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Research Article

On Integral Transforms and Matrix Functions

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The Sumudu transform of certain elementary matrix functions is obtained. These transforms are then used to solve the differential equation of a general linear conservative vibration system, a vibrating system with a special type of viscous damping.

1. Introduction

The importance of matrices and matrix problems in engineering has been clearly demonstrated during the last years [1, 2]. It has been shown that the solution of systems of ordinary and partial differential equations that arise in physics and engineering can be most efficiently formulated in the language of matrices. Boundary value problems become matrix problems after first passing through a reformulation in terms of integral equations. One of the most common problems encountered by the mathematical technologist is the solution of sets of ordinary linear differential equations with constant coefficients. It was found in [3] that the response of a linear dynamical system may be efficiently determined by formulating its response in terms of the matrix exponential function.

In the literature, there are several integral transforms and widely used in physics, astronomy as well as in engineering. In [4], Watugala introduced a new transform and named as Sumudu transform which is defined over the set of the functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0,\ \left| f(t) \right| < M e^{t/\tau_i},\ \text{if } t \in (-1)^i \times [0, \infty) \right\} \tag{1.1}$$

by the following formula:

$$G(u) = S[f(t);u] =: \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2) \tag{1.2}$$

and applied this new transform to the solution of ordinary differential equations and control engineering problems, see [4–6]. In [7], some fundamental properties of the Sumudu transform were established.

In [8], the Sumudu transform was extended to the distributions (generalized functions) and some of their properties were also studied in [9, 10]. Recently, Kılıçman et al. applied this transform to solve the system of differential equations, see [11]. The inversion of the transformed coefficients is obtained by using *Trzaska's* method [12] and the Heaviside expansion technique.

In the present paper, the intimate connection between the Sumudu transform theory and certain matrix functions that arise in the solution of systems of ordinary differential equations is demonstrated. The techniques are developed and then applied to problems in dynamics and electrical transmission lines.

Note that the Sumudu and Laplace transforms have the following relationship that interchanges the image of $\sin(x+t)$ and $\cos(x+t)$. It turns out that

$$\begin{aligned} S_2[\sin(x+t)] &= \mathcal{L}_2[\cos(x+t)] = \frac{u+v}{(1+u)^2(1+v)^2}, \\ S_2[\cos(x+t)] &= \mathcal{L}_2[\sin(x+t)] = \frac{1}{(1+u)^2(1+v)^2}. \end{aligned} \tag{1.3}$$

Further, an interesting fact about the Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except for the factor $n!$. Thus, if size $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then $F(u) = \sum_{n=0}^{\infty} n! a_n t^n$; see [13]. Furthermore, Laplace and Sumudu transforms of the Dirac delta function and the Heaviside function satisfy

$$\begin{aligned} S_2[H(x,t)] &= \mathcal{L}_2[\delta(x,t)] = 1, \\ S_2[\delta(x,t)] &= \mathcal{L}_2[H(x,t)] = \frac{1}{uv}, \end{aligned} \tag{1.4}$$

for details, see [8, 14], where the authors generalize the concept of the Sumudu transform to distributions. Since the Sumudu transform is a convenient tool for solving differential equations in the time domain, without the need for performing an inverse Sumudu transform, see [15]. The applicability of this new interesting transform and efficiency in solving the linear ordinary differential equations with constant and nonconstant coefficients having the convolutions were also studied in [16, 17].

2. Main Results

The following theorem was proved in [5].

Theorem 2.1. *Let $f(x)$ and $g(x)$ be two functions having Sumudu transforms. Then Sumudu transform of the convolution of the $f(x)$ and $g(x)$,*

$$(f * g)(x) = \int_0^x f(\zeta)g(x-\zeta)d\zeta, \tag{2.1}$$

is given by

$$S[(f * g)(x); u] = uF(u)G(u). \tag{2.2}$$

Next, it can be extended to the double convolution as follows.

