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Research Article

Solution Properties of Linear Descriptor (Singular) Matrix Differential Systems of Higher Order with (Non-) Consistent Initial Conditions

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In some interesting applications in control and system theory, linear descriptor (singular) matrix differential equations of higher order with time-invariant coefficients and (non-) consistent initial conditions have been used. In this paper, we provide a study for the solution properties of a more general class of the Apostol-Kolodner-type equations with consistent and nonconsistent initial conditions.

1. Introduction

Linear Time-Invariant (LTI) (i.e., with constant matrix coefficients) descriptor matrix differential systems of type (1.1) with several kinds of inputs

$$FX^{(r)}(t) = AX(t) + BU(t), \tag{1.1}$$

where $F, A \in \mathcal{M}(n \times m; \mathbb{F})$, $B \in \mathcal{M}(n \times \mu; \mathbb{F})$, and $U \in C^{\infty}(\mathbb{F}, \mathcal{M}(\mu \times m; \mathbb{F}))$, often appear in control and system theory. For instance, (1.1) identifies and models effectively many physical, engineering, mechanical, as well as financial phenomena. For instance, we can provide in economy, the well-known, famous input-output *Leontief* model and its several important extensions, advice [1, 2]. Moreover, in the beginning of this introductive section, we should point out that singular perturbations arise often in systems whose dynamics have sufficiently separate slow and fast parts. Now by considering the classical proportional feedback controller

$$U(t) = -\widetilde{F}X(t), \tag{1.2}$$

we can obtain (1.3), where $G = A - B\widetilde{F}$.

Our long-term purpose is to study the solution of LTI descriptor matrix differential systems of higher order (1.1) into the mainstream of matrix pencil theory, that is,

$$FX^{(r)}(t) = GX(t), \tag{1.3}$$

where, for (1.1), (1.2), and (1.3), rth is the order of the systems, $F, G \in \mathcal{M}(n \times m; \mathbb{F})$ (where matrix F is singular), and $X \in C^{\infty}(\mathbb{F}, \mathcal{M}(m \times l; \mathbb{F})$ (note that $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$). For the sake of simplicity we set in the sequel $\mathcal{M}_n = \mathcal{M}(n \times n; \mathbb{F})$ and $\mathcal{M}_{n,m} = \mathcal{M}(n \times m; \mathbb{F})$.

Matrix pencil theory has been extensively used for the study of LTI descriptor differential equations of first order; see, for instance, [3–6]. Systems of type (1.3) are more general, including the special case when $F = I_n$, where I_n is the identity matrix of $\mathcal{M}_n$, since the well-known class of *higher order linear matrix differential of Apostol-type* equations is derived straightforwardly; see [7–10]. In the same way, system (1.1) might be considered as the more general class of *higher order linear descriptor matrix differential equations of Apostol-Kolodner type*, since Kolodner has also studied such systems in nondescriptor form; see also [8].

Recently, in [5], the *regular* case of higher order linear descriptor matrix differential equations of Apostol-Kolodner type has been investigated. The regular case is simpler, since it considers square matrix coefficients and the Weierstrass canonical form has been applied. Actually, the recent work is a nonstraight generalization of [5]. Analytically, in this article, we study the linear descriptor matrix differential equations of higher order whose coefficients are rectangular constant matrices, that is, the singular case is examined. Adopting several different methods for computing the matrix powers and exponential, new formulas representing auxiliary results are obtained. This allows us to prove properties of a large class of linear matrix differential equations of higher order; in particular results of Apostol and Kolodner are recovered; see also [5, 8].

Finally, it should be mentioned that in the classical theory of linear (descriptor) differential systems, see, for instance, [1, 2, 11–13], one of the important features is that not every initial condition X_0 admits a functional solution. Thus, we shall call X_0 *a consistent initial condition* for (1.3) at t_o if there is a solution for (1.3), which is defined on some interval $[t_o, t_o + \gamma]$, $\gamma > 0$ such that $X(t_o) = X_0$.

On the other hand, it is not rare to appear in some practical significant applications that the assumption of the *initial conditions* for (1.3) can be *nonconsistent*, that is, $X(t_o) \neq X_0$.

2. Mathematical Background and Notations

In this preliminary section, some well-known concepts and definitions for matrix pencils are introduced. This discussion is highly important, in order to understand better the results of Section 3.

Definition 2.1. Given $F, G \in \mathcal{M}_{nm}$ and an indeterminate $s \in \mathbb{F}$, the matrix pencil $sF - G$ is called regular when $m = n$ and $\det(sF - G) \neq 0$ (where 0 is the zero element of $\mathcal{M}(1, \mathbb{F})$). In any other case, the pencil is called singular.

In this paper, as we are going to see in the next paragraph, we consider the case that the pencil is *singular*. The next definition is very important, since the notion of strict equivalence between two pencils is presented.

Definition 2.2. The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}_n$ and $Q \in \mathcal{M}_m$ such that

$$P(sF - G)Q = s\tilde{F} - \tilde{G}. \tag{2.1}$$

The characterization of singular pencils requires the definition of additional sets of invariants known as the minimal indices.

Let us assume that $r = \operatorname{rank}_{\mathbb{F}(s)}(sF - G)$, where $\mathbb{F}(s)$ denotes the field of rational functions in s having coefficients in the field $\mathbb{F}$. The equations

$$(sF - G)\underline{x}(s) = \underline{0}, \qquad \underline{\psi}^T(s)(sF - G) = \underline{0}^T \tag{2.2}$$

have nonzero solutions $\underline{x}(s)$ and $\underline{\psi}(s)$ which are vectors in the rational vector spaces

$$\mathcal{N}_{\text{right}}(s) \triangleq \mathcal{N}_{\text{right}}(sF - G), \qquad \mathcal{N}_{\text{left}}(s) \triangleq \mathcal{N}_{\text{left}}(sF - G), \tag{2.3}$$

respectively, where

$$\begin{aligned} \mathcal{N}_{\text{right}}(s) &\triangleq \left\{\underline{x}(s) \in \mathbb{F}(s)^m : (sF - G)\underline{x}(s) = \underline{0}_n\right\}, \\ \mathcal{N}_{\text{left}}(s) &= \left\{\underline{\psi}(s) \in \mathbb{F}(s)^n : \underline{\psi}^T(s)(sF - G) = \underline{0}_m^T\right\}. \end{aligned} \tag{2.4}$$

The sets of the minimal degrees $\{v_i,\ 1 \leq i \leq m-r\}$ and $\{u_j,\ 1 \leq j \leq n-r\}$ are known as *column minimal indices* (c.m.i.) and *row minimal indices* (r.m.i.) of $sF - G$, respectively. Furthermore, if $r = \operatorname{rank}_{\mathbb{F}(s)}(sF - G)$, it is evident such that

$$r = \sum_{i=g+1}^{m-r} v_i + \sum_{j=h+1}^{n-r} u_j + \operatorname*{rank}_{\mathbb{F}(s)}(sF_w - G_w), \tag{2.5}$$

where $sF_w - G_w$ is the *complex Weierstrass canonical form*; see [3].

Let $B_1, B_2, \ldots, B_n$ be elements of $\mathcal{M}_n$.

The direct sum of them denoted by $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is the block $\operatorname{diag}\{B_1, B_2, \ldots, B_n\}$.

Thus, there exists $P \in \mathcal{M}_m$ and $Q \in \mathcal{M}_n$ such that the *complex Kronecker form* $sF_k - G_k$ of the singular pencil $sF - G$ is defined as follows:

$$sF_k - G_k \triangleq \mathbb{O}_{h,g} \oplus s\Lambda_v - \lambda_v \oplus s\Lambda_u^T - \lambda_u^T \oplus sI_p - J_p \oplus sH_q - I_q, \tag{2.6}$$

where $v = \sum_{i=g+1}^{m-r} v_i$, $u = \sum_{j=h+1}^{n-r} u_j$, $p = \sum_{j=1}^{\kappa} p_j$, and $q = \sum_{j=1}^{\sigma} q_j$ (see below). In more details, the following are given.

