Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 167160, 10 pages
doi:10.1155/2011/167160

Research Article

New Properties of Complex Functions with Mean Value Conditions

Yuzhen Bai[1] and Lei Wu[1,2]

[1] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*
[2] *Xingtan College, Qufu Normal University, Qufu 273100, China*

Correspondence should be addressed to Yuzhen Bai, baiyu99@126.com

Received 6 August 2011; Accepted 24 September 2011

Academic Editor: Marcia Federson

We apply mollifiers to study the properties of real functions which satisfy mean value conditions and present new equivalent conditions for complex analytic functions. New properties of complex functions with mean value conditions are given.

1. Introduction

There are many good properties of complex analytic function. In the references on complex function theory (see [1] and the references therein), we see that analytic function satisfies mean value theorem but the converse is wrong. Hence, mean value condition is weaker than analytic condition.

The mean value problem has been a very active area in recent years. The mean value theorem for real-valued differentiable functions defined on an interval is one of the most fundamental results in analysis. However, the theorem is incorrect for complex-valued functions even if the function is differentiable throughout the complex plane. Qazi [2] illustrated that by examples and presented three results of a positive nature. A mean value theorem for continuous vector functions was introduced by mollified derivatives and smooth approximations in [3]. Crespi et al. [4] and La Torre [5] gave some characterizations of convex functions by means of second-order mollified derivatives. Second-order necessary optimality conditions for nonsmooth vector optimization problems were given by smooth approximations in [6]. Eberhard and Mordukhovich [7] mainly concerned deriving first-order and second-order necessary (and partly sufficient) optimality conditions for a general class of constrained optimization problems via convolution smoothing. Eberhard et al. [8]

demonstrated that second-order subdifferentials were constructed via the accumulation of local Hessian information provided by an integral convolution approximation of the function. In [9], Aimar et al. showed the parabolic mean value formula.

In this paper, we will apply mollifiers to study the properties of real functions which satisfy mean value conditions and present new equivalent conditions for complex analytic functions. New properties of complex functions with mean value conditions will be given.

We introduce the notations: $z = x + iy$, $\overline{z} = x - iy$, $p_0 = (x_0, y_0)$, $p = (x, y)$, $B_r(p_0) = \{p \mid \operatorname{dist}(p, p_0) \leq r\}$, $\partial B_r(p_0) = \{p \mid \operatorname{dist}(p, p_0) = r\}$. Using the chain rule of derivation, we have

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \Delta = \frac{\partial}{\partial z \partial \overline{z}} = \frac{\partial}{\partial \overline{z} \partial z} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.1}$$

The Cauchy-Riemann equation of analytic function $f(z) = u(x, y) + iv(x, y)$ can be written as

$$\frac{\partial f(z)}{\partial \overline{z}} = 0. \tag{1.2}$$

We will use the following classical definitions and results of functional analysis.

Definition 1.1 (see [10]). The functions

$$\varphi_\varepsilon(x) = \begin{cases} \dfrac{c}{\varepsilon^n} \exp\left(\dfrac{\varepsilon^2}{|x|^2 - \varepsilon^2}\right), & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon, \end{cases} \tag{1.3}$$

with $c \in R$ such that $\int_{R^n} \varphi_\varepsilon(x) dx = 1$, are called standard mollifiers.

From the definition, we see the functions φ_ε are C^∞.

Definition 1.2 (see [3]). Give a locally integrable function $f : R^n \to R^m$ and a sequence of bounded mollifiers, and define the functions f_ε by the convolution

$$f_\varepsilon(x) := \int_{R^n} f(x - y)\varphi_\varepsilon(y) dy = \int_{R^n} f(y)\varphi_\varepsilon(x - y) dy. \tag{1.4}$$

The sequence $f_\varepsilon(x)$ is said to be a sequence of mollified functions.

Proposition 1.3 (Properties of mollifiers, see [10]). *Suppose that $\Omega \subset R^n$ is open, $\varepsilon > 0$, write $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Then,*

(i) $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$,

(ii) $f_\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$,

(iii) *if* $f \in C(\Omega)$, *then* $f_\varepsilon \to f$ *uniformly on compact subsets of* Ω,

(iv) *If* $1 \leq p < \infty$ *and* $f \in L^p_{\text{loc}}(\Omega)$ *then* $f_\varepsilon \to f$ *in* $L^p_{\text{loc}}(\Omega)$.

