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Research Article

The Local Strong and Weak Solutions for a Nonlinear Dissipative Camassa-Holm Equation

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Using the Kato theorem for abstract differential equations, the local well-posedness of the solution for a nonlinear dissipative Camassa-Holm equation is established in space $C([0,T), H^s(R)) \cap C^1([0,T), H^{s-1}(R))$ with $s > 3/2$. In addition, a sufficient condition for the existence of weak solutions of the equation in lower order Sobolev space $H^s(R)$ with $1 \leq s \leq 3/2$ is developed.

1. Introduction

Camassa and Holm [1] used the Hamiltonian method to derive a completely integrable wave equation

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. Its alternative derivation as a model for water waves can be found in Constantin and Lannes [2] and Johnson [3]. Equation (1.1) also models wave current interaction [4], while Dai [5] derived it as a model in elasticity (see Constantin and Strauss [6]). Moreover, it was pointed out in Lakshmanan [7] that the Camassa-Holm equation (1.1) could be relevant to the modeling of tsunami waves (see Constantin and Johnson [8]).

In fact, a huge amount of work has been carried out to investigate the dynamic properties of (1.1). For $k = 0$, (1.1) has traveling wave solutions of the form $c\,e^{-|x-ct|}$, called peakons, which capture the main feature of the exact traveling wave solutions of greatest height of the governing equations (see [9–11]). For $k > 0$, its solitary waves are stable solitons [6, 11]. It was shown in [12–14] that the inverse spectral or scattering approach was a powerful tool to handle Camassa-Holm equation. Equation (1.1) is a completely integrable infinite-dimensional Hamiltonian system (in the sense that for a large class of initial data, the flow is equivalent to a linear flow at constant speed [15]). It should be emphasized that (1.1) gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group (see [16, 17]), and this geometric illustration leads to a proof that the Least Action Principle holds. It is worthwhile to mention that Xin and Zhang [18] proved that the global existence of the weak solution in the energy space $H^1(R)$ without any sign conditions on the initial value, and the uniqueness of this weak solution is obtained under some conditions on the solution [19]. Coclite et al. [20] extended the analysis presented in [18, 19] and obtained many useful dynamic properties to other equations (also see [21–24]). Li and Olver [25] established the local well-posedness in the Sobolev space $H^s(R)$ with $s > 3/2$ for (1.1) and gave conditions on the initial data that lead to finite time blowup of certain solutions. It was shown in Constantin and Escher [26] that the blowup occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. After wave breaking, the solution can be continued uniquely either as a global conservative weak solution [21] or a global dissipative solution [22]. For peakons, these possibilities are explicitly illustrated in the paper [27]. For other methods to handle the problems relating to various dynamic properties of the Camassa-Holm equation and other shallow water models, the reader is referred to [10, 28–32] and the references therein.

In this paper, motivated by the work in [25, 33], we study the following generalized Camassa-Holm equation

$$u_t - u_{txx} + 2ku_x + au^m u_x = 2u_x u_{xx} + uu_{xxx} + \beta\partial_x\left[(u_x)^N\right], \tag{1.2}$$

where $m \geq 1$ and $N \geq 1$ are natural numbers, and a, k, and β are arbitrary constants. Obviously, (1.2) reduces to (1.1) if we set $a = 3$, $m = 1$, and $\beta = 0$. Actually, Wu and Yin [34] consider a nonlinearly dissipative Camassa-Holm equation which includes a nonlinearly dissipative term $L(u)$, where L is a differential operator or a quasidifferential operator. Therefore, we can regard the term $\beta\partial_x[(u_x)^N]$ as a nonlinearly dissipative term for the dissipative Camassa-Holm equation (1.2).