(S1) Matrix $\mathbb{O}_{h,g}$ is uniquely defined by the sets $\underbrace{\{0,0,\ldots,0\}}_{g}$ and $\underbrace{\{0,0,\ldots,0\}}_{h}$ of zero column and row minimal indices, respectively.

(S2) The second normal block $s\Lambda_v - \lambda_v$ is uniquely defined by the *set of nonzero column minimal indices* (a new arrangement of the indices of v must be noted in order to simplify the notation) $\{v_{g+1} \leq \cdots \leq v_{m-r}\}$ of $sF - Q$ and has the form

$$s\Lambda_v - \lambda_v \triangleq s\Lambda_{v_{g+1}} - \lambda_{v_{g+1}} \oplus \cdots \oplus s\Lambda_{v_i} - \lambda_{v_i} \oplus \cdots \oplus s\Lambda_{v_{m-r}} - \lambda_{v_{m-r}}, \tag{2.7}$$

where $\Lambda_{v_i} = [I_{v_i} \vdots \underline{0}] \in \mathcal{M}_{v_i, v_i+1}$, $\lambda_{v_i} = [H_{v_i} \vdots \underline{\varepsilon}_{v_i}] \in \mathcal{M}_{v_i, v_i+1}$ for every $i = g+1, g+2, \ldots, m-r$, and I_{v_i} and H_{v_i} denote the $v_i \times v_i$ identity and the nilpotent (with index of nilpotency v_i) matrix, respectively. $\underline{0}$ and $\underline{\varepsilon}_{v_i} = [0 \quad \cdots \quad 0 \quad 1]^T$ are the zero column and the column with element 1 at the $v_i$place, respectively.

(S3) The third normal block $s\Lambda_u^T - \lambda_u^T$ is uniquely determined by the *set of nonzero row minimal indices* (a new arrangement of the indices of u must be noted in order to simplify the notation) $\{u_{h+1} \leq \cdots \leq u_{n-r}\}$ of $sF - G$ and has the form

$$s\Lambda_u^T - \lambda_u^T \triangleq s\Lambda_{u_{h+1}}^T - \lambda_{u_{h+1}}^T \oplus \cdots \oplus s\Lambda_{u_j}^T - \lambda_{u_j}^T \oplus \cdots \oplus s\Lambda_{u_{n-r}}^T - \lambda_{u_{n-r}}^T, \tag{2.8}$$

where $\Lambda_{u_j}^T = \begin{bmatrix} \underline{e}_{u_j}^T \\ \cdots \\ H_{u_j} \end{bmatrix} \in \mathcal{M}_{u_j+1, u_j}$, $\lambda_{u_j}^T = \begin{bmatrix} \underline{0}^T \\ \cdots \\ I_{u_j} \end{bmatrix} \in \mathcal{M}_{u_j+1, u_j}$ for every $j = h+1, h+2, \ldots, m-r$, and I_{u_j} and H_{u_j} denote the $u_j \times u_j$ identity and nilpotent (with index of nilpotency u_j) matrix, respectively. $\underline{0}$ and $\underline{e}_{u_j} = [1 \quad \cdots \quad 0 \quad 0]^T$ are the zero column and the column with element 1 at the first place, respectively.

(S4-S5) The forth and the fifth normal matrix block is the complex Weierstrass form $sF_w - G_w$ of the singular pencil $sF - G$ which is defined by

$$sF_w - Q_w \triangleq sI_p - J_p \oplus sH_q - I_q, \tag{2.9}$$

where the first normal Jordan-type element is uniquely defined by the set of *finite elementary divisors* (f.e.d.)

$$(s - a_1)^{p_1}, \ldots, (s - a_\kappa)^{p_\kappa}, \qquad \sum_{j=1}^{\kappa} p_j = p \tag{2.10}$$

of $sF - G$ and has the form

$$sI_p - J_p \triangleq sI_{p_1} - J_{p_1}(a_1) \oplus \cdots \oplus sI_{p_\kappa} - J_{p_\kappa}(a_\kappa). \tag{2.11}$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the *infinite elementary divisors* (i.e.d.)

$$\widehat{s}^{q_1}, \ldots, \widehat{s}^{q_\sigma}, \qquad \sum_{j=1}^{\sigma} q_j = q \tag{2.12}$$

of $sF - G$ and have the form

$$sH_q - I_q \triangleq sH_{q_1} - I_{q_1} \oplus \cdots \oplus sH_{q_\sigma} - I_{q_\sigma}. \tag{2.13}$$

Thus H_q is a nilpotent element of $\mathcal{M}_n$ with index $\widetilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_q^{\widetilde{q}} = \mathbb{O}, \tag{2.14}$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are the matrices

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{p_j}, \qquad J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & 0 & \cdots & 0 \\ 0 & a_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j},$$
$$H_{q_j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}. \tag{2.15}$$

In the last part of this introductive section, some elements for the analytic computation of $e^{A(t-t_o)}$, $t \in [t_o, \infty)$ are provided. To perform this computation, many theoretical and numerical methods have been developed.

Thus, the interested reader might consult papers in [1, 2, 7–10, 14–16], and the references therein. In order to obtain more analytic formulas, the following known results should be mentioned.

Lemma 2.3 (see [15]).

$$e^{J_{p_j}(a_j)(t-t_o)} = [d_{k_1 k_2}]_{p_j}, \tag{2.16}$$

where

$$d_{k_1k_2} = \begin{cases} e^{a_j(t-t_o)} \dfrac{(t-t_o)^{k_2-k_1}}{(k_2-k_1)!}, & 1 \le k_1 \le k_2 \le p_j, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.17}$$

Another expression for the exponential matrix of Jordan block, see (2.18), is provided by the following lemma.

Lemma 2.4 (see [15]).

$$e^{J_{p_j}(a_j)(t-t_o)} = \sum_{i=0}^{p_j-1} f_i(t-t_o)\left[J_{p_j}(a_j)\right]^i, \tag{2.18}$$

where the $f_k(t-t_o)$*'s satisfy the following system of* p_j *equations:*

$$\begin{gathered} \sum_{i=k}^{p_j-1} \binom{i}{k} a_j^{i-k} f_i(t-t_o) = \frac{(t-t_o)^k}{k!} e^{a_j t}, \quad k = 1, 2, \ldots, p_j, \\ \left[J_{p_j}(a_j)\right]^i = \left[\beta_{k_1k_2}^{(i)}\right]_{p_j}, \quad \textit{for } 1 \le k_1, k_2 \le p_j, \end{gathered} \tag{2.19}$$

where $\beta_{k_1k_2}^{(i)} = \binom{i}{k_2-k_1} a_j^{i-(k_2-k_1)}$.

3. Solution Space for Consistent Initial Conditions

In this section, the main results for consistent initial conditions are analytically presented for the singular case. The whole discussion extends the existing literature; see, for instance [8]. Now, in order to obtain a solution, we deal with consistent initial value problem. More analytically, we consider the system

$$FX^{(r)}(t) = GX(t), \tag{3.1}$$

with known

$$X(t_o), X'(t_o), \ldots, X^{(r-1)}(t_o), \tag{3.2}$$

where $F, G \in \mathcal{M}_{n,m}$ (where matrix F is singular), and $X \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}_{m,l})$.