This paper is organized as follows. In Section 2, we give the definitions of mean value conditions and their equivalent forms. Applying mollifiers, we show some properties of real functions with mean value conditions in Section 3. Section 4 contains our main results for complex functions satisfying mean value condition, that is, the new equivalent condition of complex analytic function and the new properties of complex functions. At last, we present two problems with their answers.

2. Mean Value Conditions

Definition 2.1 (Mean value condition). Let Ω be a domain in complex number field (bounded or unbounded) and $f(z) = u(x,y) + iv(x,y)$ a continuous complex function defined in Ω. For any $z_0 \in \Omega$ and $\{z \mid |z - z_0| \leq r\} \subset \Omega$, if

$$f(z_0) = \frac{1}{2\pi r}\int_{|z-z_0|=r} f(z)ds, \tag{2.1}$$

we say that $f(z)$ satisfies the mean value condition in domain Ω.

Remark 2.2. If $f(z)$ is an analytic function in domain Ω, then $f(z)$ satisfies the mean value condition in domain Ω (see [1]), the converse is wrong. For example, $f(z) = 1 + iy$ satisfies the mean value condition in the complex number field, but it is not analytic. Hence, mean value condition is weaker than analytic condition.

Definition 2.3 (Mean value condition). Set $w(p) \in C(\Omega)$.

(i) For any $B_r(p_0) \subset \Omega$, if

$$w(p_0) = \frac{1}{2\pi r}\int_{\partial B_r(p_0)} w(p)ds, \tag{2.2}$$

we say that $w(p)$ satisfies the first mean value condition.

(ii) For any $B_r(p_0) \subset \Omega$, if

$$w(p_0) = \frac{1}{\pi r^2}\int_{B_r(p_0)} w(p)dp, \tag{2.3}$$

we say that $w(p)$ satisfies the second mean value condition.

Proposition 2.4. *(1) The first and the second mean value conditions of $w(p)$ are equivalent.*

(2)

(i) *The first mean value condition of $w(p)$ can be written as*

$$w(p_0) = \frac{1}{2\pi}\int_{|\omega|=1} w(p_0 + r\omega)ds. \tag{2.4}$$

(ii) *The second mean value condition of $w(p)$ can be written as*

$$w(p_0) = \frac{1}{\pi}\int_{|\omega|\le 1} w(p_0 + r\omega)d\omega. \tag{2.5}$$

(3) The mean value condition of complex function $f(z)$ can be written as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)d\theta. \tag{2.6}$$

(4) The complex function $f(z) = u(x,y) + iv(x,y)$ satisfies mean value condition if and only if real functions $u(x,y)$ and $v(x,y)$ satisfy mean value conditions.

Proof. (1) Differentiating both sides of

$$w(p_0) = \frac{1}{\pi r^2}\int_{B_r(p_0)} w(p)dp = \frac{1}{\pi r^2}\int_0^r d\rho\int_{\partial B_\rho(p_0)} w(p)ds, \tag{2.7}$$

with respect to r, we have

$$0 = -\frac{2}{\pi r^3}\int_0^r d\rho\int_{\partial B_\rho(p_0)} w(p)ds + \frac{1}{\pi r^2}\int_{\partial B_r(p_0)} w(p)ds, \tag{2.8}$$

that is,

$$\frac{1}{2\pi}\int_{\partial B_r(p_0)} w(p)ds = \frac{1}{\pi r^2}\int_{B_r(p_0)} w(p)dp = w(p_0). \tag{2.9}$$

We write the first mean value condition as

$$w(p_0)\rho = \frac{1}{2\pi}\int_{\partial B_\rho(p_0)} w(p)ds \tag{2.10}$$

and get the second mean value condition by integrating the both sides of (2.10) with respect to ρ on $[0,r]$.