Due to the term $\beta\partial_x[(u_x)^N]$ in (1.2), the conservation laws in previous works [10, 25] for (1.1) lose their powers to obtain some bounded estimates of the solution for (1.2). A new conservation law different from those presented in [10, 25] will be established to prove the local existence and uniqueness of the solution to (2.3) subject to initial value $u_0(x) \in H^s(R)$ with $s > 3/2$. We should address that all the generalized versions of the Camassa-Holm equation in previous works (see [17, 25, 34]) do not involve the nonlinear term $\partial_x[(u_x)^N]$. Lai and Wu [33] only studied a generalized Camassa-Holm equation in the case where $\beta \geq 0$ and N is an odd number. Namely, (1.2) with $\beta < 0$ and arbitrary positive integer N was not investigated in [33].

The main tasks of this paper are two-fold. Firstly, by using the Kato theorem for abstract differential equations, we establish the local existence and uniqueness of solutions for (1.2) with any β and arbitrary positive integer N in space $C([0,T), H^s(R)) \bigcap C^1([0,T), H^{s-1}(R))$ with $s > 3/2$. Secondly, it is shown that the existence of weak solutions in lower order Sobolev space $H^s(R)$ with $1 \leq s \leq 3/2$. The ideas of proving the second result come from those presented in Li and Olver [25].

2. Main Results

Firstly, we give some notation.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times R$ is denoted by C_0^∞. $L^p = L^p(R)$ $(1 \le p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1+|\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi\right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1-\partial_x^2)^{1/2}$.

In order to study the existence of solutions for (1.2), we consider its Cauchy problem in the form

$$\begin{aligned} u_t - u_{txx} &= -2ku_x - \frac{a}{m+1}\left(u^{m+1}\right)_x + 2u_x u_{xx} + uu_{xxx} + \beta\partial_x\left[(u_x)^N\right] \\ &= -ku_x - \frac{a}{m+1}\left(u^{m+1}\right)_x + \frac{1}{2}\partial_x^3 u^2 - \frac{1}{2}\partial_x\left(u_x^2\right) + \beta\partial_x\left[(u_x)^N\right], \\ u(0,x) &= u_0(x), \end{aligned} \tag{2.2}$$

which is equivalent to

$$\begin{gathered} u_t + uu_x = \Lambda^{-2}\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x + \Lambda^{-2}(uu_x) - \frac{1}{2}\Lambda^{-2}\partial_x\left(u_x^2\right) + \beta\Lambda^{-2}\partial_x\left[(u_x)^N\right], \\ u(0,x) = u_0(x). \end{gathered} \tag{2.3}$$

Now, we state our main results.

Theorem 2.1. *Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then problem* (2.2) *or problem* (2.3) *has a unique solution $u(t,x) \in C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$ where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.*

Theorem 2.2. *Suppose that $u_0(x) \in H^s$ with $1 \le s \le 3/2$ and $\|u_{0x}\|_{L^\infty} < \infty$. Then there exists a $T > 0$ such that* (1.2) *subject to initial value $u_0(x)$ has a weak solution $u(t,x) \in L^2([0,T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0,T] \times R)$.*

3. Local Well-Posedness

We consider the abstract quasilinear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v), \quad t \ge 0, \ v(0) = v_0. \tag{3.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ be the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. We state the following conditions in which $\rho_1,\ \rho_2,\ \rho_3$, and ρ_4 are constants depending on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\left\|(A(y) - A(z))w\right\|_X \leq \rho_1 \left\|y - z\right\|_X \|w\|_Y, \quad y, z, w \in Y, \tag{3.2}$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left\|(B(y) - B(z))w\right\|_X \leq \rho_2 \left\|y - z\right\|_Y \|w\|_X, \quad y, z \in Y,\ w \in X. \tag{3.3}$$

(iii) $f : Y \to Y$ extends to a map from X into X is bounded on bounded sets in Y, and satisfies

$$\begin{aligned} \left\|f(y) - f(z)\right\|_Y &\leq \rho_3 \left\|y - z\right\|_Y, \quad y, z \in Y, \\ \left\|f(y) - f(z)\right\|_X &\leq \rho_4 \left\|y - z\right\|_X, \quad y, z \in Y. \end{aligned} \tag{3.4}$$

Kato Theorem (see [35])

