4.2. A positive measure μ on U is called a Carleson measure (in U) if there is a constant $K < \infty$ such that $\mu(S(b,h)) < Kh$ for all $b \in \partial U$ and $0 < h < 1$, where $S(b,h) = \{z \in U : |z - b| < h\}$.

Proposition 4.3. *Let $1 \le p < \infty$ and $P \subseteq R_F$. If μ is a finite, positive Borel measure on U, then the following conditions are equivalent.*

(a) *μ is a Carleson measure in U.*

(b) *There is a constant $C < \infty$ such that*

$$\int_U |F(f)|^p d\mu \le C\|f\|_{H_{F,p}}^p \tag{4.2}$$

for all f in $H_{F,p}$.

Proof. By [2, Theorem 2.33], (a)⇒(b) is clear. Now we prove that (b) implies (a). For each $p_0 \in P$, there is a $q_{p_0} \in H_{F,p}$ such that $F(q_{p_0}) = p_0$ and

$$\int_U |p_0|^p d\mu \le C\|p_0\|_p^p. \tag{4.3}$$

If $g \in H^p$, then there is a sequence $\{p_n\}$ in P such that $\|p_n - g\|_p \to 0$ as $n \to \infty$. So

$$\|p_n\|_p^p \longrightarrow \|g\|_p^p, \quad \text{as } n \longrightarrow \infty. \tag{4.4}$$

By (4.3), we obtain

$$\int_U |p_n|^p d\mu \le C\|p_n\|_p^p \tag{4.5}$$

for each $n \in \mathbb{N}$. The right-hand side of (4.5) is bounded, hence there is a subsequence $\{p_{n_k}\}$ such that $\int_U |p_{n_k}|^p d\mu$ converges. Since point evaluations at the points of U are all continuous on H^p, the Principle of Uniform Boundedness implies that $\{p_{n_k}\}$ converges to g uniformly on compact subsets of U. In particular, for each $0 < r < 1$, the convergence is uniform on rU. Therefore, for each $0 < r < 1$, $\int_{rU} |p_{n_k}|^p d\mu \to \int_{rU} |g|^p d\mu$ as $k \to \infty$. Since $\{\int_U |p_{n_k}|^p d\mu\}$ is a bounded sequence, there is an $M > 0$ such that $\int_{rU} |g|^p d\mu \le M$ for each $0 < r < 1$. Hence $\int_U |g|^p d\mu \le M$ and $g \in L^p(\mu)$. Now let $\{r_m\}$ be an increasing sequence which converges to 1 and let $\epsilon > 0$. By [31, page 32], there is an integer m_0 such that $\int_A |g|^p d\mu < \epsilon$, where $A = U - r_{m_0} U$. Let $B = r_{m_0} U$. Then by [31, page 73], we have $\limsup_{k \to \infty} \int_A |p_{n_k}|^p d\mu \le \epsilon$. So there is a K_0 such that for each $k > K_0$, $\int_A |p_{n_k}|^p d\mu \le \epsilon$. Again apply [31, page 32] for $|p_{n_1}|^p, |p_{n_2}|^p, \ldots, |p_{n_{K_0}}|^p$. Therefore, there are $m_1, m_2, \ldots, m_{K_0}$ such that

$$\int_{U - r_{m_k} U} |p_{n_k}|^p d\mu \le \epsilon \tag{4.6}$$

for each $1 \leq k \leq K_0$. Let $m' = \max\{m_0, m_1, \ldots, m_{K_0}\}$. Therefore, for each $m > m'$ and $k \in \mathbb{N}$, $\int_{U - r_m U} |p_{n_k}|^p d\mu \leq \epsilon$. So by [32, Theorem 19.10], we have

$$\begin{aligned}
\lim_{k \to \infty} \int_U |p_{n_k}|^p d\mu &= \lim_{k \to \infty} \lim_{m \to \infty} \int_{r_m U} |p_{n_k}|^p d\mu \\
&= \lim_{m \to \infty} \lim_{k \to \infty} \int_{r_m U} |p_{n_k}|^p d\mu \\
&= \lim_{m \to \infty} \int_{r_m U} |g|^p d\mu \\
&= \int_U |g|^p d\mu.
\end{aligned} \tag{4.7}$$