(2)

(i) Let $p = p_0 + r\omega$. Then, by integral transform formula, we get

$$w(p_0) = \frac{1}{2\pi r}\int_{\partial B_r(p_0)} w(p)ds = \frac{1}{2\pi}\int_{|\omega|=1} w(p_0 + r\omega)ds. \tag{2.11}$$

(ii) In the same way, we get

$$w(p_0) = \frac{1}{\pi r^2}\int_{B_r(p_0)} w(p)dp = \frac{1}{\pi}\int_{|\omega|\le 1} w(p_0 + r\omega)d\omega. \tag{2.12}$$

(3) Let $z = z_0 + re^{i\theta}$. Then, by integral transform formula, we get

$$f(z_0) = \frac{1}{2\pi r}\int_{|z-z_0|=r} f(z)ds = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)d\theta. \tag{2.13}$$

(4) Let $z = z_0 + r(\omega_1 + i\omega_2)$ and $\omega = (\omega_1, \omega_2)$. Then, by integral transform formula, we see

$$\begin{aligned} u(x_0, y_0) + iv(x_0, y_0) = f(z_0) &= \frac{1}{2\pi r}\int_{|z-z_0|=r} [u(x,y) + iv(x,y)]ds \\ &= \frac{1}{2\pi}\int_{|\omega|=1} [u(p_0 + r\omega) + iv(p_0 + r\omega)]ds, \end{aligned} \tag{2.14}$$

which implies

$$u(x_0, y_0) = \frac{1}{2\pi}\int_{|\omega|=1} u(p_0 + r\omega)ds, \qquad v(x_0, y_0) = \frac{1}{2\pi}\int_{|\omega|=1} v(p_0 + r\omega)ds. \tag{2.15}$$

□

3. Preliminaries

In this section, we give the properties of real functions satisfying the mean value conditions. These properties will be used to prove our main results in Section 4.

Lemma 3.1. *If $\Delta w(p) = 0$, $p = (x, y) \in \Omega$, then, $w(x, y)$ satisfies the mean value condition in domain Ω.*

Proof. For any $B_r(p_0) \subset \Omega$, using Green formula, we have

$$\int_{B_\rho(p_0)} \Delta w(p)dp = \int_{\partial B_\rho(p_0)} \frac{\partial w}{\partial \nu}ds = \rho\int_{|\omega|=1} \frac{\partial w}{\partial \rho}(p_0 + \rho\omega)ds = \rho\frac{\partial}{\partial \rho}\int_{|\omega|=1} w(p_0 + \rho\omega)ds. \tag{3.1}$$

Since w is harmonic in Ω, we obtain from (3.1) that

$$\frac{\partial}{\partial \rho}\int_{|\omega|=1} w(p_0 + \rho\omega)ds = 0. \tag{3.2}$$

Integrating both sides of (3.2) with respect to ρ on $[0, r]$, we get

$$\int_{|\omega|=1} w(p_0 + r\omega)ds = \int_{|\omega|=1} w(p_0)ds = 2\pi w(p_0), \tag{3.3}$$

that is,

$$w(p_0) = \frac{1}{2\pi} \int_{|\omega|=1} w(p_0 + r\omega) ds = \frac{1}{2\pi r} \int_{\partial B_r(p_0)} w(p) ds. \tag{3.4}$$

□

Lemma 3.2. *Assume (i) $\psi(\rho) \in C[0, r]$ and $A(r) := \int_{B_r(p_0)} \psi(|p - p_0|) dp \neq 0$; (ii) $w(p)$ satisfies the mean value condition in $B_r(p_0)$ and $w(p) \in C(\overline{B}_r(p_0))$. Then,*

$$w(p_0) = \frac{1}{A(r)} \int_{B_r(p_0)} w(p) \psi(|p - p_0|) dp. \tag{3.5}$$

Proof. From (i) and (ii), we have

$$A(r) := \int_{B_r(p_0)} \psi(|p - p_0|) dp = \int_0^r d\rho \int_{\partial B_\rho(p_0)} \psi(|p - p_0|) ds = 2\pi \int_0^r \psi(\rho) \rho \, d\rho, \tag{3.6}$$

$$w(p_0) = \frac{1}{2\pi\rho} \int_{\partial B_\rho(p_0)} w(p) ds. \tag{3.7}$$

Multiplying the both sides of (3.7) by $2\pi\rho\psi(\rho)$ and integrating the result with respect to ρ on $[0, r]$, we have

$$2\pi w(p_0) \int_0^r \psi(\rho) \rho \, d\rho = \int_0^r \left(\int_{\partial B_\rho(p_0)} w(p) \psi(\rho) ds \right) d\rho = \int_{B_r(p_0)} w(p) \psi(|p - p_0|) dp. \tag{3.8}$$