Assume that (i), (ii), and (iii) hold. If $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to problem (3.1) such that

$$v = v(\cdot, v_0) \in C([0, T); Y) \bigcap C^1([0, T); X). \tag{3.5}$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is a continuous map from Y to the space

$$C([0, T); Y) \bigcap C^1([0, T); X). \tag{3.6}$$

For problem (2.3), we set $A(u) = u\partial_x$, $Y = H^s(R)$, $X = H^{s-1}(R)$, $\Lambda = (1 - \partial_x^2)^{1/2}$,

$$f(u) = \Lambda^{-2}\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x + \Lambda^{-2}(uu_x) - \frac{1}{2}\Lambda^{-2}\partial_x\left(u_x^2\right) + \beta\Lambda^{-2}\partial_x\left[(u_x)^N\right], \tag{3.7}$$

and $Q = \Lambda$. In order to prove Theorem 2.1, we only need to check that $A(u)$ and $f(u)$ satisfy assumptions (i)–(iii).

Lemma 3.1. *The operator $A(u) = u\partial_x$ with $u \in H^s(R)$, $s > 3/2$ belongs to $G(H^{s-1}, 1, \beta)$.*

Lemma 3.2. *Let $A(u) = u\partial_x$ with $u \in H^s$ and $s > 3/2$. Then $A(u) \in L(H^s, H^{s-1})$ for all $u \in H^s$. Moreover,*

$$\|(A(u) - A(z))w\|_{H^{s-1}} \le \rho_1 \|u - z\|_{H^{s-1}} \|w\|_{H^s}, \quad u, z, w \in H^s(R). \tag{3.8}$$

Lemma 3.3. *For $s > 3/2$, $u, z \in H^s$ and $w \in H^{s-1}$, it holds that $B(u) = [\Lambda, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s$ and*

$$\|(B(u) - B(z))w\|_{H^{s-1}} \le \rho_2 \|u - z\|_{H^s} \|w\|_{H^{s-1}}. \tag{3.9}$$

Proofs of the above Lemmas 3.1–3.3 can be found in [29] or [31].

Lemma 3.4 (see [35]). *Let r and q be real numbers such that $-r < q \le r$. Then*

$$\begin{aligned} \|uv\|_{H^q} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ \|uv\|_{H^{r+q-1/2}} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}. \end{aligned} \tag{3.10}$$

Lemma 3.5. *Let $u, z \in H^s$ with $s > 3/2$, then $f(u)$ is bounded on bounded sets in H^s and satisfies*

$$\|f(u) - f(z)\|_{H^s} \le \rho_3 \|u - z\|_{H^s}, \tag{3.11}$$

$$\|f(u) - f(z)\|_{H^{s-1}} \le \rho_4 \|u - z\|_{H^{s-1}}. \tag{3.12}$$

Proof. Using the algebra property of the space H^{s_0} with $s_0 > 1/2$, we have

$$\begin{aligned} &\|f(u) - f(z)\|_{H^s} \\ &\quad \le c\left[\left\|\Lambda^{-2}\left(\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x - \left[-kz - \frac{a}{m+1}\left(z^{m+1}\right)\right]_x\right)\right\|_{H^s}\right. \\ &\qquad \left. + \left\|\Lambda^{-2}(uu_x - zz_x)\right\|_{H^s} + \left\|\Lambda^{-2}\partial_x\left(u_x^2 - z_x^2\right)\right\|_{H^s} + \left\|\Lambda^{-2}\partial_x\left[(u_x)^N\right] - \Lambda^{-2}\partial_x\left[(z_x)^N\right]\ \right\|_{H^s}\right] \\ &\quad \le c\left[\|u - z\|_{H^{s-1}} + \left\|u^{m+1} - z^{m+1}\right\|_{H^{s-1}} + \|uu_x - zz_x\|_{H^{s-1}} + \left\|u_x^2 - z_x^2\right\|_{H^{s-1}}\right. \\ &\qquad \left. + \left\|(u_x)^N - (z_x)^N\right\|_{H^{s-1}}\right] \\ &\quad \le c\|u - z\|_{H^s}\left[1 + \sum_{j=0}^{m} \|u\|_{H^s}^{m-j}\|z\|_{H^s}^{j} + \|u\|_{H^s} + \|z\|_{H^s} + \sum_{j=0}^{N-1} \|u_x\|_{H^{s-1}}^{N-j}\|z_x\|_{H^{s-1}}^{j}\right] \\ &\quad \le \rho_3\|u - z\|_{H^s}, \end{aligned} \tag{3.13}$$