Theorem 2.2. *Let $f(t,x)$ and $g(t,x)$ have double Sumudu transform. Then, double Sumudu transform of the double convolution of f and g,*

$$(f * *g)(t,x) = \int_0^t \int_0^x f(\zeta,\eta)g(t-\zeta, x-\eta)d\zeta\, d\eta, \tag{2.3}$$

exists and is given by

$$S_2[(f * *g)(t,x); v, u] = uvF(v,u)G(v,u). \tag{2.4}$$

Proof. By using the definition of double Sumudu transform and double convolution, we have

$$\begin{aligned} S_2[(f * *g)(t,x); v, u] &= \frac{1}{uv}\iint_0^\infty e^{-((t/v)+(x/u))}(f * *g)(t,x)dt\, dx \\ &= \frac{1}{uv}\iint_0^\infty e^{((t/v)+(x/u))}\left(\int_0^t\int_0^x f(\zeta,\eta)g(t-\zeta,x-\eta)d\zeta\, d\eta\right)dt\, dx. \end{aligned} \tag{2.5}$$

Let $\alpha = t-\zeta$ and $\beta = x-\eta$, and using the valid extension of upper bound of integrals to $t \to \infty$ and $x \to \infty$, we have

$$S_2[(f * *g)(t,x); v, u] = \frac{1}{uv}\iint_0^\infty e^{-((\zeta/v)-(\eta/u))}d\zeta\, d\eta \int_{-\zeta}^\infty \int_{-\eta}^\infty e^{-((\alpha/v)-(\beta/u))}g(\alpha,\beta)d\alpha\, d\beta. \tag{2.6}$$

Since both functions $f(t,x)$ and $g(t,x)$ are zero, for $t<0$, and $x<0$, it follows with respect to lower limit of integrations that

$$S_2[(f**g)(t,x);v,u]=\frac{1}{uv}\iint_0^\infty e^{-((\zeta/v)-(\eta/u))}d\zeta\,d\eta\iint_0^\infty e^{-((\alpha/v)-(\beta/u))}g(\alpha,\beta)d\alpha\,d\beta. \tag{2.7}$$

Then, it is easy to see that

$$S_2[(f**g)(t,x);v,u]=uvF(v,u)G(v,u). \tag{2.8}$$

See the further details in [14]. □

Mathematical models of many physical biological and economic processes are involved with system of linear constant coefficient of ordinary differential

$$\frac{df}{dx}=Af,\qquad f(0)=I. \tag{2.9}$$

Equation (2.9) was studied in [18] by using by Laplace transform where f and A are square matrices of the nth order, and the elements of A are known constants, and also in control theory A is known as the state of companion matrix. The initial condition satisfied by the matrix $f(x)$ is $f(0)=I$, where I is the nth order unit matrix. It is well known that (2.9) as the solution with the given initial condition is

$$f(x)=\sum_{k=0}^{\infty}\left(\frac{(Ax)^k}{k!}\right)=e^{Ax}, \tag{2.10}$$

where e^{Ax} is the matrix exponential function. To obtain the solution of (2.9) by Sumudu transform, we use the following definition:

$$Sf(x)=\int_0^\infty e^{-x/u}f(x)dx=F(u),\quad \operatorname{Re} u>0 \tag{2.11}$$

and Sumudu transform of derivatives

$$S\left[\frac{df}{dx}\right]=\frac{1}{u}F(u)-\frac{1}{u}f(0)=\frac{1}{u}F(u)-\frac{1}{u}I. \tag{2.12}$$

Then Sumudu transform of (2.9) is, therefore, given by

$$[I-uA]F(u)=I. \tag{2.13}$$

Hence

$$F(u)=\frac{I}{[I-uA]}. \tag{2.14}$$

In the next, we give some applications.