Combining (3.6) and (3.8), we obtain the conclusion. □

Lemma 3.3. *If $w(p) \in C(\Omega)$ satisfies the mean value condition, then, (i) $w(p) \in C^\infty(\Omega)$; (ii) $\Delta w(p) = 0$.*

Proof. (i) Method 1: choose $\varphi(p) \in C_0^\infty(B_1(0))$ with

$$\int_{B_1(0)} \varphi(p) dp = 1, \quad \varphi(p) = \psi(|p|). \tag{3.9}$$

Using integral transform formulas, we have

$$2\pi \int_0^1 r\psi(r) dr = 1. \tag{3.10}$$

Define $\varphi_\varepsilon(p) = (1/\varepsilon^2)\varphi(p/\varepsilon)$, with $\varepsilon < \text{dist}(p, \partial\Omega)$, $p \in \Omega$. Using integral transform formulas and (3.10), we get

$$\begin{aligned}
\int_\Omega w(p)\varphi_\varepsilon(p-p_0)dp &= \frac{1}{\varepsilon^2}\int_{|p|<\varepsilon} w(p_0+p)\varphi\left(\frac{p}{\varepsilon}\right)dp = \int_{|p|<1} w(p_0+\varepsilon p)\varphi(p)dp \\
&= \int_0^1 dr \int_{\partial B_r(p_0)} w(p_0+\varepsilon p)\varphi(p)ds \\
&= \int_0^1 r dr \int_{\partial B_1(p_0)} w(p_0+\varepsilon r\omega)\varphi(r\omega)ds \\
&= \int_0^1 \varphi(r) r dr \int_{|\omega|=1} w(p_0+\varepsilon r\omega)ds = 2\pi w(p_0)\int_0^1 \varphi(r) r dr = w(p_0),
\end{aligned} \tag{3.11}$$

that is,

$$w(p_0) = (\varphi_\varepsilon * w)(p_0), \quad \forall p_0 = (x_0, y_0) \in \Omega_\varepsilon = \{p_0 \mid p_0 \in \Omega,\ d(p_0, \partial\Omega) > \varepsilon\}. \tag{3.12}$$

Applying (3.12) and Proposition 1.3, noticing the arbitrariness of ε, we conclude that $w(p) \in C^\infty(\Omega)$.

Method 2: choose $\varphi(p)$ as above. Define $\varphi_\varepsilon(p) = (1/\varepsilon^2)\varphi(p/\varepsilon)$, with $\varepsilon < \text{dist}(p, \partial\Omega)$, $p \in \Omega$, then,

$$\int_{B_r(p_0)} \varphi_\varepsilon(p)dp = 1. \tag{3.13}$$

Using Lemma 3.2, we obtain

$$w(p_0) = (\varphi_\varepsilon * w)(p_0), \quad \forall p_0 = (x_0, y_0) \in \Omega_\varepsilon = \{p_0 \mid p_0 \in \Omega, d(p_0, \partial\Omega) > \varepsilon\}. \tag{3.14}$$

Applying Proposition 1.3, noticing the arbitrariness of ε, we conclude that $w(p) \in C^\infty(\Omega)$.

(ii) Using (3.1) and Proposition 2.4, we get

$$\int_{B_r(p_0)} \Delta w(p)dp = r\frac{\partial}{\partial r}\int_{|\omega|=1} w(p_0+r\omega)ds = r\frac{\partial}{\partial r}(2\pi w(p_0)) = 0, \quad \forall B_r(p_0) \subset \Omega, \tag{3.15}$$

which implies $\Delta w(p) = 0$, $p = (x, y) \in \Omega$. □

4. Main Results

In this section, we give the main results for the complex functions which satisfy the mean value conditions.