From the singularity of $s^rF - G$, there exist nonsingular matrices $P \in \mathcal{M}_n$ and $Q \in \mathcal{M}_m$ such that (see also Section 2)

$$\begin{gathered} PFQ = F_k = \mathbb{O}_{h,g} \oplus \Lambda_v \oplus \Lambda_u^T \oplus I_p \oplus H_q, \\ PGQ = G_k = \mathbb{O}_{h,g} \oplus \lambda_v \oplus \lambda_u^T \oplus J_p \oplus I_q, \end{gathered} \tag{3.3}$$

where Λ_v,λ_v, Λ_u^T, λ_u^T, I_p, J_p, H_q, and I_q are given by

$$\begin{aligned}
\Lambda_v &= \Lambda_{v_{g+1}} \oplus \cdots \oplus \Lambda_{v_i} \oplus \cdots \oplus \Lambda_{v_{m-r}}, \\
\lambda_v &\triangleq \lambda_{v_{g+1}} \oplus \cdots \oplus \lambda_{v_i} \oplus \cdots \oplus \lambda_{v_{m-r}}, \\
\Lambda_u^T &\triangleq \Lambda_{u_{h+1}}^T \oplus \cdots \oplus \Lambda_{u_j}^T \oplus \cdots \oplus \Lambda_{u_{n-r}}^T, \\
\lambda_u^T &\triangleq \lambda_{u_{h+1}}^T \oplus \cdots \oplus \lambda_{u_j}^T \oplus \cdots \oplus \lambda_{u_{n-r}}^T, \\
I_p &= I_{p_1} \oplus \cdots \oplus I_{p_\kappa}, \\
J_p &= J_{p_1}(a_1) \oplus \cdots \oplus J_{p_\kappa}(a_\kappa), \\
H_q &\triangleq H_{q_1} \oplus \cdots \oplus H_{q_\sigma}, \\
I_q &= I_{q_1} \oplus \cdots \oplus I_{q_\sigma}.
\end{aligned} \tag{3.4}$$

By using the Kronecker canonical form, we might rewrite system (1.3), as the following lemma denotes.

Lemma 3.1. *System* (1.3) *may be divided into five subsystems:*

$$\mathbb{O}_{h,g} Y_g^{(r)}(t) = \mathbb{O}_{h,g} Y_g(t), \tag{3.5}$$

$$\Lambda_v Y_v^{(r)}(t) = \lambda_v Y_v(t), \tag{3.6}$$

$$\Lambda_u^T Y_u^{(r)}(t) = \lambda_u^T Y_u(t), \tag{3.7}$$

the so-called slow subsystem

$$Y_p^{(r)}(t) = J_p Y_p(t), \tag{3.8}$$

and the relative fast subsystem

$$H_q Y_q^{(r)}(t) = Y_q(t). \tag{3.9}$$

Proof. Consider the transformation

$$X(t) = QY(t), \tag{3.10}$$

where $Q \in \mathcal{M}_m$ and $Y \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}_{m,l})$. Substituting the previous expression into (1.3), we obtain

$$FQY^{(r)}(t) = GQY(t). \tag{3.11}$$

Whereby, multiplying by P, we arrive at

$$F_k Y^{(r)}(t) = G_k Y(t). \tag{3.12}$$

Moreover, we can write $Y(t)$ as

$$Y(t) = \begin{bmatrix} Y_g^T(t) & Y_v^T(t) & Y_u^T(t) & Y_p^T(t) & Y_q^T(t) \end{bmatrix}^T \in \mathcal{M}_{m,l}, \tag{3.13}$$

where $Y_g(t) \in C^\infty(\mathbb{F}, \mathcal{M}_{g,l})$, $Y_v(t) \in C^\infty(\mathbb{F}, \mathcal{M}_{v,l})$, $Y_v(t) \in C^\infty(\mathbb{F}, \mathcal{M}_{u,l})$, $Y_p(t) \in C^\infty(\mathbb{F}, \mathcal{M}_{p,l})$, and $Y_q(t) \in C^\infty(\mathbb{F}, \mathcal{M}_{q,l})$. Note that g is the number of zero column entries, $v = \sum_{i=g+1}^{m-r} v_i$, $u = \sum_{j=h+1}^{n-r} u_j$, $p = \sum_{j=1}^{\kappa} p_j$, and $q = \sum_{j=1}^{\sigma} q_j$.

And taking into account the above expressions, we arrive easily at (3.5)–(3.9). □

Proposition 3.2. *For system* (3.5), *the elements of the matrix* $Y_g(t) \in C^\infty(\mathbb{F}, \mathcal{M}(g \times l; \mathbb{F}))$ *can be chosen arbitrarily.*

Proof. Since $\mathbb{O}_{h,g}$, it is profound that any g-column vector can be chosen. □

Proposition 3.3. *The analytic solution of system*

$$\Lambda_{v_i} Y_{v_i}^{(r)}(t) = \lambda_{v_i} Y_{v_i}(t) \tag{3.14}$$

is given by the expression

$$Y_{v_i}(t) = \begin{bmatrix} \underline{Y}_1(t) & \underline{Y}_2(t) & \cdots & \underline{Y}_l(t) \end{bmatrix} = \left[y_{\lambda,j}(t) \right]_{\substack{\lambda=1,2,\ldots,v_i \\ j=1,2,\ldots,l}}, \tag{3.15}$$

where

$$y_{\lambda,j}(t) = \int \cdots \int y_{\lambda+1,j}(t) \underbrace{dt \cdots dt}_{r\text{-times}} + \sum_{\xi=1}^{r} c_{\lambda,r-\xi+1} \frac{t^{\xi-1}}{(\xi-1)!}, \tag{3.16}$$

where $y_{\lambda+1,j}(t)$ *is an arbitrary function, for every* $\lambda = 1, 2, \ldots, v_i$, $i = g+1, \ldots, m-r$, *and* $j = 1, 2, \ldots, l$. *(Note that* $c_{\lambda,r-\xi+1}$ *should be uniquely determined via the given initial conditions.)*

Proof. System (3.14) is rewritten as

$$\begin{bmatrix} I_{v_i} \vdots \underline{0} \end{bmatrix} Y_{v_i}^{(r)}(t) = \begin{bmatrix} H_{v_i} \vdots \underline{\varepsilon}_{v_i} \end{bmatrix} Y_{v_i}(t), \tag{3.17}$$

for every $i = g+1, g+2, \ldots, m-r$. Now, we denote

$$Y_{v_i}(t) = \begin{bmatrix} \Psi_{v_i}(t) \\ \underline{y}_1(t) \end{bmatrix}, \tag{3.18}$$

where $\Psi_{v_i}(t) \in \mathcal{M}_{v_i l}$, $\Psi_{v_i}(t) = [\underline{Y}_1(t) \;\; \underline{Y}_2(t) \;\; \cdots \;\; \underline{Y}_l(t)]$ with $\underline{Y}_j(t) = [y_{1,j}(t) \;\; y_{2,j}(t) \;\; \cdots \;\; y_{v_i,j}(t)]^T$, and $\underline{y}_1(t) \in \mathcal{M}_{1l}$ (vector, $1 \times l$).

