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Research Article

Nonsquareness in Musielak-Orlicz-Bochner Function Spaces

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The criteria for nonsquareness in the classical Orlicz function spaces have been given already. However, because of the complication of Musielak-Orlicz-Bochner function spaces, at present the criteria for nonsquareness have not been discussed yet. In the paper, the criteria for nonsquareness of Musielak-Orlicz-Bochner function spaces are given. As a corollary, the criteria for nonsquareness of Musielak-Orlicz function spaces are given.

1. Introduction

A lot of nonsquareness concepts in Banach spaces are known. Nonsquareness are the important notion in geometry of Banach space. One of reasons is that the property is strongly related to the fixed point property (see [1]). The criteria for nonsquareness in the classical Orlicz function spaces have been given in [2] already. However, because of the complication of Musielak-Orlicz-Bochner function spaces, at present the criteria for nonsquareness have not been discussed yet. The aim of this paper is to give criteria nonsquareness of Musielak-Orlicz-Bochner function spaces. As a corollary, the criteria for nonsquareness of Musielak-Orlicz function spaces are given. The topic of this paper is related to the topic of [3–8].

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively. By X^*, denote the dual space of X. Let N, R, and R^+ denote the set natural number, reals, and nonnegative reals, respectively. Let us recall some geometrical notions concerning nonsquareness. A Banach space X is said to be nonsquare space if for any $x, y \in S(X)$ we have $\min\{\|(1/2)(x+y)\|, \|(1/2)(x-y)\|\} < 1$. A Banach space X is said to be uniformly nonsquare space if for any $x, y \in S(X)$, there exists $\delta > 0$ such that $\min\{\|(1/2)(x+y)\|, \|(1/2)(x-y)\|\} < 1-\delta$.

Let $(T, \sum, \mu)$ be nonatomic measure space. Suppose that a function $M : T \times [0, \infty) \to [0, \infty]$ satisfies the following conditions:

(1) for μ-a.e, $t \in T$, $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$,

(2) for μ-a.e, $t \in T$, $M(t, u)$ is convex on $[0, \infty)$ with respect to u,

(3) for each $u \in [0, \infty)$, $M(t, u)$ is a μ-measurable function of t on T.

Let $e(t) = \sup\{u > 0 : M(t, u) = 0\}$. It is well known that $e(t)$ is μ-measurable (see [2]).

Moreover, for a given Banach space $(X, \|\cdot\|)$, we denote by X_T, the set of all strongly μ-measurable function from T to X, and for each $u \in X_T$, define the modular of u by

$$\rho_M(u) = \int_T M(t, \|u(t)\|)dt. \tag{1.1}$$

Put

$$L_M(X) = \{u \in X_T : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}. \tag{1.2}$$

It is well known that Musielak-Orlicz-Bochner function space $L_M(X)$ is Banach spaces equipped with the Luxemburg norm

$$\|u\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \le 1\right\} \tag{1.3}$$

or Orlicz's norm

$$\|u\|^0 = \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(ku)\right]. \tag{1.4}$$

In particular, $L_M(R)$ and $L_M^0(R)$ are said to be Musielak-Orlicz function space. Set

$$\operatorname{supp} u = \{t \in T : \|u(t)\| \ne 0\}, \qquad K(u) = \left\{k > 0 : \frac{1}{k}(1 + \rho_M(ku)) = \|u\|^0\right\}. \tag{1.5}$$

In particular, the set $K(u)$ can be nonempty. To show that, we give a proposition.

Proposition 1.1 (see [9]). *If* $\lim_{u\to\infty}(M(t, u)/u) = \infty$ *μ-a.e. $t \in T$, then $K(u) \ne \phi$ for any $u \in L_M^0(X)$.*

We define a function

$$\sigma(t) = \sup\left\{u \ge 0 : M\left(t, \frac{1}{2}u\right) = \frac{1}{2}M(t, u)\right\}. \tag{1.6}$$

Function $\sigma(t)$ will be used in the further part of the paper. Moreover, $\sigma(t)$ is μ-measurable. To show that, we give a proposition.