Proposition 4.1. *$f(z)$ satisfies the mean value condition in Ω if and only if $\Delta f(z) = 0$ in Ω.*

Proof. Proposition 2.4, Lemmas 3.1, and 3.3 yield the assertion. □

Theorem 4.2. *$f(z)$ satisfies the mean value condition in Ω and $\partial f(z)/\partial\overline{z} = 0$ if and only if $f(z)$ is analytic in Ω.*

Proof. Denote $f(z) = u(x, y) + iv(x, y)$.

Firstly, we prove the necessary condition. Employing the assumption and Lemma 3.3, we can assert $u(x, y), v(x, y) \in C^\infty(\Omega)$; hence, the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous in Ω. Combining with Cauchy-Riemann equation $\partial f(z)/\partial\overline{z} = 0$, we conclude that $f(z)$ is analytic in Ω.

Secondly, we prove the sufficient condition. The assumption that $f(z)$ is analytic in Ω implies

(i) $u_x(x, y) = v_y(x, y)$, $u_y(x, y) = -v_x(x, y)$, that is, $\partial f(z)/\partial\overline{z} = 0$,

(ii) $\Delta f(z) = 0$. Combining with Proposition 4.1 implies $f(z)$ satisfies the mean value condition in Ω. □

Theorem 4.3. *Suppose that $f(z)$ satisfies the mean value condition in Ω, and $|f(z)|$ is bounded. Then $f(z)$ is a constant in Ω.*

Proof. Since $f(z) = u(x, y) + iv(x, y)$ satisfies the mean value condition in Ω, using Proposition 2.4 and Lemma 3.3, we get $\Delta u = 0$, $\Delta v = 0, (x, y) \in \Omega$.

Since $|f(z)|$ is bounded, we obtain u and v are bounded, respectively, Without loss of generality, we assume that $u \geq 0$. For all $M_0 \in \mathbb{R}^2$, one can choose $B_R(O)$ with $M_0 \in B_R(O)$. Denote $R_0 = d(M_0, O)$. The Harnack inequality (see [11]) implies

$$\frac{R - R_0}{R + R_0} u(O) \leq u(M_0) \leq \frac{R + R_0}{R - R_0} u(O). \tag{4.1}$$

Letting $R \to +\infty$, we conclude $u(M_0) = u(O)$. In the similar way, we conclude $v(M_0) = v(O)$. Since M_0 is arbitrary, we conclude $f(z) = u(x, y) + iv(x, y)$ is a constant in Ω. □

Remark 4.4. This theorem may be proved by the local estimates for harmonic functions too. On the local estimates for harmonic functions, one can see [10].

Theorem 4.5. *Suppose that $f(z)$ satisfies the mean value condition in Ω, and $|f(z)|$ is a constant. Then, $f(z)$ is a constant in Ω.*

Proof. Since $f(z) = u + iv$ satisfies the mean value condition in Ω, using Proposition 2.4 and Lemma 3.3, we obtain

$$\Delta u(x, y) = \Delta v(x, y) = 0. \tag{4.2}$$

Since $|f(z)|$ is a constant, we get the following in Ω

$$u^2(x, y) + v^2(x, y) \equiv \text{constant}. \tag{4.3}$$

From (4.3), we get $2uu_x + 2vv_x = 0$ and

$$u_x^2 + v_x^2 + uu_{xx} + vv_{xx} = 0. \tag{4.4}$$

In the similar way, we have

$$u_y^2 + v_y^2 + uu_{yy} + vv_{yy} = 0. \tag{4.5}$$

Adding (4.4) to (4.5) and noting (4.2), we obtain

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 = 0, \tag{4.6}$$

which implies u and v are constants, that is, $f(z)$ is a constant in Ω. □

Theorem 4.6. *Suppose* (1) $f(z) = u(x,y) + iv(x,y)$ *satisfies the mean value condition in* Ω; (2) $f(z)$ *is continuous on* $\overline{\Omega}$; (3) $f(z)$ *is not a constant. Then,* $\max_{\overline{\Omega}}|f(z)|$ *can be obtained only on the boundary of* Ω.