Thus,

$$\left[I_{v_i} \vdots \underline{0}\right] \begin{bmatrix} \Psi_{v_i}^{(r)}(t) \\ \underline{y}_1^{(r)}(t) \end{bmatrix} = \left[H_{v_i} \vdots \underline{\varepsilon}_{v_i}\right] \begin{bmatrix} \Psi_{v_i}(t) \\ \underline{y}_1(t) \end{bmatrix}, \tag{3.19}$$

or, equivalently, we obtain

$$\Psi_{v_i}^{(r)}(t) = H_{v_i}\Psi_{v_i}(t) + \underline{\varepsilon}_{v_i}\underline{y}_1(t). \tag{3.20}$$

Note that $\underline{\varepsilon}_{v_i}\underline{y}_1(t)$ is a matrix with $v_i \times v_i$-elements as follows

$$\underline{\varepsilon}_{v_i}\underline{y}_1(t) = \left[\underline{\tilde{Y}}_1(t) \;\; \underline{\tilde{Y}}_2(t) \;\; \cdots \;\; \underline{\tilde{Y}}_l(t)\right] \triangleq \begin{bmatrix} & \mathbb{O}_{v_i-1,l} & \\ y_{v_i+1,1}(t) & y_{v_i+1,2}(t) \;\; \cdots & y_{v_i,+1l}(t) \end{bmatrix} \tag{3.21}$$

where $\underline{\tilde{Y}}_j(t) = [0 \;\; 0 \;\; \cdots \;\; y_{v_i+1,j}(t)]^T$, for $j = 1, 2, \ldots, l$.

Consequently, (3.20) is rewritten as follows:

$$\begin{aligned} \left[\underline{Y}_1^{(r)}(t) \;\; \underline{Y}_2^{(r)}(t) \;\; \cdots \;\; \underline{Y}_l^{(r)}(t)\right] &= \left[H_{v_i}\underline{Y}_1(t) \;\; H_{v_i}\underline{Y}_2(t) \;\; \cdots \;\; H_{v_i}\underline{Y}_l(t)\right] \\ &\quad + \left[\underline{\tilde{Y}}_1(t) \;\; \underline{\tilde{Y}}_2(t) \;\; \cdots \;\; \underline{\tilde{Y}}_l(t)\right], \end{aligned} \tag{3.22}$$

or, equivalently,

$$\underline{Y}_j^{(r)}(t) = H_{v_i}\underline{Y}_j(t) + \underline{\tilde{Y}}_j(t), \tag{3.23}$$

and eventually, as a scalar system, we obtain

$$\begin{gathered} y_{1,j}^{(r)}(t) = y_{2,j}(t), y_{2,j}^{(r)}(t) = y_{3,j}(t), \ldots, \\ y_{v_i-1,j}^{(r)}(t) = y_{v_i,j}(t), y_{v_i,j}^{(r)}(t) = y_{v_i+1,j}(t). \end{gathered} \tag{3.24}$$

Denote that element $y_{v_i+1,j}(t)$ is an arbitrary function; then the solution is given iteratively, as follows.

Firstly, we take the equation $y^{(r)}_{v_i,j}(t) = y_{v_i+1,j}(t)$ for every $j = 1, 2, \ldots, l$,

$$y^{(r-1)}_{v_i,j}(t) = \int y_{v_i+1,j}(t)dt + c_{v_i,1}, \ldots,$$
$$y_{v_i,j}(t) = \int \cdots \int y_{v_i+1,j}(t)\underbrace{dt \cdots dt}_{r\text{-times}} + \sum_{\xi=1}^{r} c_{v_i,r-\xi+1} \frac{t^{\xi-1}}{(\xi-1)!}. \tag{3.25}$$

We continue the procedure, for $y^{(r)}_{v_i-1,j}(t) = y_{v_i,j}(t)$, and so forth. Thus, we finally obtain (3.15). □

With the following remark, we obtain the solution of subsystem (3.6).

Remark 3.4. The solution of subsystem (3.6) is given by

$$Y_v(t) = Y_{v_{g+1}}(t) \oplus \cdots \oplus Y_{v_i}(t) \oplus \cdots \oplus Y_{v_{m-r}}(t), \tag{3.26}$$

where the results of Proposition 3.3 are also considered.

Remark 3.5. Considering the solution (3.14), and therefore the system (3.6), it should be pointed out that the solution is not unique, since the last component of the solution vector is chosen arbitrary. Moreover, it is worth to be emphasized here that the solution of the singular system (1.3) is not unique.

Proposition 3.6. *The system*

$$\Lambda^T_{u_j} Y^{(r)}_{u_j}(t) = \lambda^T_{u_j} Y_{u_j}(t) \tag{3.27}$$

has only the zero solution.

Proof. Consider that system (3.27) can be rewritten as follows:

$$\begin{bmatrix} \underline{e}^T_{u_j} \\ \cdots \\ H_{u_j} \end{bmatrix} Y^{(r)}_{u_j}(t) = \begin{bmatrix} \underline{0}^T \\ \cdots \\ I_{u_j} \end{bmatrix} Y_{u_j}(t) \tag{3.28}$$

for every $j = h+1, h+2, \ldots, m-r$.

Afterwards, we obtain straightforwardly the following system:

$$\begin{bmatrix} \underline{e}^T_{u_j} Y^{(r)}_{u_j}(t) \\ H_{u_j} Y^{(r)}_{u_j}(t) \end{bmatrix} = \begin{bmatrix} \underline{0}^T \\ Y_{u_j}(t) \end{bmatrix} \Longleftrightarrow \begin{cases} \underline{e}^T_{u_j} Y^{(r)}_{u_j}(t) = \underline{0}^T, \\ H_{u_j} Y^{(r)}_{u_j}(t) = Y_{u_j}(t). \end{cases} \tag{3.29}$$

Now, by successively taking rth derivatives with respect to t on both sides of

$$H_{u_j} Y_{u_j}^{(r)}(t) = Y_{u_j}(t), \tag{3.30}$$

and left multiplying by the matrix H_{u_j}, $u_j - 1$ times (where u_j is the index of the nilpotent matrix H_{u_j}, i.e., $H_{u_j}^{u_j} = \mathbb{O}$), we obtain the following equations:

$$H_{u_j}^2 Y_{u_j}^{(2r)}(t) = H_{u_j} Y_{u_j}^{(r)}(t), \dots, H_{u_j}^{u_j} Y_{u_j}^{(u_j r)}(t) = H_{u_j}^{u_j-1} Y_{u_j}^{((u_j-1)r)}(t). \tag{3.31}$$

Thus, we conclude to the following expression:

$$Y_{u_j}(t) = H_{u_j} Y_{u_j}^{(r)}(t) = H_{u_j}^2 Y_{u_j}^{(2r)}(t) = \cdots = H_{u_j}^{u_j} Y_{u_j}^{(u_j r)}(t) = \mathbb{O}. \tag{3.32}$$

□

Remark 3.7. Consequently, the subsystem (3.7) has also the zero solution.

Proposition 3.8 (see [5]). (a) *The analytic solution of the so-called slow subsystem* (3.8) *is given by*

$$Y_p(t) = LR \bigoplus_{j=1}^{\kappa} \bigoplus_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)} R^{-1} \mathbf{Z}(t_o), \tag{3.33}$$

where $L = [I_p \quad \mathbb{O} \quad \cdots \quad \mathbb{O}] \in \mathcal{M}_{p,\mathrm{pr}}$*;* $R \in \mathcal{M}_{\mathrm{pr}}$ *such that* $\mathbf{J} = R^{-1}\mathbf{A}R$.

Note that $\mathbf{J} \in \mathcal{M}_{\mathrm{pr}}$ *is the Jordan Canonical form of matrix*

$$\mathbf{A} = \begin{bmatrix} \mathbb{O} & I_p & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I_p & \cdots & \mathbb{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & I_p \\ J_p & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{bmatrix}, \tag{3.34}$$

and $\mathbf{Z}(t_o) = \left[Y_p^T(t_o) \quad Y'^T_p(t_o) \quad \cdots \quad Y_p^{(r-1)^T}(t_o)\right]^T \in \mathcal{M}_{\mathrm{pr},l}$.