Proposition 1.2. *Function $\sigma(t)$ is μ-measurable.*

Proof. Pick a dense set $\{r_i\}_{i=1}^{\infty}$ in $[0,\infty)$ and set

$$B_k = \left\{t \in T : M\left(t, \frac{1}{2}r_k\right) = \frac{1}{2}M(t, r_k)\right\}, \quad q_k(t) = r_k \chi_{B_k}(t) \quad (k \in N). \tag{1.7}$$

It is easy to see that for all $k \in N$, $\sigma(t) \geq q_k(t)$ μ-a.e on T. Hence, $\sup_{k\geq 1} q_k(t) \leq \sigma(t)$. For μ-a.e $t \in T$, arbitrarily choose $\varepsilon \in (0, \sigma(t))$. Then, there exists $r_k \in (\sigma(t) - \varepsilon, \sigma(t))$ such that $M(t, (1/2)r_k) = (1/2)M(t, r_k)$, that is, $q_k(t) \geq r_k > \sigma(t) - \varepsilon$. Since ε is arbitrary, we find $\sup_{k\geq 1} q_k(t) \geq \sigma(t)$. Thus, $\sup_{k\geq 1} q_k(t) = \sigma(t)$. □

It is easy to prove the following proposition.

Proposition 1.3. *For any $\alpha \in (0,1)$, if $u(t) \leq \sigma(t)$, then $M(t, \alpha u(t)) = \alpha M(t, u(t))$.*

Proposition 1.4. *For any $\alpha \in (0,1)$, if $u(t) < \sigma(t) < v(t)$, then $M(t, \alpha u(t) + (1-\alpha)v(t)) < \alpha M(t, u(t)) + (1-\alpha)M(t, v(t))$.*

Proposition 1.5. *For any $\alpha \in (0,1)$, if $\sigma(t) < v(t)$, then $M(t, \alpha v(t)) < \alpha M(t, v(t))$.*

Definition 1.6 (see [2]). We say that $M(t,u)$ satisfies condition $\Delta (M \in \Delta)$ if there exist $K \geq 1$ and a measureable nonnegative function $\delta(t)$ on T such that $\int_T M(t, \delta(t))dt < \infty$ and $M(t, 2u) \leq KM(t,u)$ for almost all $t \in T$ and all $u \geq \delta(t)$.

First, we give some results that will used in the further part of the paper.

Lemma 1.7 (see [2]). *Suppose $M \in \Delta$. Then $\rho_M(u) = 1 \Leftrightarrow \|u\| = 1$.*

Lemma 1.8 (see [9]). *Let $L_M^0(X)$ be Musielak-Orlicz-Bochner function spaces, then, if $K(u) = \phi$, one has $\|u\|^0 = \int_T A(t) \cdot \|u(t)\| dt$, where $A(t) = \lim_{u\to\infty}(M(t,u)/u)$.*

2. Main Results

Theorem 2.1. *$L_M(X)$ is nonsquare space if and only if*

(a) *$M \in \Delta$,*

(b) *for any $u, v \in S(L_M(X))$, one has $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\} > 0$ or $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\| > \sigma(t)\}) > 0$,*

(c) *X is nonsquare space.*

In order to prove the theorem, we give a lemma.

Lemma 2.2. *Let X be nonsquare space, then for any $x, y \neq 0$, one has*

$$\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} > 0. \tag{2.1}$$

Proof. For any $x, y \neq 0$, without loss of generality, we may assume $\|x\| \leq \|y\|$. Since X is nonsquare space, we have

$$\|x\| + \|x\| = \|x\| + \left\| \frac{\|x\|}{\|y\|} y \right\| > min\left\{ \left\| x + \frac{\|x\|}{\|y\|} y \right\|, \left\| x - \frac{\|x\|}{\|y\|} y \right\| \right\}. \tag{2.2}$$