from which we obtain (3.11).

Applying Lemma 3.4, $uu_x = (1/2)(u^2)_x$, $s > 3/2$, $\|u\|_{L^\infty} \le c\|u\|_{H^{s-1}}$ and $\|u_x\|_{L^\infty} \le c\|u\|_{H^s}$, we get

$$\begin{aligned}
&\|f(u) - f(z)\|_{H^{s-1}} \\
&\quad \le c\left[\|u-z\|_{H^{s-2}} + \left\|u^{m+1} - z^{m+1}\right\|_{H^{s-2}} + \left\|u^2 - z^2\right\|_{H^{s-2}}\right. \\
&\qquad \left. + \|(u_x - z_x)(u_x + z_x)\|_{H^{s-2}} + \left\|(u_x - z_x)\sum_{j=0}^{N-1} u_x^{N-1-j} z_x^j\right\|_{H^{s-2}}\right] \\
&\quad \le c\|u-z\|_{H^{s-1}}\left[1 + \sum_{j=0}^{m} \|u\|_{H^{s-1}}^{m-j}\|z\|_{H^{s-1}}^{j} + \|u\|_{H^{s-1}} + \|z\|_{H^{s-1}}\right. \\
&\qquad \left. + \|u\|_{H^s} + \|z\|_{H^s} + \sum_{j=0}^{N-1} \|u_x\|_{H^{s-1}}^{N-j}\|z_x\|_{H^{s-1}}^{j}\right] \\
&\quad \le \rho_4\|u-z\|_{H^{s-1}},
\end{aligned} \tag{3.14}$$

which completes the proof of (3.12). □

Proof of Theorem 2.1. Using the Kato Theorem, Lemmas 3.1–3.3, and 3.5, we know that system (2.2) or problem (2.3) has a unique solution

$$u(t,x) \in C([0,T); H^s(R)) \bigcap C^1\left([0,T); H^{s-1}(R)\right). \tag{3.15}$$

□

4. Existence of Weak Solutions

For $s \ge 2$, using the first equation of system (2.2) derives

$$\frac{d}{dt}\int_R \left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1} d\tau\right) dx = 0, \tag{4.1}$$

from which we have the conservation law

$$\int_R \left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1} d\tau\right) dx = \int_R \left(u_0^2 + u_{0x}^2\right) dx. \tag{4.2}$$

Lemma 4.1 (Kato and Ponce [36]). *If $r > 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_r \le c(\|u\|_{L^\infty}\|v\|_r + \|u\|_r\|v\|_{L^\infty}), \tag{4.3}$$

where c is a constant depending only on r.

Lemma 4.2 (Kato and Ponce [36]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \leq c\left(\|\partial_x u\|_{L^\infty} \left\|\Lambda^{r-1} v\right\|_{L^2} + \|\Lambda^r u\|_{L^2} \|v\|_{L^\infty}\right). \tag{4.4}$$

Lemma 4.3. *Let $s \geq 2$ and the function $u(t,x)$ is a solution of problem (2.2) and the initial data $u_0(x) \in H^s(R)$. Then the following inequality holds*

$$\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq \|u_0\|_{H^1} e^{|\beta| \int_0^t \|u_x\|_{L^\infty}^{N-1} d\tau}. \tag{4.5}$$