2.1. Resolvent of A

The matrix $F(u) = [I - uA]$ is the characteristic matrix of A. The matrix $Q(u) = (I - uA)^{-1}$ is called the resolvent of A. If λ is the eigenvalue of A with maximum modulus, then we have the geometric progression expansion,

$$Q(u) = (I - uA)^{-1} = \sum_{k=0}^{\infty}\left((Au)^k\right) = F(u), \tag{2.15}$$

provided that, $|u| > |\lambda|$. Sumudu transform variable u may be taken large enough so that $|u| > |\lambda|$ is satisfied in the infinite geometric series in (2.15). The inverse Sumudu transform of (2.15) may now be taken in order to obtain

$$S^{-1}[F(u)] = S^{-1}\left[(I - uA)^{-1}\right] = S^{-1}\left[\sum_{k=0}^{\infty}\left((Au)^k\right)\right] = f(x), \tag{2.16}$$

and, therefore, (2.16) can be written in the form of

$$S^{-1}[F(u)] = f(x) = \sum_{k=0}^{\infty}\left(\frac{(Ax)^k}{k!}\right) = e^{Ax}. \tag{2.17}$$

Thus we have, useful result,

$$S^{-1}\left[(I - uA)^{-1}\right] = e^{Ax} \tag{2.18}$$

as a well-known scalar inverse Sumudu transform,

$$S^{-1}\left[(1 - au)^{-1}\right] = e^{ax}. \tag{2.19}$$

The partial fractional exponential of the resolvent, if $G(A)$ is a rational function of A, then

$$G(A) = G(\lambda_1)L_1 + G(\lambda_2)L_2 + \cdots + G(\lambda_n)L_n, \tag{2.20}$$

where n eigenvalues of $A, \lambda_k,\ k = 1, 2, 3, \ldots, n$ and

$$L_k = \frac{B(\lambda_k)}{\Phi'(\lambda_k)}, \tag{2.21}$$

in (2.21), $B(\lambda)$ and $\Phi(\lambda)$ are defined by

$$B(\lambda) = \operatorname{adj}(I - \lambda A), \qquad \Phi(\lambda) = \det(I - \lambda A), \qquad \Phi'(\lambda_k) = \left(\frac{d\Phi}{d\lambda}\right)_{\lambda=\lambda_k}. \tag{2.22}$$

The matrices $L_k,\ k = 1,2,3,\ldots,n$ are called Sylvester matrices of A. It is also well known that the Sylvester matrices have the following properties:

$$\begin{gathered} L_1 + L_2 + \cdots + L_n = I, \\ L_s L_t = 0 \quad \text{if } s \neq t, \text{ orthogonal.} \end{gathered} \tag{2.23}$$

In order to obtain the partial fraction of resolvent $Q(u) = (I - uA)^{-1}$, we let $G(A) = (I - uA)^{-1}$ in (2.20); we obtain

$$Q(u) = (I - uA)^{-1} = \frac{L_1}{(1 - \lambda_1 u)} + \frac{L_2}{(1 - \lambda_2 u)} + \cdots + \frac{L_n}{(1 - \lambda_n u)}. \tag{2.24}$$

Now by taking the inverse Sumudu transform of (2.24), we have

$$S^{-1}[Q(u)] = L_1 e^{\lambda_1 x} + L_2 e^{\lambda_2 x} + \cdots + L_n e^{\lambda_n x} = e^{Ax}. \tag{2.25}$$

In the next example, we apply inverse Sumudu transform as follows: let

$$F(u) = \frac{1}{I - A^2 u^2}, \tag{2.26}$$

where A nth is order square matrix as defined above, u is the Sumudu transform variable, and I is the nth order unit matrix; by using partial fractional form and inverse Sumudu transform, we have

$$S^{-1}[F(u)] = S^{-1}\left[\frac{1}{2(I - Au)}\right] + S^{-1}\left[\frac{1}{2(I + Au)}\right] = \frac{1}{2}\left(e^{Ax} + e^{-Ax}\right) = \cosh(Ax). \tag{2.27}$$