Therefore, by (2.2), we obtain

$$\begin{aligned} \|x + y\| &\leq \left\| x + \frac{\|x\|}{\|y\|} \cdot y \right\| + \left(1 - \frac{\|x\|}{\|y\|}\right) \cdot \|y\| \\ &< \|x\| + \|x\| + \|y\| - \|x\| \\ &= \|x\| + \|y\| \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} \|x - y\| &\leq \left\| x - \frac{\|x\|}{\|y\|} \cdot y \right\| + \left(1 - \frac{\|x\|}{\|y\|}\right) \cdot \|y\| \\ &< \|x\| + \|x\| + \|y\| - \|x\| \\ &= \|x\| + \|y\|. \end{aligned} \tag{2.4}$$

This implies $\|x\| + \|y\| - \min\{\|x + y\|, \|x - y\|\} > 0$. This completes the proof. □

Proof of Theorem 2.1. Necessity. (a) If $L_M(X)$ is nonsquare space, then $L_M(R)$ is nonsquare space, because $L_M(R)$ is isometrically embedded into $L_M(X)$. Since $L_M(R)$ is nonsquare space, then $M \in \Delta$ which follows from the theorem proved in more general case (see [10, 11]). Namely, if $M \notin \Delta$, then $L_M(R)$ contains isometric copy of l^∞.

If (b) is not true, then there exist $u, v \in S(L_M(X))$ such that $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\} = 0$ and $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\|) > \sigma(t)\}) = 0$. Let $G = \operatorname{supp} u \cap \operatorname{supp} v$. We have

$$\begin{aligned} &\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v) \\ &\quad = \frac{1}{2}\int_T M(t, \|u(t)\|)dt + \frac{1}{2}\int_T M(t, \|v(t)\|)dt \\ &\quad = \int_T \frac{1}{2}M(t, \|u(t)\|) + \frac{1}{2}M(t, \|v(t)\|)dt \\ &\quad \geq \int_G M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt + \int_{T\setminus G} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt \end{aligned}$$

$$\begin{aligned} &\geq \int_G M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt + \int_{T\setminus G} M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt \\ &= \rho_M\left(\frac{u+v}{2}\right). \end{aligned} \tag{2.5}$$

By $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\} = 0$ and $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\|) > \sigma(t)\}) = 0$, we obtain that two inequalities of (2.5) are equations. This implies $(1/2)\rho_M(u) + (1/2)\rho_M(v) = \rho_M((u+v)/2)$. By Lemma 1.7, we have $\rho_M((u+v)/2) = 1$. Thus, $\|(1/2)(u+v)\| = 1$. Similarly, we have $\|(1/2)(u-v)\| = 1$, a contradiction!

(c) Pick $h(t) \in S(L_M(X))$, then there exists $d > 0$ such that $\mu E > 0$, where $E = \{t \in T : \|h(t)\| \geq d\}$. Put $h_1(t) = d \cdot x_0 \cdot \chi_E(t)$, where $x_0 \in S(X)$. It is easy to see that $h_1(t) \in L_M(X) \setminus \{0\}$. Hence, there exists $k > 0$ such that $k \cdot h_1(t) \in S(L_M(X))$. By Lemma 1.7, we have

$$1 = \int_T M(t, \|k \cdot h_1(t)\|)dt = \int_E M(t, \|k \cdot d \cdot x_0\|)dt. \tag{2.6}$$

Let $\alpha = k \cdot d$. Then, $\int_E M(t,\alpha)dt = 1$. The necessity of (c) follows from the fact that X is isometrically embedded into $L_M(X)$. Namely, defining the operator $I : X \to L_M(X)$ by

$$I(x) = \alpha \cdot x \cdot \chi_E(t), \quad x \in X. \tag{2.7}$$

Hence, for any $x \in X \setminus \{0\}$, we have

$$\rho_M\left(\frac{I(x)}{\|x\|}\right) = \int_E M\left(t, \left\|\frac{I(x)}{\|x\|}\right\|\right)dt = \int_E M\left(t, \frac{\alpha\|x\|}{\|x\|}\right)dt = \int_E M(t,\alpha)dt = 1. \tag{2.8}$$

By Lemma 1.7, we have $\|I(x)/\|x\|\|_{L_M(X)} = 1$, hence $\|I(x)\|_{L_M(X)} = \|x\|$.

Sufficiency. The proof requires the consideration of two cases separately.