The eigenvalues of the matrix $\mathbf{A}$ *are given by*

$$\lambda_{jk} = \sqrt[r]{|a_j|}\left(\cos\frac{2k\pi + \varphi_j}{r} + z\sin\frac{2k\pi + \varphi_j}{r}\right), \tag{3.35}$$

where $a_j = |a_j|(\cos\varphi_j + z\sin\varphi_j)$ *(*a_j *finite elementary divisors) and* $z^2 = -1$ *for every* $j = 1, 2, \dots, \kappa$ *and* $k = 0, 1, 2, \dots, r-1$.

(b) *However, the relative fast subsystem* (3.9) *has only the zero solution.*

It is worth to say that the results of Ben Taher and Rachidi [14] can be compared with the results of Proposition 3.8, which has been discussed extensively in [5].

Remark 3.9. The characteristic polynomial of $\mathbf{A}$ is $\varphi(\lambda) = \prod_{j=1}^{\kappa}(\lambda^r - a_j)^{p_j}$, with $a_i \neq a_j$ for $i \neq j$ and $\sum_{j=1}^{\kappa} p_j = p$. Without loss of generality, we define that

$$d_1 = \tau_1, \quad d_2 = \tau_2, \dots, d_l = \tau_l, \text{and } d_{l+1} < \tau_{l+1}, \dots, d_\kappa < \tau_\kappa, \tag{3.36}$$

where $d_j, \tau_j, j = 1, 2, \dots, \kappa$, are the geometric and algebraic multiplicities of the given eigenvalues a_j, respectively.

(i) Consequently, when $d_j = \tau_j$, then

$$J_{jk}(\lambda_{j\kappa}) = \begin{bmatrix} \lambda_{jk} & & & \\ & \lambda_{jk} & & \\ & & \ddots & \\ & & & \lambda_{jk} \end{bmatrix} \in \mathcal{M}_{\tau_j} \tag{3.37}$$

is also a diagonal matrix with diagonal elements of the eigenvalue λ_{jk}, for $j = 1, \dots, l$.

(ii) When $d_j < \tau_j$, then

$$J_{jk,z_j} = \begin{bmatrix} \lambda_{jk} & 1 & & & \\ & \lambda_{jk} & 1 & & \\ & & \lambda_{jk} & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_{jk} \end{bmatrix} \in \mathcal{M}_{z_j} \tag{3.38}$$

for $j = l+1, l+2, \dots, \kappa$, and $z_j = 1, 2, \dots, d_j$.

Hence, the set of consistent initial conditions for system

$$F_k Y^{(r)}(t) = G_k Y(t) \tag{3.39}$$

has the following form:

$$Y^{(k)}(t_o) = \left\{ \left[Y_g^{(k)T}(t_o) \;\; Y_v^{(k)T}(t_o) \;\; \mathbb{O}_u^T \;\; Y_p^{(k)T}(t_o) \;\; \mathbb{O}_q^T \right]^T ; k = 0, \dots, r-1 \right\}. \tag{3.40}$$

In more details, since we have considered (3.14) and we can denote

$$Q = \left[Q_{n,g} \;\; Q_{n,v} \;\; Q_{n,u} \;\; Q_{n,p} \;\; Q_{n,q} \right], \tag{3.41}$$

then we can derive the following expression:

$$\begin{aligned} X(t_o) &= \left[Q_{n,g}\ \ Q_{n,v}\ \ Q_{n,u}\ \ Q_{n,p}\ \ Q_{n,q}\right]\left[Y_g^T(t_o)\ \ Y_v^T(t_o)\ \ \mathbb{O}_u^T\ \ Y_p^T(t_o)\ \ \mathbb{O}_q^T\right]^T \\ &= Q_{n,g}Y_g(t_o) + Q_{n,v}Y_v(t_o) + Q_{n,p}Y_p(t_o). \end{aligned} \tag{3.42}$$

Then, the set of consistent initial conditions for (1.3) is given by

$$\begin{aligned} &\Big\{Q_{n,g}Y_g(t_o) + Q_{n,v}Y_v(t_o) + Q_{n,p}Y_p(t_o)Q_{n,g}Y_g'(t_o) + Q_{n,v}Y_v'(t_o) + Q_{n,p}Y_p'(t_o)\cdots \\ &\quad Q_{n,g}Y_g^{(r-1)}(t_o) + Q_{n,v}Y_v^{(r-1)}(t_o) + Q_{n,p}Y_p^{(r-1)}(t_o)\Big\}. \end{aligned} \tag{3.43}$$

Now, taking into consideration (3.2) and (3.43), we conclude to

$$\begin{aligned} X(t_o) &= Q_{n,g}Y_g(t_o) + Q_{n,v}Y_v(t_o) + Q_{n,p}Y_p(t_o), \\ X'(t_o) &= Q_{n,g}Y_g'(t_o) + Q_{n,v}Y_v'(t_o) + Q_{n,p}Y_p'(t_o), \\ &\vdots \\ X^{(r-1)}(t_o) &= Q_{n,g}Y_g^{(r-1)}(t_o) + Q_{n,v}Y_v^{(r-1)}(t_o) + Q_{n,p}Y_p^{(r-1)}(t_o). \end{aligned} \tag{3.44}$$

Theorem 3.10. *The analytic solution of* (3.2) *is given by*

$$X(t) = Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) + Q_{n,p}LR\bigoplus_{j=1}^{\kappa}\bigoplus_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)}R^{-1}\mathbf{Z}(t_o) \tag{3.45}$$

for $Y_v(t) = Y_{v_{g+1}}(t) \oplus \cdots \oplus Y_{v_i}(t) \oplus \cdots \oplus Y_{v_{m-r}}(t)$, *where* $Y_{v_i}(t) = [y_{\lambda,j}]_{\lambda=1,2,\ldots,v_i, j=1,2,\ldots,l}$, *for* $i = g+1,\ldots,m-r$ *and*

$$y_{\lambda,j}(t) = \int\cdots\int y_{\lambda+1,j}(t)\underbrace{dt\cdots dt}_{r\text{-times}} + \sum_{\xi=1}^{r} c_{\lambda,r-\xi+1}\frac{t^{\xi-1}}{(\xi-1)!}. \tag{3.46}$$

(Note that $c_{\lambda,r-\xi+1}$ *should be uniquely determined via the given initial conditions.)*

The matrix $Y_g(t)$ *is arbitrarily chosen. Moreover*

$$L = \left[I_p\ \ \mathbb{O}\ \ \cdots\ \ \mathbb{O}\right] \in \mathcal{M}_{p,\mathrm{pr}}, \quad R \in \mathcal{M}_{\mathrm{pr}} \tag{3.47}$$

such that $\mathbf{J} = R^{-1}\mathbf{A}R$, *where* $\mathbf{J} \in \mathcal{M}_{\mathrm{pr}}$ *is the Jordan Canonical form of matrix* $\mathbf{A}$, *and*

$$\mathbf{Z}(t_o) = \left[Y_p^T(t_o)\ \ Y_p'^T(t_o)\ \ \cdots\ \ Y_p^{(r-1)^T}(t_o)\right]^T \in \mathcal{M}_{\mathrm{pr},l}. \tag{3.48}$$

Proof. Using the results of Lemma 3.1, Propositions 3.2–3.8, Remarks 3.4 and 3.7 and (3.10) then we obtain

$$\begin{aligned} X(t) = QY(t) &= \begin{bmatrix} Q_{n,g} & Q_{n,v} & Q_{n,u} & Q_{n,p} & Q_{n,q} \end{bmatrix} \cdot \begin{bmatrix} Y_g^T(t) & Y_v^T(t) & \mathbb{O}_u^T & Y_p^T(t) & \mathbb{O}_q^T \end{bmatrix}^T \\ &= Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) + Q_{n,p}Y_p(t). \end{aligned} \tag{3.49}$$

Finally, (3.45) is derived. □

The next remark connects the solution with the set of initial condition for the system (1.3).