Proof. Denoting $M = \max_{\overline{\Omega}}|f(z)|$, then, we have $0 < M < +\infty$. Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| = M$. For any $B_\rho(z_0) \subset \Omega$, the mean value condition implies that, for all $z \in \partial B_\rho$, $|f(z)| = M$. Hence $|f(z)|$ is a constant in the neighborhood of M_0. Theorem 4.3 implies $f(z)$ is a constant in this neighborhood of M_0. Applying the circular chain method, we have $f(z)$ is a constant in Ω, which is a contradiction. □

In the following, we present two problems.

Problem 1. Suppose (1) $f(z) = u(x,y) + iv(x,y)$ satisfies the mean value condition in Ω; (2) $f(z)$ is continuous on $\overline{\Omega}$; (3) $f(z)$ is not a constant; (4) for all $z \in \Omega$, $f(z) \neq 0$. Can one confirm that $\min_{\overline{\Omega}}|f(z)|$ is obtained only on the boundary of Ω?

Answer

One can't confirm. For example, $f(z) = 1 + iy$ in $\Omega = \{z \mid |z| < 1\}$. This example shows that the minimal module principle doesn't hold for complex function satisfying mean value condition. But analytic complex function has minimal module principle.

Problem 2. If $f(z)$ satisfies the mean value condition in Ω, can one confirm that $f(z)$ is infinitely differentiable in Ω?

Answer

One can not confirm. For example, $f(z) = 1 + iy$ in $\Omega = \{z \mid |z| < 1\}$ does not satisfy the Cauchy-Riemann equation. This example shows that mean value condition can not imply the differential property of complex function. But analytic complex function is infinitely differentiable.

Acknowledgments

The authors gratefully acknowledge the referees who pointed out the important references and gave some valuable suggestions which improved the results in this paper. Project is supported by the Natural Science Foundation of Shandong Province of China (ZR2011AQ006, ZR2011AM008) and STPF of University in Shandong Province of China (J10LA13, J09LA04).

References

[1] S. Lang, *Complex Analysis*, vol. 103 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 4th edition, 2003.

[2] M. A. Qazi, "The mean value theorem and analytic functions of a complex variable," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 30–38, 2006.

[3] D. la Torre and M. Rocca, "Mean value theorem for continuous vector functions by smooth approximations," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 791–794, 2004.

[4] G. P. Crespi, D. la Torre, and M. Rocca, "Mollified derivatives and second-order optimality conditions," *Journal of Nonlinear and Convex Analysis*, vol. 4, no. 3, pp. 437–454, 2003.

[5] D. la Torre, "Optimality conditions for convex vector functions by mollified derivatives," in *Generalized Convexity and Related Topics*, vol. 583 of *Lecture Notes in Economics and Mathematical Systems*, pp. 327–335, Springer, Berlin, Germany, 2007.

[6] G. P. Crespi, D. la Torre, and M. Rocca, "Second order optimality conditions for nonsmooth multiobjective optimization problems," in *Generalized Convexity, Generalized Monotonicity and Applications*, vol. 77 of *Nonconvex Optimization and Its Applications*, pp. 213–228, Springer, New York, NY, USA, 2005.

[7] A. C. Eberhard and B. S. Mordukhovich, "First-order and second-order optimality conditions for nonsmooth constrained problems via convolution smoothing," *Optimization*, vol. 60, no. 1-2, pp. 253–275, 2011.

[8] A. Eberhard, M. Nyblom, and R. Sivakumaran, "Second order subdifferentials constructed using integral convolutions smoothing," in *Generalized Convexity, Generalized Monotonicity and Applications*, vol. 77 of *Nonconvex Optimization and Its Applications*, pp. 229–261, Springer, New York, NY, USA, 2005.

[9] H. Aimar, I. Gómez, and B. Iaffei, "Parabolic mean values and maximal estimates for gradients of temperatures," *Journal of Functional Analysis*, vol. 255, no. 8, pp. 1939–1956, 2008.

[10] L. C. Evans, *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 1998.

[11] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes, American Mathematical Society, 2000.