Case 1. $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\| > \sigma(t)\}) > 0$. Without loss of generality, we may assume $\mu\{t \in T : \|u(t)\| > \sigma(t)\} > 0$. Let $F = \{t \in T : \|u(t)\| > \sigma(t)\}$. Put

$$\begin{aligned} F_1 &= \{t \in F : \|u(t)\| + \|v(t)\| > \|u(t)+v(t)\|\}, \\ F_2 &= \{t \in F : \|u(t)\| + \|v(t)\| > \|u(t)-v(t)\|\}, \\ F_3 &= \{t \in F : \|v(t)\| = 0\}. \end{aligned} \tag{2.9}$$

Since X is nonsquare space, we have $\mu(F_1 \cup F_3) > 0$ or $\mu(F_2 \cup F_3) > 0$ by Lemma 2.2. Without loss of generality, we may assume $\mu(F_1 \cup F_3) > 0$. Moreover, we have

$$\begin{aligned}
&\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v) - \rho_M\left(\frac{1}{2}(u+v)\right) \\
&\quad = \frac{1}{2}\int_T M(t, \|u(t)\|)dt + \frac{1}{2}\int_T M(t, \|v(t)\|)dt - \int_T M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt \\
&\quad = \int_T \left[\frac{1}{2}M(t, \|u(t)\|) + \frac{1}{2}M(t, \|v(t)\|) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)\right]dt \\
&\quad \geq \int_{F_1\cup F_3} \left[\frac{1}{2}M(t, \|u(t)\|) + \frac{1}{2}M(t, \|v(t)\|) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)\right]dt \\
&\quad \geq \int_{F_1\cup F_3} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt \\
&\quad \geq 0.
\end{aligned} \tag{2.10}$$

Let $E = \{t \in F_1 \cup F_3 : \|v(t)\| \geq \sigma(t)\}$, $E_1 = \{t \in E : \|v(t)\| = \sigma(t) = 0\}$. Then, $\mu((F_1 \cup F_3) \setminus (E \setminus E_1)) > 0$ or $\mu(E \setminus E_1) > 0$. By $F = \{t \in T : \|u(t)\| > \sigma(t)\} \supset (F_1 \cup F_3) \supset ((F_1 \cup F_3) \setminus (E \setminus E_1))$, we obtain

$$\begin{aligned}
&\int_{F_1\cup F_3} \left[\frac{1}{2}M(t, \|u(t)\|) + \frac{1}{2}M(t, \|v(t)\|) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)\right]dt \\
&\qquad > \int_{F_1\cup F_3} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt,
\end{aligned} \tag{2.11}$$

whenever $\mu((F_1 \cup F_3) \setminus (E \setminus E_1)) > 0$. By $F = \{t \in T : \|u(t)\| > \sigma(t)\} \supset (F_1 \cup F_3) \supset (E \setminus E_1)$, we obtain

$$\int_{F_1\cup F_3} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right) - M\left(t, \frac{1}{2}\|u(t)+v(t)\|\right)dt > 0, \tag{2.12}$$

whenever $\mu(E \setminus E_1) > 0$. This means that one of three inequalities of (2.10) is strict inequality. By $\rho_M(u) = \rho_M(v) = 1$, we have $\rho_M((1/2)(u+v)) < 1$. By Lemma 1.7, we have $\|(1/2)(u+v)\| < 1$.

Case 2. $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\|) > \sigma(t)\}) = 0$. By (b), we have $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\} > 0$. Let $G = \{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\}$. Put

$$\begin{aligned}
G_1 &= \{t \in G : \|u(t)\| + \|v(t)\| > \|u(t) + v(t)\|\}, \\
G_2 &= \{t \in G : \|u(t)\| + \|v(t)\| > \|u(t) - v(t)\|\}.
\end{aligned} \tag{2.13}$$

Since X is nonsquare space, we have $\mu G_1 > 0$ or $\mu G_2 > 0$ by Lemma 2.2. Without loss of generality, we may assume $\mu G_1 > 0$. Hence,

$$\int_{G_1} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt > \int_{G_1} M\left(t, \frac{1}{2}\|u(t) + v(t)\|\right)dt. \tag{2.14}$$