For $q \in (0, s-1]$, there is a constant c, which only depends on m, N, k, a, and β, such that

$$\begin{aligned} \int_R \left(\Lambda^{q+1} u\right)^2 dx \leq \int_R \left(\Lambda^{q+1} u_0\right)^2 dx + c \int_0^t \|u_x\|_{L^\infty} \|u\|_{H^{q+1}}^2 \left(1 + \|u\|_{L^\infty}^{m-1}\right) d\tau \\ + c \int_0^t \|u\|_{H^{q+1}}^2 \|u_x\|_{L^\infty}^{N-1} d\tau. \end{aligned} \tag{4.6}$$

For $q \in [0, s-1]$, there is a constant c, which only depends on m, N, k, a, and β, such that

$$\|u_t\|_{H^q} \leq c\|u\|_{H^{q+1}} \left(1 + \left(1 + \|u\|_{L^\infty}^{m-1}\right) \|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.7}$$

Proof. Using $\|u\|_{H^1}^2 = \int_R (u^2 + u_x^2) dx$ and (4.2) derives (4.5).

Using $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_R \Lambda^q u \Lambda^q \partial_x^3 \left(u^2\right) dx = -2 \int_R \left(\Lambda^{q+1} u\right) \Lambda^{q+1} (u u_x) dx + 2 \int_R (\Lambda^q u) \Lambda^q (u u_x) dx. \tag{4.8}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (2.3) and integrating with respect to x by parts, we have the identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2\right) dx = -a \int_R (\Lambda^q u) \Lambda^q (u^m u_x) dx \\ - \int_R \left(\Lambda^{q+1} u\right) \Lambda^{q+1} (u u_x) dx + \frac{1}{2} \int_R (\Lambda^q u_x) \Lambda^q \left(u_x^2\right) dx \\ + \int_R (\Lambda^q u) \Lambda^q (u u_x) dx - \beta \int_R \Lambda^q u_x \Lambda^q \left[(u_x)^N\right] dx. \end{aligned} \tag{4.9}$$

We will estimate the terms on the right-hand side of (4.9) separately. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\int_R (\Lambda^q u)\Lambda^q(u^m u_x)dx &= \int_R (\Lambda^q u)[\Lambda^q(u^m u_x) - u^m\Lambda^q u_x]dx + \int_R (\Lambda^q u)u^m\Lambda^q u_x dx \\ &\le c\|u\|_{H^q}\left(m\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{m-1}\|u\|_{H^q}\right) \\ &\quad + \frac{1}{2}\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \\ &\le c\|u\|_{H^q}^2\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}.\end{aligned} \tag{4.10}$$

Using the above estimate to the second term yields

$$\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx \le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}. \tag{4.11}$$

For the third term, using the Cauchy-Schwartz inequality and Lemma 4.1, we obtain

$$\begin{aligned}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx &\le \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q\left(u_x^2\right)\right\|_{L^2} \\ &\le c\|u\|_{H^{q+1}}(\|u_x\|_{L^\infty}\|u_x\|_{H^q} + \|u_x\|_{L^\infty}\|u_x\|_{H^q}) \\ &\le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}.\end{aligned} \tag{4.12}$$

For the last term in (4.9), using Lemma 4.1 repeatedly results in

$$\begin{aligned}\left|\int_R (\Lambda^q u_x)\Lambda^q(u_x)^N dx\right| &\le \|u_x\|_{H^q}\left\|u_x^N\right\|_{H^q} \\ &\le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}.\end{aligned} \tag{4.13}$$

It follows from (4.9) to (4.13) that there exists a constant c depending only on m, N and the coefficients of (1.2) such that

$$\frac{1}{2}\frac{d}{dt}\int_R \left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2\right]dx \le c\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2\left(1 + \|u\|_{L^\infty}^{m-1}\right) + c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}. \tag{4.14}$$