Another example, consider the case in which $F(u)$ is given by

$$F(u) = \frac{1 + ku}{\left[(I + ku)^2 - A^2 u^2\right]}, \tag{2.28}$$

where u is the Sumudu transform variable, I is the nth order unit matrix, k is scalar, and A nth is order square matrix, by using partial fractional form, we have

$$F(u) = \frac{1}{2}\left[\frac{1}{(I - (A - k)u)} + \frac{1}{(\mathrm{I} + (A + k)u)}\right]. \tag{2.29}$$

The inverse Sumudu transform of (2.29) is

$$S^{-1}[F(u)] = S^{-1}\left[\frac{1}{2}\left[\frac{1}{(I-(A-k)u)} + \frac{1}{(I+(A+k)u)}\right]\right] = e^{-kx}\cosh(Ax), \tag{2.30}$$

where $\cosh(Ax)$ is the matrix hyperbolic cos function of A.

2.2. State-Space Equation

Every linear time-invariant lumped system can be described by a set of equations in the following form:

$$f'(t) = Af(t) + Bv(t),$$
$$g(t) = Cf(t) + Dv(t). \tag{2.31}$$

Then for a system with p inputs, q outputs, and n state variables, A, B, C, and D are, respectively, $n \times n$, $n \times p$, $q \times n$ and $q \times p$ constant matrices. Applying Sumudu transform to (2.31) yields

$$\left[\frac{1}{u}F(u) - \frac{1}{u}f(0)\right] = AF(u) + BV(u),$$
$$G(u) = CF(u) + DV(u), \tag{2.32}$$

where

$$Sf(t) = \int_0^\infty e^{-t/u} f(x)dx = F(u), \qquad \text{Re } u > 0,$$
$$Sv(t) = \int_0^\infty e^{-t/u} v(t)dx = V(u), \tag{2.33}$$
$$Sg(t) = \int_0^\infty e^{-t/u} g(t)dx = G(u).$$

Hence

$$F(u) = \frac{f(0)}{(I-Au)} + \frac{BuV(u)}{(I-Au)}. \tag{2.34}$$

Thus

$$G(u) = \frac{Cf(0)}{(I-Au)} + \frac{CBuV(u)}{(I-Au)} + DV(u). \tag{2.35}$$

On using inverse Sumudu transform for (2.34) and (2.35) and the above theorem, we obtain $f(t)$ and $g(t)$ as follows:

$$f(t) = f(0)e^{At} + B\int_0^t e^{(t-\zeta)}v(\zeta)d\zeta,$$
$$g(t) = Cf(0)e^{At} + BC\int_0^t e^{(t-\zeta)}v(\zeta)d\zeta + Dv(t). \tag{2.36}$$

Now let us apply Sumudu transform to matrix differential equation as follows consider Vibrations of linear conservative system

$$Mf'' + \mu f = g(x), \tag{2.37}$$

where M is a symmetric matrix of order n called the inertia matrix; f is an nth order matrix whose elements are the n generalized coordinates of the system; μ is an nth order symmetric matrix called the stiffness matrix; $g(x)$ is an nth order column matrix of the n generalized forces acting on the system. If we multiply (2.37) by M^{-1}, the inverse of the inertia matrix M, then we have

$$f'' + M^{-1}\mu f = M^{-1}g(x). \tag{2.38}$$

Let the following notation be introduced:

$$M^{-1}\mu = V = A^2, \qquad M^{-1}g(x) = h(x), \tag{2.39}$$

with above notation (2.39) written in the form of

$$f'' + A^2 f = h(x). \tag{2.40}$$

Let Sumudu transform of

$$S[f(x)] = F(u), \qquad S[f''(x)] = \frac{F(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u}, \qquad S[h(x)] = H(u). \tag{2.41}$$