Remark 3.11. If $\tilde{Q}_{n,p}$ is the existing left inverse of $Q_{n,p}$, then considering also (3.10)

$$\mathbf{Z}(t_o) = \begin{bmatrix} Y_p(t_o) \\ Y'_p(t_o) \\ \vdots \\ Y_p^{(r-1)}(t_o) \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{p,n}X_p(t_o) \\ \tilde{Q}_{p,n}X'_p(t_o) \\ \vdots \\ \tilde{Q}_{p,n}X_p^{(r-1)}(t_o) \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{p,n} & & & \\ & \tilde{Q}_{p,n} & & \\ & & \ddots & \\ & & & \tilde{Q}_{p,n} \end{bmatrix} \begin{bmatrix} X_p(t_o) \\ X'_p(t_o) \\ \vdots \\ X_p^{(r-1)}(t_o) \end{bmatrix} = \tilde{Q}\mathbf{\Psi}(t_o). \tag{3.50}$$

Finally, the solution (3.45) is given by

$$X(t) = Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) + Q_{n,p}LR\bigoplus_{j=1}^{\kappa}\bigoplus_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)}R^{-1}\tilde{Q}\mathbf{\Psi}(t_o), \tag{3.51}$$

where $\mathbf{\Psi}(t_o) = \begin{bmatrix} X_p^T(t_o) & X'^T_p(t_o) & \cdots & X_p^{(r-1)^T}(t_o) \end{bmatrix}^T \in \mathcal{M}_{\mathrm{pr},l}$ and $\tilde{Q}_{n,p}$ is the existing left inverse of $Q_{n,p}$.

The following two expressions, that is, (3.52) and (3.54) are based on Lemmas 2.3 and 2.4. Thus, two new analytical formulas are derived which are practically very useful. Their proofs are straightforward exercise of Lemmas 2.3, 2.4, and (3.51).

Proposition 3.12. *Considering the results of Lemma 2.3, one obtains the expression*

$$\begin{aligned} X(t) &= Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) \\ &\quad + Q_{n,p}LR\left[\left(\bigoplus_{j=0}^{l}\bigoplus_{k=0}^{r-1} e^{\lambda_{jk}(t-t_o)}I_{\tau_{jk}}\right)\oplus\left(\bigoplus_{j=l+1}^{\kappa}\bigoplus_{k=0}^{r-1}\bigoplus_{z_j=1}^{d_j}(d_{k_1k_2})_{z_j}\right)\right]R^{-1}\tilde{Q}\mathbf{\Psi}(t_o), \end{aligned} \tag{3.52}$$

where

$$d_{k_1k_2} = \begin{cases} e^{\lambda_{jk}(t-t_o)} \dfrac{(t-t_o)^{k_2-k_1}}{(k_2-k_1)!}, & 1 \leq k_1 \leq k_2 \leq z_j, \\ 0, & \text{otherwise} \end{cases} \tag{3.53}$$

for $j = l+1, l+2, \ldots, \kappa$, *and* $z_j = 1, 2, \ldots, d_j$.

Another expression for the exponential matrix of Jordan block, see (2.18), is provided by the following lemma.

Proposition 3.13. *Considering the results of Lemma 2.4, one obtains the expression*

$$\begin{aligned} X(t) &= Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) \\ &\quad + Q_{n,p}LR\left[\left(\bigoplus_{j=0}^{l}\bigoplus_{k=0}^{r-1} e^{\lambda_{jk}(t-t_o)} I_{\tau_{jk}}\right) \bigoplus \left(\bigoplus_{j=l+1}^{\kappa}\bigoplus_{k=0}^{r-1}\bigoplus_{z_j=1}^{d_j}\sum_{i=0}^{z_j-1} f_i(t-t_o)\left[J_{z_j}(\lambda_{jk})\right]^i\right)\right] \\ &\quad \times R^{-1}\widetilde{Q}\mathbf{\Psi}(t_o), \end{aligned} \tag{3.54}$$

where the $f_k(t-t_o)$*'s satisfy the following system of* z_j *(for* $z_j = 1, 2, \ldots, d_j$*) equations:*

$$\sum_{i=k}^{z_j-1} \binom{i}{k} a_j^{i-k} f_i(t-t_o) = \frac{(t-t_o)^k}{k!} e^{\lambda_{jk}t}, \quad k = 1, 2, \ldots, z_j \tag{3.55}$$

and $[J_{z_j}(\lambda_{jk})]^i = [\beta_{k_1k_2}^{(i)}]_{z_j}$, *for* $1 \leq k_1, k_2 \leq z_j$ *where* $\beta_{k_1k_2}^{(i)} = \binom{i}{k_2-k_1} a_j^{i-(k_2-k_1)}$.

Analyzing more the results of this section, see Theorem 3.10 and Lemma 2.3 and 2.4, we can present briefly a symbolical algorithm for the solution of system (1.3).

Symbolical Algorithm

Step 1. Determine the pencil $sF - G$.

Step 2. Calculate the expressions (3.3). Thus, we have to find the f.e.d, i.e.d., r.m.i, c.m.i, and so forth, (i.e., the *complex Kronecker form* $sF_k - G_k$ of the singular pencil $sF - G$ here; it should be noticed that this step is not an easy task; some parts are still under research).

Step 3. Using the results of Step 2, determine the matrices Λ_v,λ_v, Λ_u^T, λ_u^T, I_p, J_p, H_q and I_q.

Step 4. Determine $Q_{n,g}, Q_{n,v}, Q_{n,p}, L, R$ (using the Jordan canonical form of matrix $\mathbf{A}$), $\widetilde{Q}$ and $\mathbf{\Psi}(t_o)$ (see Remark 3.9).

Step 5. Considering the transformation (3.10), that is, $X(t) = QY(t)$, we obtain (3.54). Then the following.

Substep 5.1. Choose an arbitrary matrix $Y_g(t)$.

Substep 5.2. Determine the matrix $Y_v(t)$, that is,

$$Y_v(t) = Y_{v_{g+1}}(t) \oplus \ldots \oplus Y_{v_i}(t) \oplus \ldots \oplus Y_{v_{m-r}}(t), \tag{3.56}$$

where $Y_{v_i}(t) = [y_{\lambda,j}]_{\lambda=1,2,\ldots,v_i, j=1,2,\ldots,l}$, for $i = g+1, \ldots, m-r$ and

$$y_{\lambda,j}(t) = \int \cdots \int y_{\lambda+1,j}(t) \underbrace{dt \cdots dt}_{r\text{-times}} + \sum_{\xi=1}^{r} c_{\lambda,r-\xi+1} \frac{t^{\xi-1}}{(\xi-1)!}. \tag{3.57}$$

Step fta

Following the results of *Lemma 4*, determine

$$\bigoplus_{j=1}^{\kappa} \bigoplus_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)} = \left(\bigoplus_{j=0}^{l} \bigoplus_{k=0}^{r-1} e^{\lambda_{jk}(t-t_o)} I_{\tau_{jk}} \right) \bigoplus \left(\bigoplus_{j=l+1}^{\kappa} \bigoplus_{k=0}^{r-1} \bigoplus_{z_j=1}^{d_j} (d_{k_1 k_2})_{z_j} \right), \tag{3.58}$$

where

$$d_{k_1 k_2} = \begin{cases} e^{\lambda_{jk}(t-t_o)} \dfrac{(t-t_o)^{k_2-k_1}}{(k_2-k_1)!}, & 1 \le k_1 \le k_2 \le z_j, \\ 0, & \text{otherwise} \end{cases} \tag{3.59}$$

for $j = l+1, l+2, \ldots, \kappa$, and $z_j = 1, 2, \ldots, d_j$.