Integrating both sides of the above inequality with respect to t results in inequality (4.6).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (2.3) to obtain the equation

$$u_t = \left(1-\partial_x^2\right)^{-1}\left[-2ku_x + \partial_x\left(-\frac{a}{m+1}u^{m+1} + \frac{1}{2}\partial_x^2\left(u^2\right) - \frac{1}{2}u_x^2\right) + \beta\partial_x\left[(u_x)^N\right]\right]. \tag{4.15}$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (4.15) for $q \in (0, s-1]$ gives rise to

$$\int_R (\Lambda^q u_t)^2 dx = \int_R (\Lambda^q u_t)\Lambda^{q-2}\left[\partial_x\left(-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{2}\partial_x^2\left(u^2\right) - \frac{1}{2}u_x^2\right) + \beta\partial_x\left[(u_x)^N\right]\right]d\tau. \tag{4.16}$$

For the right-hand side of (4.16), we have

$$\begin{aligned}
&\int_R (\Lambda^q u_t)\Lambda^{q-2}(-2ku_x)dx \le c\|u_t\|_{H^q}\|u\|_{H^q},\\
&\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(-\frac{a}{m+1}u^{m+1} - \frac{1}{2}u_x^2\right)dx\\
&\quad\le c\|u_t\|_{H^q}\left(\int_R\left(1+\xi^2\right)^{q-1} \times \left[\int_R\left[-\frac{a}{m+1}\widehat{u^m}(\xi-\eta)\widehat{u}(\eta) - \frac{1}{2}\widehat{u_x}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2}\\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\left(1+\|u\|_{L^\infty}^{m-1}\right).
\end{aligned} \tag{4.17}$$

Since

$$\int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2(uu_x)dx = -\int(\Lambda^q u_t)\Lambda^q(uu_x)dx + \int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx, \tag{4.18}$$

using Lemma 4.1, $\|uu_x\|_{H^q} \le c\|(u^2)_x\|_{H^q} \le c\|u\|_{L^\infty}\|u\|_{H^{q+1}}$ and $\|u\|_{L^\infty} \le \|u\|_{H^1}$, we have

$$\begin{aligned}
\int(\Lambda^q u_t)\Lambda^q(uu_x)dx &\le c\|u_t\|_{H^q}\|uu_x\|_{H^q}\\
&\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}},\\
\int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx &\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}.
\end{aligned} \tag{4.19}$$

Using the Cauchy-Schwartz inequality and Lemma 4.1 yields

$$\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(u_x^N\right)dx\right| \le c\|u_t\|_{H^q}\|u_x\|_{L^\infty}^{N-1}\|u\|_{H^{q+1}}. \tag{4.20}$$

Applying (4.17)–(4.20) into (4.16) yields the inequality

$$\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(1+\left(1+\|u\|_{L^\infty}^{m-1}\right)\|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.21}$$

This completes the proof of Lemma 4.3. □

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{4.22}$$

and setting $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s(R)$ and $s > 0$.

It follows from Theorem 2.1 that for each ε the Cauchy problem

$$\begin{gathered} u_t - u_{txx} = \partial_x\left(-2ku - \frac{a}{m+1}u^{m+1}\right) + \frac{1}{2}\partial_x^3\left(u^2\right) - \frac{1}{2}\partial_x\left(u_x^2\right) + \beta\partial_x\left[(u_x)^N\right], \\ u(0,x) = u_{\varepsilon 0}(x), \quad x \in R, \end{gathered} \tag{4.23}$$

has a unique solution $u_\varepsilon(t,x) \in C^\infty([0,T); H^\infty)$.

Lemma 4.4. *Under the assumptions of problem* (4.23)*, the following estimates hold for any* ε *with* $0 < \varepsilon < 1/4$ *and* $s > 0$

$$\begin{gathered} \|u_{\varepsilon 0x}\|_{L^\infty} \leq c_1\|u_{0x}\|_{L^\infty}, \\ \|u_{\varepsilon 0}\|_{H^q} \leq c_1, \quad \text{if } q \leq s, \\ \|u_{\varepsilon 0}\|_{H^q} \leq c_1\varepsilon^{(s-q)/4}, \quad \text{if } q > s, \\ \|u_{\varepsilon 0} - u_0\|_{H^q} \leq c_1\varepsilon^{(s-q)/4}, \quad \text{if } q \leq s, \\ \|u_{\varepsilon 0} - u_0\|_{H^s} = o(1), \end{gathered} \tag{4.24}$$

where c_1 *is a constant independent of* ε.