The matrix $f(0)$ is an nth order column matrix whose elements are the initial values of the generalized coordinate; $f'(0)$ is an nth order column matrix whose elements are the initial values of the generalized velocities of the system. The Sumudu transform of (2.40) is given by

$$\frac{F(u)}{u^2} + A^2F(u) = H(u) + \frac{f(0)}{u^2} + \frac{f'(0)}{u}. \tag{2.42}$$

Equation (2.42) can be written in the form of

$$F(u) = \frac{1}{(I + A^2u^2)} \left[u^2 H(u) + f(0) + u f'(0)\right]. \tag{2.43}$$

By using inverse Sumudu transform and convolution for (2.43), we have

$$f(x) = f(0)\cos(Ax) + \frac{1}{A} f'(0)\sin(Ax) + \frac{1}{A}\int_0^t \sin[A(x-t)]h(t)dt. \tag{2.44}$$

2.3. Free Oscillations of the System

If $h(x) = 0$, we have the free oscillations of the conservative system. Since $M^{-1}\mu = V = A^2$, then (2.43) can be written as

$$F(u) = \left(I + Vu^2\right)^{-1}\left[f(0) + u f'(0)\right]. \tag{2.45}$$

Representation of $F(u)$ may be obtained by substituting

$$F(V) = \left(I + Vu^2\right)^{-1}. \tag{2.46}$$

For $F(V)$ Sylvester's theorem (2.20), we have

$$Q(u) = \left(I + \lambda u^2\right)^{-1} = \left[\frac{L_1}{(1+\lambda_1 u^2)} + \frac{L_2}{(1+\lambda_2 u^2)} + \cdots + \frac{L_n}{(1+\lambda_n u^2)}\right]\left[f(0) + u f'(0)\right], \tag{2.47}$$

where L_k is the kth Sylvester's matrix of V and λ_k is the eigenvalue of V. If we let

$$\lambda_k = v_k^2, \quad k = 1, 2, 3, \ldots, n. \tag{2.48}$$

Then (2.47) took the form

$$Q(u) = \left[\frac{L_1}{(1+v_1^2 u^2)} + \frac{L_2}{(1+v_2^2 u^2)} + \cdots + \frac{L_n}{(1+v_n^2 u^2)}\right]\left[f(0) + u f'(0)\right]. \tag{2.49}$$

If we take the inverse Sumudu transform of each term of (2.49), we obtain

$$\begin{aligned} f(x) &= \left[L_1\cos(v_1 x) + L_2\cos(v_2 x) + \cdots + L_n\cos(v_n x)\right] f(0) \\ &\quad + \left[\frac{L_1\sin(v_1 x)}{v_1} + \frac{L_2\sin(v_2 x)}{v_2} + \cdots + \frac{L_n\sin(v_n x)}{v_n}\right] f'(0) \\ &= \cos(Ax) f(0) + A^{-1}\sin(Ax) f'(0), \quad A^2 = V. \end{aligned} \tag{2.50}$$

2.4. Linear Vibrations with Symmetric Damping

The solution of problems involving vibrations of linear systems with viscous damping entails some difficulty because of the presence of complex eigenvalues in the computations. In this part, the vibrations of damped linear systems that exhibit symmetry are considered. The matrix differential equation of motion equation (2.37) takes the form

$$Mf'' + 2Cf' + Kf = g(x). \tag{2.51}$$

The matrix $2C$ is the damping matrix of the system. Let us consider the free oscillations for which $g(x) = 0$. If we follows the same procedure as used above, we may obtain Sumudu transform of (2.51) at $g(x) = 0$ as follows:

$$\left(\frac{M}{u^2} + \frac{2C}{u} + K\right)F(u) = M\left(\frac{f(0)}{u^2} + \frac{f'(0)}{u}\right) + \frac{2Cf(0)}{u}, \tag{2.52}$$

where, as above, Sumudu transforms of $f(x) = F(u)$ and $f(0)$ and $f'(0)$ are the initial displacement and initial velocity vector of the system; let us consider the following cases to the matrix C. (I) If $C = \alpha M$, in this case the matrix C is proportional to the inertia matrix M, where α is scalar constant having the proper dimensions. And multipling the resulting by M^{-1}, (2.51) becomes

$$\left(I + 2\alpha I u + Vu^2\right)F(u) = \left(f(0) + f'(0)u\right) + 2\alpha f(0)u, \quad V = M^{-1}K. \tag{2.53}$$