Step ftb

Following the results of *Lemma 5*, determine

$$\bigoplus_{j=1}^{\kappa} \bigoplus_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)} = \left(\bigoplus_{j=0}^{l} \bigoplus_{k=0}^{r-1} e^{\lambda_{jk}(t-t_o)} I_{\tau_{jk}} \right) \oplus \left(\bigoplus_{j=l+1}^{\kappa} \bigoplus_{k=0}^{r-1} \bigoplus_{z_j=1}^{d_j} \sum_{i=0}^{z_j-1} f_i(t-t_o) \left[J_{z_j}(\lambda_{jk}) \right]^i \right), \tag{3.60}$$

where $f_k(t-t_o)'s$ satisfy the following system of z_j (for $z_j = 1, 2, \ldots, d_j$) equations:

$$\sum_{i=k}^{z_j-1} \binom{i}{k} a_j^{i-k} f_i(t-t_o) = \frac{(t-t_o)^k}{k!} e^{\lambda_{jk} t}, \quad k = 1, 2, \ldots, z_j, \tag{3.61}$$

and $[J_{z_j}(\lambda_{jk})]^i = [\beta^{(i)}_{k_1 k_2}]_{z_j}$, for $1 \le k_1, k_2 \le z_j$, and $\beta^{(i)}_{k_1 k_2} = \binom{i}{k_2-k_1} a_j^{i-(k_2-k_1)}$.

Example 3.14 (with *consistent* initial condition). Consider the 2nd order system

$$F\ddot{X}(t) = GX(t), \tag{3.62}$$

where $F, G \in \mathcal{M}(11 \times 12; \mathbb{R})$, and $X \in C^\infty(\mathbb{R}, \mathcal{M}(12 \times 5; \mathbb{R}))$, with

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{11,12},$$
$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{11,12}, \tag{3.63}$$

with known consistent initial conditions $X(t_o), X'(t_o)$.

From the singularity of $sF - G$, there exist nonsingular matrices

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{11}, \qquad Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{12} \tag{3.64}$$

Then, using (3.3), we obtain

$$\begin{aligned} PFQ = F_k &= \mathbb{O}_{1,2} \oplus \Lambda_3 \oplus \Lambda_2^T \oplus I_2 \oplus H_2, \\ PGQ = G_k &= \mathbb{O}_{1,2} \oplus \lambda_3 \oplus \lambda_2^T \oplus J_2 \oplus I_2, \end{aligned} \tag{3.65}$$

where Λ_2,λ_2, Λ_2^T, λ_2^T, I_2, J_2, and H_2 are given by

$$\begin{gathered} \Lambda_3 = \left[I_3 \vdots \underline{0} \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{3,4}, \qquad \lambda_3 = \left[H_3 \vdots \underline{\varepsilon}_3 \right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{2,3}, \\ \Lambda_2^T = \begin{bmatrix} \underline{e}_2^T \\ \cdots \\ H_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{3,2}, \qquad \lambda_2^T = \begin{bmatrix} \underline{0}^T \\ \cdots \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_{3,2}, \\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J_2(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{gathered} \tag{3.66}$$

Considering the transformation (3.10), that is, $X(t) = QY(t)$, we obtain the results of Lemma 2.3 (see also **J**, below):

$$X(t) = Q_{12,g=2}Y_{g=2}(t) + Q_{12,v=2}Y_{v=2}(t) + Q_{12,p=2}LR\left[\bigoplus_{j=1}^{2}\bigoplus_{z_j=1}^{2}\left(d_{k_1k_2}\right)_{z_j}\right]R^{-1}\tilde{Q}\mathbf{\Psi}(t_o), \tag{3.67}$$

where

$$Q_{12,g=2} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$Q_{12,v=2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.68}$$
$$Q_{12,p=2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(i) $Y_{g=2}(t) \in \mathcal{M}_{2,5}$ is an arbitrarily chosen matrix.

(ii) $Y_{v=2}(t) \in \mathcal{M}_{2,5}$ with $Y_2(t) = [y_{\lambda,j}]_{\lambda=1,2,j=1,2,\ldots,5}$, and

$$y_{\lambda,j}(t) = \iint y_{\lambda+1,j}(t)dt\,dt + \sum_{\xi=1}^{2} c_{\lambda,3-\xi}\frac{t^{\xi-1}}{(\xi-1)!}. \tag{3.69}$$

(iii)

$$L = \begin{bmatrix} I_2 & \mathbb{O}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{2,4}. \tag{3.70}$$

(iv)

$$R = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \qquad R^{-1} = \begin{bmatrix} 2 & 1 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$
$$\mathbf{J} = R^{-1}\mathbf{A}R = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.71}$$

(We have only two eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 1$.)

(v)

$$\tilde{Q}\boldsymbol{\Psi}(t_o) = \begin{bmatrix} \tilde{Q}_{p=2,12} & \\ & \tilde{Q}_{p=2,12} \end{bmatrix} \begin{bmatrix} X_p(t_o) \\ X'_p(t_o) \end{bmatrix}, \tag{3.72}$$

where $X(t_o), X'(t_o)$ are known, and

$$\tilde{Q}_{p=2,12} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.73}$$

(vi) With

$$\overset{2}{\underset{j=1}{\oplus}} \overset{2}{\underset{z_j=1}{\oplus}} (d_{k_1k_2})_{z_j} = \begin{bmatrix} e^{-(t-t_o)} & e^{-(t-t_o)}(t-t_o) & 0 & 0 \\ 0 & e^{-(t-t_o)} & 0 & 0 \\ 0 & 0 & e^{(t-t_o)} & e^{(t-t_o)}(t-t_o) \\ 0 & 0 & 0 & e^{(t-t_o)} \end{bmatrix} \tag{3.74}$$

Combining the above arithmetic results, the analytic solution of system (3.62) is given by considering (3.58).

4. Solution Space Form of Nonconsistent Initial Conditions

In this short section, we would like to describe briefly the impulse behaviour of the solution of the original system (1.3), at time t_o, see also [11–13]. In that case, we reformulate Proposition 3.6, so that the impulse solution is finally obtained. Note that in this part of the paper the condition that $X \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(m \times l; \mathbb{F}))$ does not hold anymore, since we are interesting for solutions with impulsive behaviour (again $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$).

Moreover, we assume that the space of nonconsistent initial conditions is denoted by $\mathcal{C}_0^*$, which is called also *redundancy space*. Then, considering also Lemma 3.1, and especially (3.7) and (3.9), we have the nonconsistent initial condition that is

$$\begin{gathered} Y_{u_j}^{(k)}(t_o) \neq Y_{u_j,o}, \quad \text{for } j = h+1, h+2, \ldots, m-r, \\ Y_{q_z}^{(k)}(t_o) \neq Y_{q_z,o}, \quad \text{for } z = 1, 2, \ldots, \sigma, \end{gathered} \tag{4.1}$$

for $k = 0, 1, \ldots, r-1$.

In order to be able to find a solution, we use the classical method of Laplace transformation. This method has been applied several times in descriptor system theory; see for instance, [4, 5, 11].

Proposition 4.1. *The analytic solution of the system* (3.27) *is given by*

$$Y_{u_j}(t) = -\sum_{k=0}^{r-1}\sum_{\zeta=0}^{u_j-2}\delta^{(r\zeta+r-1-k)}(t)H_{u_j}^{\zeta+1}Y_{u_j,o}^{(k)}, \tag{4.2}$$

where δ and $\delta^{(k)}$ are the delta function of Dirac and its derivatives, respectively.