So by the preceding relation and (4.4) and (4.5), we have

$$\int_U |g|^p d\mu \leq C \|g\|_p^p \tag{4.8}$$

for every $g \in H^p$. Thus the result follows from [2, Theorem 2.33]. □

Definition 4.4. For b on the unit circle and $0 < h < 1$, let

$$\mathcal{S}(b, h) = \left\{ z \in \overline{U} : |z - b| < h \right\}. \tag{4.9}$$

From now on, unless otherwise stated, we assume that $H_{F,p}$ is a Banach space.

Proposition 4.5. *Let $1 \leq p < \infty$ and $P \subseteq R_F$. If μ is a finite, positive Borel measure on $\overline{U}$, then the following conditions are equivalent.*

(a) *There is a constant $K < \infty$ such that $\mu(\mathcal{S}(b, h)) < Kh$ for all $b \in \partial U$ and $0 < h < 1$.*

(b) *There is a constant $C < \infty$ such that*

$$\int_{\overline{U}} |F(f)|^p d\mu \leq C \|f\|_{H_{F,p}}^p \tag{4.10}$$

for all $f \in H_{F,p}$.

Proof. The implication (a)⇒(b) follows by Proposition 4.3 and is exactly the same as the proof of [2, Theorem 2.35(1)⇒(2)]. For the other direction, by Theorem 2.2, $H^p \subseteq R_F$. Hence for each $g \in H^p$, we have

$$\int_{\overline{U}} |g|^p d\mu \leq C \|g\|_p^p. \tag{4.11}$$

Then the result follows from [2, Theorem 2.35(2)⇒(1)]. □

In the following theorem, we use the techniques used in [2, Theorem 3.12 part (1)].

Theorem 4.6. *Suppose $1 \leq p < \infty$, φ and φ_0 are analytic self-maps of U. Let $P \subseteq R_F$ and for each $f \in H_{F,p}$, $F(f \circ \varphi) = F(f) \circ \varphi_0$. Then C_φ is a bounded operator on $H_{F,p}$ if and only if $\mu(\mathcal{S}(\zeta, h)) = \bigcirc(h)$ for all ζ in ∂U and $0 < h < 1$, where $\mu(E) = \sigma((\varphi_0^*)^{-1}(E))$, σ is normalized Lebesgue measure on the unit circle, and E is a subset of the closed disk $\overline{U}$.*

Proof. Let $f \in H_{F,p}$. By [2, Theorem 2.25] and [33, page 163], we obtain

$$\begin{aligned} \int_{\partial U} \left|(F(f \circ \varphi))^*\right|^p d\sigma &= \int_{\partial U} \left|(F(f) \circ \varphi_0)^*\right|^p d\sigma \\ &= \int_{\partial U} \left|(F(f))^* \circ \varphi_0^*\right|^p d\sigma \\ &= \int_{\overline{U}} \left|(F(f))^*\right|^p d\mu. \end{aligned} \tag{4.12}$$

So the result follows from Proposition 4.5. □

Remark 4.7. If p, φ, φ_0 and F satisfy the hypotheses in Theorem 4.6, then by [2, Theorem 3.12, part (1)], C_φ is a bounded operator on $H_{F,p}$ if and only if C_{φ_0} is a bounded operator on H^p. Since we know that the composition operators are always bounded on each of the Hardy spaces H^p, C_φ is a bounded operator on $H_{F,p}$.

5. Compactness of the Composition Operators on $H_{F,p}$

In this section, we investigate the compactness of the composition operators on $H_{F,p}$.

The idea of the proof of the following theorem is similar to the proof of Proposition 3.11 in [2]. A detailed proof of another similar result can be found, for example, in [18].