The proof of this Lemma can be found in Lai and Wu [33].

Lemma 4.5. *If* $u_0(x) \in H^s(R)$ *with* $s \in [1, 3/2]$ *such that* $\|u_{0x}\|_{L^\infty} < \infty$*. Let* $u_{\varepsilon 0}$ *be defined as in system* (4.23)*. Then there exist two positive constants* T *and* c*, which are independent of* ε*, such that the solution* u_ε *of problem* (4.23) *satisfies* $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ *for any* $t \in [0,T)$.

Proof. Using notation $u = u_\varepsilon$ and differentiating both sides of the first equation of problem (4.23) or (4.15) with respect to x give rise to

$$\begin{aligned} u_{tx} + \frac{1}{2}\partial_x^2 u^2 - \frac{1}{2}u_x^2 = 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{2}u^2 - \beta u_x^N \\ - \Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{2}u^2 + \frac{1}{2}u_x^2 - \beta u_x^N\right]. \end{aligned} \tag{4.25}$$

Letting $p > 0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\begin{aligned}
&\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx + \frac{p}{2p+2}\int_R (u_x)^{2p+3}dx \\
&\quad= \int_R (u_x)^{2p+1}\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{2}u^2 - \beta u_x^N\right)dx \\
&\quad- \int_R (u_x)^{2p+1}\Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]dx.
\end{aligned} \tag{4.26}$$

Applying the Hölder's inequality yields

$$\begin{aligned}
\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx \le{}& \left\{|2k|\left(\int_R |u|^{2p+2}dx\right)^{1/(2p+2)} + \frac{a}{m+1}\left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)}\right. \\
&+ \frac{1}{2}\left(\int_R \left|u^2\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} \\
&\left.+\left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\}\left(\int_R |u_x|^{2p+2}dx\right)^{(2p+1)/(2p+2)} \\
&+ \frac{p}{2p+2}\|u_x\|_{L^\infty}\int_R |u_x|^{2p+2}dx,
\end{aligned} \tag{4.27}$$

or

$$\begin{aligned}
\frac{d}{dt}\left(\int_R (u_x)^{2p+2}dx\right)^{1/(2p+2)} \le{}& |2k|\left(\int_R |u|^{2p+2}dx\right)^{1/(2p+2)} + \frac{a}{m+1}\left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)} \\
&+ \frac{1}{2}\left(\int_R \left|u^2\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} \\
&+ \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)} + \frac{p}{2p+2}\|u_x\|_{L^\infty}\left(\int_R |u_x|^{2p+2}dx\right)^{1/(2p+2)},
\end{aligned} \tag{4.28}$$

where

$$G = \Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]. \tag{4.29}$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating both sides of the inequality (4.28) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + \int_0^t c\left[\left(\|u\|_{L^\infty} + \left\|u^2\right\|_{L^\infty} + \left\|u^{m+1}\right\|_{L^\infty} + \beta\|u_x\|_{L^\infty}^N + \|G\|_{L^\infty}\right) + \frac{1}{2}\|u_x\|_{L^\infty}^2\right]d\tau. \tag{4.30}$$