Proof. Let us start by observing that—as it is well known—there exists a $u_j \in \mathbb{N}$ such that $H_{u_j}^{u_j} = \mathbb{O}$, that is, the index of nilpotency equals u_j of H_{u_j}.

Moreover, system (3.27) can be rewritten as follows; see also proof of Proposition 3.6,

$$\begin{bmatrix} \underline{e}_{u_j}^T Y_{u_j}^{(r)}(t) \\ H_{u_j} Y_{u_j}^{(r)}(t) \end{bmatrix} = \begin{bmatrix} \underline{0}^T \\ Y_{u_j}(t) \end{bmatrix} \Longleftrightarrow \begin{cases} \underline{e}_{u_j}^T Y_{u_j}^{(r)}(t) = \underline{0}^t, \\ H_{u_j} Y_{u_j}^{(r)}(t) = Y_{u_j}(t). \end{cases} \tag{4.3}$$

Where by taking the Laplace transformation of $H_{u_j}Y_{u_j}^{(r)}(t) = Y_{u_j}(t)$, the following expression derives:

$$H_{u_j}\Im\left\{Y_{u_j}^{(r)}(t)\right\} = \Im\left\{Y_{u_j}(t)\right\}, \tag{4.4}$$

and by defining $\Im\{Y_{u_j}(t)\} = \mathcal{X}_{u_j}(s)$, we obtain

$$\left(s^r H_{u_j} - I_{u_j}\right)\mathcal{X}_{u_j}(s) = H_{u_j}\sum_{k=0}^{r-1} s^{r-1-k} Y_{u_j,o}^{(k)}. \tag{4.5}$$

Since u_j is the index nilpotency of H_{u_j}, it is known that

$$\left(s^r H_{u_j} - I_{u_j}\right)^{-1} = -\sum_{\zeta=0}^{u_j-1}\left(s^r H_{u_j}\right)^{\zeta}, \tag{4.6}$$

where $H_{u_j}^0 = I_{u_j}$; see, for instance [4, 10]. Thus, substituting the above expression into (4.5), the following equation is being taken:

$$\mathcal{X}_{u_j}(s) = -\sum_{k=0}^{r-1}\sum_{\zeta=0}^{u_j-2} s^{r\zeta+r-1-k} H_{u_j}^{\zeta+1} Y_{u_j,o}^{(k)}. \tag{4.7}$$

Since $\Im\{\delta^{(k)}(t)\} = s^k$, (4.7) is transformed into (4.8):

$$\mathcal{X}_{u_j}(s) = -\sum_{k=0}^{r-1}\sum_{\zeta=0}^{u_j-2}\Im\left\{\delta^{(r\zeta+r-1-k)}\right\}(t)H_{u_j}^{\zeta+1}Y_{u_j,o}^{(k)}. \tag{4.8}$$

Now, by applying the inverse *Laplace* transformation into (4.8), the equation (4.2) is derived. □

Remark 4.2. The analytic solution of the subsystem (3.7) is given by

$$Y_u(t) = Y_{u_{h+1}}(t) \oplus \cdots \oplus Y_{u_j}(t) \oplus \cdots \oplus Y_{u_{n-r}}(t), \tag{4.9}$$

where the results of Proposition 4.1 are also considered.

Similarly to Proposition 4.1, we can prove the following proposition.

Proposition 4.3. *The analytic solution of the system* (3.9) *is given by*

$$Y_q(s) = -\sum_{k=0}^{r-1}\sum_{z=0}^{q^*-2}\delta^{(rz+r-1-k)}(t)H_q^{z+1}Y_{q,o}^{(k)}. \tag{4.10}$$

Theorem 4.4. *The analytic solution of* (1.3) *is given by*

$$\begin{aligned} X(t) = {}& Q_{n,g}Y_g(t) + Q_{n,v}Y_v(t) + Q_{n,u}Y_u(t) + Q_{n,p}LR \mathop{\oplus}_{j=0}^{\kappa}\mathop{\oplus}_{k=0}^{r-1} e^{J_{jk}(\lambda_{jk})(t-t_o)}R^{-1}Z(t_o) \\ & - Q_{n,q}\sum_{k=0}^{r-1}\sum_{z=0}^{q^*-2}\delta^{(rz+r-1-k)}(t)H_q^{z+1}Y_{q,o}^{(k)}. \end{aligned} \tag{4.11}$$

The matrix $Y_g(t)$ is arbitrarily chosen. For $Y_v(t) = Y_{v_{g+1}}(t) \oplus \cdots \oplus Y_{v_i}(t) \oplus \cdots \oplus Y_{v_{m-r}}(t)$ where $Y_{v_i}(t) = [y_{\lambda,j}]_{\lambda=1,2,\ldots,v_i, j=1,2,\ldots,l}$, for $i = g+1,\ldots,m-r$, and

$$y_{\lambda,j}(t) = \int\cdots\int y_{\lambda+1,j}(t)\underbrace{dt\cdots dt}_{r\text{-times}} + \sum_{\xi=1}^{r}c_{\lambda,r-\xi+1}\frac{t^{\xi-1}}{(\xi-1)!}. \tag{4.12}$$

For

$$Y_u(t) = Y_{u_{h+1}}(t) \oplus \cdots \oplus Y_{u_j}(t) \oplus \cdots \oplus Y_{u_{n-r}}(t), \tag{4.13}$$

where $Y_{u_j}(t) = -\sum_{k=0}^{r-1}\sum_{\zeta=0}^{u_j-2}\delta^{(r\zeta+r-1-k)}(t)H_{u_j}^{\zeta+1}Y_{u_j,o}^{(k)}$.

Moreover $L = [I_p \quad \mathbb{O} \quad \cdots \quad \mathbb{O}] \in \mathcal{M}_{p,\mathrm{pr}}$; $R \in \mathcal{M}_{\mathrm{pr}}$ such that $\mathbf{J} = R^{-1}\mathbf{A}R$, where $\mathbf{J} \in \mathcal{M}_{\mathrm{pr}}$ is the Jordan Canonical form of matrix $\mathbf{A}$, and

$$\mathbf{Z}(t_o) = \left[Y_q^t(t_o) \quad Y'^t_q(t_o) \quad \cdots \quad Y_p^{(r-1)t}(t_o)\right]^t \in \mathcal{M}_{\mathrm{pr},l}. \tag{4.14}$$

Proof. Combining the results of Theorem 3.10 and the above discussion, the solution is provided by (4.11). □

Remark 4.5. For $t > t_o$, it is obvious that (3.45) is satisfied. Thus, we should stress that the system (1.3) has the above impulse behaviour at time instant where a nonconsistent initial value is assumed, while it returns to smooth behaviour at time instant t_o.

5. Conclusions

In this paper, we study the class of LTI descriptor (singular) matrix descriptor differential equations of higher order whose coefficients are rectangular constant matrices. By taking into consideration that the relevant pencil is singular, we get affected by the Kronecker canonical form in order to decompose the differential system into five subsystems. Afterwards, we provide analytical formulas for this general class of Apostol-Kolodner type of equations when we have consistent and nonconsistent initial conditions.

As a future research, we would like to compare more our results with those of the preceding papers of Kolodner [9], Ben Taher and Rachidi [8, 14], and Geerts [12, 13]. Analytically, following the work in [9], we want to investigate whether it is possible to apply our results to find the dynamical solution of LTI matrix descriptor (singular) differential systems. Moreover, it also our wish to apply some combinatorial method for computing the matrix powers and exponential, as it is appeared in [8]. Finally, it is worth to extend the results of Geerts [13] to the general class of LTI matrix descriptor (regular and singular) differential systems.

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