Now let us define the following identity:

$$\left(I + 2\alpha I + Vu^2\right)F(u) = f(0) + f'(0)u + 2\alpha I u f(0), \quad V = A^2 + \alpha^2 I. \tag{2.54}$$

By using (2.53) and (2.54), we have

$$F(u) = \frac{(1 + \alpha u)f(0)}{\left((I + \alpha u)^2 + A^2u^2\right)} + \frac{u\left(f'(0) + \alpha f(0)\right)}{\left((I + \alpha u)^2 + A^2u^2\right)}. \tag{2.55}$$

On using inverse Sumudu transform for (2.55), we obtain

$$f(x) = e^{-\alpha x}\cos(Ax)f(0) + \frac{1}{A}e^{-\alpha x}\sin(Ax)\left(f'(0) + \alpha f(0)\right). \tag{2.56}$$

(II) If $C = \beta K$, the matrix C is proportional to the stiffness matrix K of the system so that, $C = \beta K$ where β is a scalar constant of proper dimensions. By substituting C in (2.52) and multiplying the results by M^{-1}, we obtain

$$\begin{gathered}\left(I + 2u\beta V + u^2\left(A^2 + \beta^2V^2\right)\right)F(u) = f(0) + f'(0)u + 2\beta Vuf(0), \\ V = M^{-1}K, \qquad A^2 + \beta^2V^2 = V.\end{gathered} \tag{2.57}$$

By simplifying (2.57),

$$F(u) = \frac{(I + \beta V u) f(0)}{\left((I + \beta V u)^2 + A^2 u^2\right)} + \frac{u(\beta V f(0) + f'(0))}{\left((I + \beta V u)^2 + A^2 u^2\right)}. \tag{2.58}$$

On using the inverse Sumudu transform for (2.58), we obtain

$$f(x) = e^{-\alpha V x} \cos(Ax) f(0) + e^{-\alpha V x} \sin(Ax)(\beta V f(0) + f'(0)). \tag{2.59}$$

2.5. Oscillations of the Foucault Pendulum

The use of Sumudu transforms of functions of matrices is demonstrated. As a concrete example, the motion of Foucault's pendulum is considered. The equations of motion for small oscillations of the Foucault pendulum are given by the following system:

$$\begin{aligned} \ddot{x} - 2\eta\dot{y} + \rho^2 x = 0, \\ \ddot{y} + 2\eta\dot{x} + \rho^2 y = 0, \end{aligned} \tag{2.60}$$

where the following notations are used: x is the deflection of the pendulum toward the south, y is the deflection of the pendulum toward the east, $\eta = \omega \sin\theta$, ω is the angular velocity of the earth, and θ is the angle of latitude. Equation (2.60) can be written in the matrix form as follows:

$$I\ddot{f} + 2\eta J\dot{f} + I\rho^2 f = 0, \tag{2.61}$$

where i is second order unit matrix and the coordinate vector has the form

$$f = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.62}$$

where J is matrix 2×2, where $J^2 = -I$. Now by taking Sumudu transform for (2.61), and after arrangement, we have

$$\left(I + 2\eta J u + I\rho^2 u^2\right) F(u) = (I + 2\eta J u) f(0) + I u f'(0), \tag{2.63}$$

where $f(0)$ and $f'(0)$ represent the initial coordinate and initial velocity vector, respectively; in order to use inverse Sumudu transform, we need the following identity:

$$\begin{aligned} \left(I + 2\eta J u + I\rho^2 u^2\right) &= (I + \eta J u)^2 + A^2 u^2 \\ &= I + 2\eta J u + (\eta J u)^2 + A^2 u^2, \end{aligned} \tag{2.64}$$

where

$$(\eta J u)^2 + A^2 u^2 = \left(A^2 - \eta^2 I\right) u^2 = I\rho^2 u^2. \tag{2.65}$$