Therefore, by (2.14), we have

$$\begin{aligned}
&\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v) \\
&\quad = \int_T \frac{1}{2}M(t, \|u(t)\|) + \frac{1}{2}M(t, \|v(t)\|)dt \\
&\quad \geq \int_{G_1} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt + \int_{T\setminus G_1} M\left(t, \frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt \\
&\quad > \int_{G_1} M\left(t, \frac{1}{2}\|u(t) + v(t)\|\right)dt + \int_{T\setminus G_1} M\left(t, \frac{1}{2}\|u(t) + v(t)\|\right)dt \\
&\quad = \rho_M\left(\frac{u+v}{2}\right).
\end{aligned} \tag{2.15}$$

By $\rho_M(u) = \rho_M(v) = 1$, we have $\rho_M((u+v)/2) < 1$. By Lemma 1.7, we have $\|(1/2)(u+v)\| < 1$. This completes the proof. □

Corollary 2.3. *$L_M(R)$ is nonsquare space if and only if*

(a) $M \in \Delta$,

(b) *for any $u, v \in S(L_M(R))$, one has $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : |u(t)| + |v(t)| > 2e(t)\} > 0$ or $\mu(\{t \in T : |u(t)| > \sigma(t)\} \cup \{t \in T : |v(t)| > \sigma(t)\}) > 0$.*

Theorem 2.4. *Let $e(t) = 0$ μ-a.e on T. Then, $L_M(\mathrm{X})$ is nonsquare space if and only if*

(a) $M \in \Delta$,

(b) $\rho_M(\sigma) < 2$,

(c) X *is nonsquare space.*

Proof. Necessity. By Theorem 2.1, (a) and (c) are obvious. Suppose that $\rho_M(\sigma) \geq 2$. Then, there exists $E \in \Sigma$ such that $\mu E > 0$, $\rho_M(\sigma \cdot \chi_{T\setminus E}) = 1$ and $\rho_M(\sigma \cdot \chi_D) = 1$, where $D \subset T \setminus E$. Set

$$u(t) = x \cdot \sigma(t) \cdot \chi_{T\setminus E}(t), \qquad v(t) = x \cdot \sigma(t) \cdot \chi_D(t), \tag{2.16}$$

where $x \in S(\mathrm{X})$. It is easy to see that $\|u\| = \|v\| = 1$, $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > 2e(t)\} = 0$ and $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\| > \sigma(t)\}) = 0$. Contradicting Theorem 2.1.

Sufficiency. We only need to prove that for any $u, v \in S(L_M(X))$, if $\mu(\operatorname{supp} u \cap \operatorname{supp} v) = 0$, then $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\| > \sigma(t)\}) > 0$. Suppose that there exist $u, v \in S(L_M(X))$ such that $\mu(\operatorname{supp} u \cap \operatorname{supp} v) = 0$, $\mu(\{t \in T : \|u(t)\| > \sigma(t)\} \cup \{t \in T : \|v(t)\| > \sigma(t)\}) = 0$. By $\mu(\operatorname{supp} u \cap \operatorname{supp} v) = 0$, we have $\rho_M(u) + \rho_M(v) = \rho_M(u+v) \leq \rho_M(\sigma) < 2$. This implies $\rho_M(u) < 1$ or $\rho_M(v) < 1$. Hence, $\|u\| < 1$ or $\|v\| < 1$, a contradiction! This completes the proof. □

Theorem 2.5. $L_M^0(X)$ *is nonsquare space if and only if*

(a) *for any* $u \in L_M^0(X) \setminus \{0\}$, *one has* $K(u) \neq \phi$,

(b) *at least one of the conditions*

(b1) $kl/(k+l) \notin K(u+v) \cap K(u-v)$,

(b2) $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) > 0$,

(b3) $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : kl/(k+l)(\|u(t)\| + \|v(t)\|) > e(t)\} > 0$