Theorem 5.1. *Suppose $1 \leq p < \infty$ and for each $w \in U$, e_w is continuous. Also if $f_n \to f$ as $n \to \infty$ uniformly on compact subsets of U, then $F(f_n) \to F(f)$ in $H(U)$ as $n \to \infty$. Then the following conditions are equivalent.*

(a) *C_φ is compact on $H_{F,p}$.*

(b) *If $\{f_n\}$ is a bounded sequence in $H_{F,p}$ and $f_n \to 0$ as $n \to \infty$ uniformly on compact subsets of U, then $\|f_n \circ \varphi\|_{H_{F,p}} \to 0$ as $n \to \infty$.*

Proof. The implication (a)⇒(b) follows exactly as the proof of [2, Theorem 3.11].

Now we show that (b)⇒(a). Let $\{f_n\}$ be a bounded sequence in $H_{F,p}$. Since e_w is continuous for each $w \in U$, $\{f_n\}$ is a normal family. So there is a function $f \in H(U)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ as $k \to \infty$ uniformly on compact subsets of U. Hence $F(f_{n_k}) \to F(f)$ in $H(U)$ as $k \to \infty$. It is easy to see that $f \in H_{F,p}$. Therefore, $\{f_{n_k} - f\}$ is a bounded sequence in $H_{F,p}$ such that it converges uniformly to 0 on compact subsets of U. By the hypotheses, we conclude that $f_{n_k} \circ \varphi \to f \circ \varphi$ in $H_{F,p}$ as $k \to \infty$. Thus C_φ is a compact operator. □

Corollary 5.2. *Let $1 \leq p < \infty$, $g \in H^p$, $g \not\equiv 0$, and $F(f) = fg$ for each $f \in H(U)$. Let φ be an analytic self-map of U and $\overline{\varphi(U)} \subseteq U$. Then C_φ is a compact operator on $H_{F,p}$.*

In the rest of this section, we investigate the relation between compactness of C_φ on $H_{F,p}$ and H^p.

Theorem 5.3. *Let p, φ_0, φ, F, and μ satisfy the hypotheses in Theorem 4.6. Then C_φ is compact on $H_{F,p}$ if and only if $\mu(\mathcal{S}(\zeta, h)) = \circ(h)$ as $h \to 0$ uniformly in ζ in ∂U.*

Proof. Let $\mu(\mathcal{S}(\zeta, h)) = \circ(h)$ as $h \to 0$ uniformly in ζ in ∂U and $\{f_n\}$ be a bounded sequence in $H_{F,p}$. Then by [2, Theorem 3.12, part (2)], there is a subsequence $\{F(f_{n_k})\}$ such that $\{C_{\varphi_0}(F(f_{n_k}))\}$ converges in H^p. Since $F(f_{n_k} \circ \varphi) = F(f_{n_k}) \circ \varphi_0$, $\{f_{n_k} \circ \varphi\}$ converges in $H_{F,p}$. Therefore, C_φ is a compact operator on $H_{F,p}$.

Conversely, let $\{f_n\}$ be a bounded sequence in H^p. By Theorem 2.2, $H^p \subseteq R_F$, so there is a sequence $\{g_n\}$ in $H_{F,p}$ such that $F(g_n) = f_n$. Since C_φ is compact on $H_{F,p}$, we may extract a subsequence $\{g_{n_k}\}$ such that $\{C_\varphi(g_{n_k})\}$ converges in $H_{F,p}$. So C_{φ_0} is a compact operator on H^p and [2, Theorem 3.12, part (2)] implies the result. □

Remark 5.4. If p, φ, φ_0, and F satisfy the hypotheses in Theorem 5.3, then by [2, Theorem 3.12, part (2)], C_φ is a compact operator on $H_{F,p}$ if and only if C_{φ_0} is a compact operator on H^p.

Now we present an example for Remarks 4.7 and 5.4.

Example 5.5. Let $p = 2$, $\varphi_0(z) = \varphi(z) = z^2$, and $F(\Sigma_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty} a_j c_j z^j$, where $\Sigma_{j=0}^{\infty} a_j z^j \in H(U)$ and for every odd positive integer j and every $n \in \mathbb{N}$, $c_{2^n j} = c_j = 1/j$ and $c_0 = 1$. It is easy to see that by Remark 4.7, C_φ is a bounded operator on $H_{F,2}$ and by Remark 5.4 and [2, Corollary 3.14] C_φ is not a compact operator on $H_{F,2}$.

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