Therefore, we obtain

$$A^2 u^2 = \left(\eta^2 + \rho^2\right) I u^2 = I B^2 u^2. \tag{2.66}$$

On using the above identity (2.63), it becomes

$$F(u) = \frac{(I + \eta J u) f(0)}{\left((I + \eta J u)^2 + I B^2 u^2\right)} + \frac{(\eta J f(0) + I f'(0)) u}{\left((I + \eta J u)^2 + I B^2 u^2\right)}. \tag{2.67}$$

By taking inverse Sumudu transform for (2.67), we obtain

$$f(t) = e^{-\eta J t} \cos(IBt) f(0) + \frac{e^{-\eta J t}}{B} \sin(IBt)\left(\eta J f(0) + I f'(0)\right). \tag{2.68}$$

Thus consider the following system:

$$\begin{aligned} a_{11} y_1' + b_{11} y_1 + a_{12} y_2' + b_{12} y_2 &= f_1(t), \\ a_{21} y_1' + b_{21} y_1 + a_{22} y_2' + b_{22} y_2 &= f_2(t). \end{aligned} \tag{2.69}$$

We presume the existence of the limits of the excitations as $t \to +0$, $f_1(0^+)$ and $f_2(0^+)$ deferring further specifications concerning these functions. Let the system be anomalous; that is,

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{2.70}$$

Initial values y_1^0 and y_2^0 of y_1 and y_2 are given as limits as $t \to +0$:

$$y_1(0^+) = y_1^0, \qquad y_2(0^+) = y_2^0. \tag{2.71}$$

Because of (2.70), we can eliminate y_1' and y_2' from (2.69). Since (2.69) represents a system of differential equations, at least one of the coefficients a_{ik} must have a nonzero value; without loss of generality, let $a_{11} \neq 0$. To accomplish the attempted elimination, multiply the first equation by a_{21} and the second equation by a_{11}, and then subtract the first from the second. With

$$B = \begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix}, \qquad C = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}, \tag{2.72}$$

thus we can write the result compactly

$$By_1(t) + Cy_2(t) = -a_{21}f_1(t) + a_{11}f_2(t). \tag{2.73}$$

The compatibility condition is obtained from (2.73), by the limiting process $t \to +0$

$$By_1^0 + Cy_2^0 = -a_{21}f_1(0^+) + a_{11}f_2(0^+). \tag{2.74}$$

If not only the determinant A but also the determinants B and C are each zero, then we must conclude that the coefficients of the second equation (2.69) are fixed multiples of the coefficients of the first equation of (2.69). In this case, either the second equation is equivalent to the first if f_2 too is the same fixed multiple of f_1, or else the equations would contradict one another. Hence, B and C cannot both be zero. Now we apply the Sumudu transformation to the system (2.69); we obtain

$$\begin{aligned}&(a_{11} + b_{11}u)Y_1(u) + (a_{12} + b_{12}u)Y_2(u) = uF_1(u) + a_{11}y_1^0 + a_{11}y_2^0,\\&(a_{21} + b_{21}u)Y_1(u) + (a_{22} + b_{22}u)Y_2(u) = uF_2(u) + a_{21}y_1^0 + a_{22}y_2^0.\end{aligned} \tag{2.75}$$

With (2.70) and (2.72), we introduce short notations for three determinants of the matrix of coefficients of (2.69), here as follows:

$$D = \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}, \qquad E = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}, \qquad G = \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix}. \tag{2.76}$$