is true, where $k \in K(u)$, $l \in K(v)$ *and* $u, v \in S(L_M^0(X))$,

(c) X *is nonsquare space.*

Proof. Necessity. (a) Suppose that there exists $u \in L_M^0(X) \setminus \{0\}$ such that $K(u) = \phi$, then $\|u\|^0 = \int_T A(t) \cdot \|u(t)\| dt$ by Lemma 1.8. Decompose T into disjoint sets T_1 and T_2 such that $\int_{T_1} A(t) \cdot \|u(t)\| dt = \int_{T_2} A(t) \cdot \|u(t)\| dt$. Put

$$u_1(t) = 2u(t)\chi_{T_1}, \qquad u_2(t) = 2u(t)\chi_{T_2}. \tag{2.17}$$

Obviously, $u = (1/2)(u_1 + u_2)$. Pick sequence $\{k_n\}_{n=1}^{\infty} \subset R^+$ such that $k_n \to \infty$ as $n \to \infty$. Let $T_0 = \operatorname{supp} u_1$. By Levi theorem, we have

$$\begin{aligned} \|u_1\|^0 &\leq \lim_{n\to\infty} \frac{1}{k_n}\left[1 + \int_T M(t, \|k_n u_1(t)\|) dt\right] \\ &= \lim_{n\to\infty} \int_{T_0} \frac{M(t, \|k_n u_1(t)\|)}{\|k_n u_1(t)\|} \|u_1(t)\| dt \\ &= \int_{T_0} \lim_{n\to\infty} \frac{M(t, \|k_n u_1(t)\|)}{\|k_n u_1(t)\|} \|u_1(t)\| dt \\ &= \int_T A(t) \|u_1(t)\| dt \\ &= \int_{T_1} A(t) \|2u(t)\| dt. \end{aligned} \tag{2.18}$$

Therefore,

$$\begin{aligned} \|u_1\|^0 &\leq \int_{T_1} A(t) \cdot \|2u(t)\| dt \\ &= \int_{T_1} A(t) \cdot \|u(t)\| dt + \int_{T_1} A(t) \cdot \|u(t)\| dt + \int_{T_2} A(t) \cdot \|u(t)\| dt - \int_{T_2} A(t) \cdot \|u(t)\| dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{T_1} A(t)\cdot \|u(t)\|dt + \int_{T_2} A(t)\cdot \|u(t)\|dt \\
&= \int_{T} A(t)\cdot \|u(t)\|dt = \|u\|^0.
\end{aligned}
\tag{2.19}
$$

Similarly, we have $\|u_2\|^0 \le \|u\|^0$. By $u = (1/2)(u_1+u_2)$, we obtain $\|u\|^0 = \|u_1\|^0 = \|u_2\|^0 = \|(1/2)(u_1+u_2)\|^0$. By $u_1(t)-u_2(t) = 2u(t)\chi_{T_1} + (-2u(t))\chi_{T_2}$, we have $\|u\|^0 = \|(1/2)(u_1-u_2)\|^0$. Therefore,

$$
\|u_1\|^0 = \|u_2\|^0 = \left\|\frac{1}{2}(u_1+u_2)\right\|^0 = \left\|\frac{1}{2}(u_1-u_2)\right\|^0 = \|u\|^0. \tag{2.20}
$$

This implies

$$
\|u_1\|^0 + \|u_2\|^0 = \min\left\{\|u_1+u_2\|^0, \|u_1-u_2\|^0\right\}, \tag{2.21}
$$

a contradiction!

If (b) is not true, then there exist $u,v \in S(L_M^0(X))$ such that $kl/(k+l) \in K(u+v)\cap K(u-v)$, $\mu(\{t\in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) = 0$ and $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : kl/(k+l)(\|u(t)\| + \|v(t)\|) > e(t)\} = 0$, where $k\in K(u)$, $l \in K(v)$. Let $E = T\setminus(\operatorname{supp} u\cap \operatorname{supp} v)$. It is easy to see that if $t\in E$, then $\|u(t)\|\cdot\|v(t)\| = 0$ on E. This implies

$$
M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right) = M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right) \quad t\in E. \tag{2.22}
$$

Therefore, by (2.22), we have

$$
\int_E M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt = \int_E M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt. \tag{2.23}
$$