Using the algebra property of $H^{s_0}(R)$ with $s_0 > 1/2$ yields ($\|u_\varepsilon\|_{H^{(1/2)+}}$ means that there exists a sufficiently small $\delta > 0$ such that $\|u_\varepsilon\|_{(1/2)+} = \|u_\varepsilon\|_{H^{1/2+\delta}}$)

$$\begin{aligned}
\|G\|_{L^\infty} &\le c\|G\|_{H^{(1/2)+}} \\
&\le c\left\|\Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]\right\|_{H^{(1/2)+}} \\
&\le c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \left\|\Lambda^{-2}(u_x^2)\right\|_{H^{(1/2)+}} + \left\|\Lambda^{-2}(u_x^N)\right\|_{H^{(1/2)+}}\right) \\
&\le c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \left\|u_x^2\right\|_{H^0} + \left\|u_x^N\right\|_{H^0}\right) \\
&\le c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \|u_x\|_{L^\infty}\|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\|u\|_{H^1}\right) \\
&\le ce^{c\int_0^t \|u_x\|_{L^\infty}^{N-1}d\tau}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right),
\end{aligned} \tag{4.31}$$

in which Lemma 4.3 is used. Therefore, we get

$$\int_0^t \|G\|_{L^\infty}d\tau \le c\int_0^t e^{c\int_0^\tau \|u_x\|_{L^\infty}^{N-1}d\xi}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right)d\tau. \tag{4.32}$$

From (4.30) and (4.32), one has

$$\begin{aligned}
\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + c\int_0^t \Big[&\|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^N + e^{c\int_0^t \|u_x\|_{L^\infty}^{N-1}d\tau} \\
&+ e^{c\int_0^\tau \|u_x\|_{L^\infty}^{N-1}d\xi}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right)\Big]d\tau.
\end{aligned} \tag{4.33}$$

From Lemma 4.4, it follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\begin{aligned}
\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t \Big[&\|W\|_{L^\infty}^2 + \|W\|_{L^\infty}^N + e^{c\int_0^t \|W\|_{L^\infty}^{N-1}d\tau} \\
&+ e^{c\int_0^\tau \|W\|_{L^\infty}^{N-1}d\xi}\left(1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^{N-1}\right)\Big]d\tau
\end{aligned} \tag{4.34}$$

has a unique solution $W \in C[0, T]$. Using the Theorem presented at page 51 in [25] or Theorem 2 in Section 1.1 presented in [37] yields that there are constants $T > 0$ and $c > 0$

independent of ε such that $\|u_x\|_{L^\infty} \le W(t)$ for arbitrary $t \in [0,T]$, which leads to the conclusion of Lemma 4.5.

Using Lemmas 4.3 and 4.5, notation $u_\varepsilon = u$ and Gronwall's inequality results in the inequalities

$$\begin{aligned} &\|u_\varepsilon\|_{H^q} \le C_T e^{C_T}, \\ &\|u_{\varepsilon t}\|_{H^r} \le C_T e^{C_T}, \end{aligned} \tag{4.35}$$

where $q \in (0,s]$, $r \in (0,s-1]$ and C_T depends on T. It follows from Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t,x)$ and its derivative u_t in $L^2([0,T],H^s)$ and $L^2([0,T],H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T],H^q(-R_1,R_1))$ for $q \in [0,s)$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T],\ H^r(-R_1,R_1))$ for $r \in [0,s-1]$. Thus, we can prove the existence of a weak solution to (2.2). □

Proof of Theorem 2.2. From Lemma 4.5, we know that $\{u_{\varepsilon_n x}\}$ ($\varepsilon_n \to 0$) is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$ and $\{u_{\varepsilon_n x}\}$ are weakly convergent to u and u_x in $L^2[0,T], H^r(-R,R)$ for any $r \in [0,s-1)$, respectively. Therefore, u satisfies the equation

$$\begin{aligned} -\int_0^T\int_R u(g_t - g_{xxt})dx\,dt = \int_0^T\int_R \Bigg[&\left(2ku + \frac{a}{m+1}u^{m+1} + \frac{1}{2}\left(u_x^2\right)\right)g_x \\ &-\frac{1}{2}u^2 g_{xxx} - \beta(u_x)^N g_x\Bigg]dx\,dt, \end{aligned} \tag{4.36}$$

with $u(0,x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T]\times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T]\times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T]\times R)$. It derives from the $\{u_{\varepsilon_n x}\}$ weakly convergent to u_x in $L^2([0,T]\times R)$ that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T]\times R)$. □

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