On using the notation of (2.76), we obtain

$$\Delta(u) = (C + D) + E \cdot u. \tag{2.77}$$

Then (2.77) follows,

$$\Delta(u)Y_1(u) = \begin{vmatrix} uF_1(u) & a_{12} + b_{12}u \\ uF_2(u) & a_{22} + b_{22}u \end{vmatrix} + \begin{vmatrix} a_{11}y_1^0 + a_{12}y_2^0 & a_{12} + b_{12}u \\ a_{21}y_1^0 + a_{22}y_2^0 & a_{22} + b_{22}u \end{vmatrix}. \tag{2.78}$$

Since $A = 0$, then

$$\begin{aligned}&\Delta(u)Y_1(u) = uF_1(u)(a_{22} + b_{22}u) - uF_2(u)(a_{12} + b_{12}u) + Cy_1^0 + Gy_2^0,\\&\Delta(u)Y_2(u) = -uF_1(u)(a_{21} + b_{21}u) + uF_2(u)(a_{11} + b_{11}u) - By_1^0 + Dy_2^0.\end{aligned} \tag{2.79}$$

For brevity, we set

$$C + D = H. \tag{2.80}$$

We presume here $H \neq 0$, that is, $\Delta(u)$ is a linear function; we divide (2.79) by the coefficient $\Delta(u)$; then we have

$$Y_1(u) = \frac{uF_1(u)(a_{22}+b_{22}u)}{H+Eu} - \frac{uF_2(u)(a_{12}+b_{12}u)}{H+Eu} + \frac{Cy_1^0}{H+Eu} + \frac{Gy_2^0}{H+Eu}, \tag{2.81}$$

$$\Delta(u)Y_2(u) = -\frac{uF_1(u)(a_{21}+b_{21}u)}{H+Eu} + \frac{uF_2(u)(a_{11}+b_{11}u)}{H+Eu} - \frac{By_1^0}{H+Eu} + \frac{Dy_2^0}{H+Eu}. \tag{2.82}$$

The first term of (2.81) can be modified as follows:

$$\frac{uF_1(u)(a_{22}+b_{22}u)}{H+Eu} = uF_1(u)\left[\frac{b_{22}}{E} + \frac{\Gamma}{1+(E/H)u}\right], \quad \text{where } \Gamma = \frac{a_{22}}{H} - \frac{b_{22}}{E}. \tag{2.83}$$

The inverse Sumudu transform of above function is given by

$$S^{-1}\left[uF_1(u)\left[\frac{b_{22}}{E} + \frac{\Gamma}{I+(E/H)u}\right]\right] = I * \frac{b_{22}}{E} f_1(t) + f_1(t) * \Gamma e^{-(E/H)t}. \tag{2.84}$$

The rest terms of (2.81) are similarly modified. Then we obtain the solution of (2.81) as

$$\begin{aligned} y_1(t) = {} & I * \frac{b_{22}}{E} f_1(t) + f_1(t) * \Gamma e^{-(E/H)t} - I * \frac{b_{12}}{E} f_2(t) - f_2(t) * \Psi e^{-(E/H)t} \\ & + \frac{C}{H} y_1^0 e^{-(E/H)t} + \frac{G}{H} y_2^0 e^{-(E/H)t}, \end{aligned} \tag{2.85}$$

where $\Psi = (a_{12}/H) - (b_{12}/E)$; similarly one can find the solution of (2.82)

$$\begin{aligned} y_2(t) = {} & -I * \frac{b_{21}}{E} f_1(t) - f_1(t) * \Phi e^{-(E/H)t} + I * \frac{b_{11}}{E} f_2(t) + f_2(t) * \Omega e^{-(E/H)t} \\ & - \frac{B}{H} y_1^0 e^{-(E/H)t} + \frac{D}{H} y_2^0 e^{-(E/H)t}, \end{aligned} \tag{2.86}$$

where $\Phi = (b_{21}/H) - (a_{21}/E)$ and $\Omega = (b_{11}/H) - (a_{11}/E)$.

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