By $\mu\{t \in \operatorname{supp} u\cap \operatorname{supp} v : (kl/(k+l))(\|u(t)\|+\|v(t)\|) > e(t)\} = 0$, we have

$$
\begin{aligned}
0 &= \int_{T\setminus E} M\left(t, \frac{kl}{k+l}\|u(\mathrm{t})\| + \frac{kl}{k+l}\|v(t)\|\right)dt \\
&= \int_{T\setminus E} M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt.
\end{aligned}
\tag{2.24}
$$

By (2.23) and (2.24), we have

$$
\int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt = \int_T M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt. \tag{2.25}
$$

By $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) = 0$, we have

$$\begin{aligned}&\int_T \left(\frac{l}{k+l} M(t, k\|u(t)\|) + \frac{k}{k+l} M(t, l\|v(t)\|)\right) dt \\ &\quad = \int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right) dt.\end{aligned} \tag{2.26}$$

Therefore, by (2.25) and (2.26), we have

$$\begin{aligned}\|u\|^0 + \|v\|^0 &= \frac{1}{k}\left[1 + \rho_M(ku)\right] + \frac{1}{l}\left[1 + \rho_M(lv)\right] \\ &= \frac{k+l}{kl}\left[1 + \frac{l}{k+l}\rho_M(ku) + \frac{k}{k+l}\rho_M(lv)\right] \\ &= \frac{k+l}{kl}\left[1 + \frac{l}{k+l}\int_T M(t, k\|u(t)\|)dt + \frac{k}{k+l}\int_T M(t, l\|v(t)\|)dt\right] \\ &= \frac{k+l}{kl}\left[1 + \int_T \left(\frac{l}{k+l} M(t, k\|u(t)\|) + \frac{k}{k+l} M(t, l\|v(t)\|)\right) dt\right] \\ &= \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right) dt\right] \\ &= \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t) + v(t)\|\right) dt\right] \\ &= \frac{k+l}{kl}\left[1 + \rho_M\left(\frac{kl}{k+l}(u+v)\right)\right] \\ &= \|u+v\|^0.\end{aligned} \tag{2.27}$$

Similarly, we have $\|u\|^0 + \|v\|^0 = \|u - v\|^0$. This implies $\|u\|^0 + \|v\|^0 = \|u+v\|^0 = \|u-v\|^0$, a contradiction!

(c) Pick $h(t) \in S(L_M^0(X))$, then there exists $d > 0$ such that $\mu E > 0$, where $E = \{t \in T : \|h(t)\| \geq d\}$. Put $h_1(t) = d \cdot x_0 \cdot \chi_E(t)$, where $x_0 \in S(X)$. It is easy to see that $h_1(t) \in L_M^0(X) \setminus \{0\}$. Hence, there exists $l > 0$ such that $l \cdot h_1(t) \in S(L_M^0(X))$. The necessity of (c) follows from the fact that X is isometrically embedded into $L_M^0(X)$. Namely, defining the operator $I : X \to L_M^0(X)$ by

$$I(x) = ld \cdot x \cdot \chi_E(t), \quad x \in X. \tag{2.28}$$

It is easy to see that $I(x_0) \in S(L_M^0(X))$. Hence, for any $x \in X \setminus \{0\}$, we have

$$\begin{aligned}
\|I(x)\|^0 &= \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(k \cdot I(x))\right] \\
&= \inf_{k>0} \frac{1}{k}\left[1 + \int_E M(t, k \cdot ld\|x\|)dt\right] \\
&= \inf_{k>0} \frac{1}{k}\left[1 + \int_E M(t, k \cdot \|x\| ld\|x_0\|)dt\right] \\
&= \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(k \cdot \|x\| I(x_0))\right] \\
&= \|\|x\| \cdot I(x_0)\|^0 \\
&= \|x\| \cdot \|I(x_0)\|^0 \\
&= \|x\|.
\end{aligned} \tag{2.29}$$

Sufficiency. Suppose that there exists $u, v \in S(L_M^0(X))$ such that $\|u\|^0 = \|v\|^0 = \|(1/2)(u+v)\|^0 = \|(1/2)(u-v)\|^0 = 1$. Let $k \in K(u)$, $l \in K(v)$. We will derive a contradiction for each of the following two cases.

Case 1. $\mu[(\{t \in T : \|u(t)\| \neq 0\} \cup \{t \in T : \|v(t)\| \neq 0\}) \setminus \{t \in T : \sigma(t) > 0\}] = 0$. By Lemma 2.2, we have $\|u(t)\| + \|v(t)\| > min\{\|u(t) + v(t)\|, \|u(t) - v(t)\|\} t \in \operatorname{supp} u \cap \operatorname{supp} v$. Put

$$\begin{aligned}
T_1 &= \left\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > \|u(t) + v(t)\|\right\}, \\
T_2 &= \left\{t \in \operatorname{supp} u \cap \operatorname{supp} v : \|u(t)\| + \|v(t)\| > \|u(t) - v(t)\|\right\}.
\end{aligned} \tag{2.30}$$

Moreover, we have

$$\begin{aligned}
\|u\|^0 + \|v\|^0 &= \frac{1}{k}\left[1 + \rho_M(ku)\right] + \frac{1}{l}\left[1 + \rho_M(lv)\right] \\
&= \frac{k+l}{kl}\left[1 + \frac{l}{k+l}\int_T M(t, k\|u(t)\|)dt + \frac{k}{k+l}\int_T M(t, l\|v(t)\|)dt\right] \\
&= \frac{k+l}{kl}\left[1 + \int_T \left(\frac{l}{k+l} M(t, k\|u(t)\|) + \frac{k}{k+l} M(t, l\|v(t)\|)\right)dt\right] \\
&\geq \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt\right]
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{k+l}{kl}\left[1+\int_{T_1} M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt\right] \\
&\quad + \frac{k+l}{kl}\left[1+\int_{T\setminus T_1} M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt\right] \\
&= \frac{k+l}{kl}\left[1+\rho_M\left(\frac{kl}{k+l}(u+v)\right)\right] \\
&\geq \|u+v\|^0.
\end{aligned} \tag{2.31}$$

By $\|u\|^0 + \|v\|^0 = \|u+v\|^0$, three inequalities of (2.31) are equation. This implies

$$\begin{aligned}
&\frac{k+l}{kl}\left[1+\int_T\left(\frac{l}{k+l}M(t, k\|u(t)\|) + \frac{k}{k+l}M(t, l\|v(t)\|)\right)dt\right] \\
&\quad = \frac{k+l}{kl}\left[1+\int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt\right].
\end{aligned} \tag{2.32}$$

Next, we will prove $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) = 0$. Suppose that $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) > 0$. Without loss of generality, we may assume $\mu\{t \in T : k\|u(t)\| - \sigma(t) > 0\} > 0$. Therefore, by (2.32), we have $l\|v(t)\| - \sigma(t) \geq 0$ μ-a.e. on $\{t \in T : k\|u(t)\| - \sigma(t) > 0\}$. Hence, $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cap \{t \in T : l\|v(t)\| - \sigma(t) \geq 0\}) > 0$. Since three inequalities of (2.31) are equation, we deduce

$$\int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt = \int_T M\left(t, \frac{kl}{k+l}\|u(t)+v(t)\|\right)dt. \tag{2.33}$$

Moreover, it is easy to see

$$\frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\| > \sigma(t) \geq e(t) \tag{2.34}$$

on $\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cap \{t \in T : l\|v(t)\| - \sigma(t) \geq 0\}$. Therefore, by (2.33) and (2.34), we have

$$\frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\| = \frac{kl}{k+l}\|u(t)+v(t)\| \tag{2.35}$$

μ-a.e. on $\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cap \{t \in T : l\|v(t)\| - \sigma(t) \geq 0\}$. Since X is nonsquare space, we have

$$\frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\| > min\left\{\frac{kl}{k+l}\|u(t)+v(t)\|,\ \frac{kl}{k+l}\|u(t)-v(t)\|\right\} \tag{2.36}$$

on $\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cap \{t \in T : l\|v(t)\| - \sigma(t) \geq 0\}$. Thus,

$$\frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\| > \frac{kl}{k+l}\|u(t) - v(t)\| \tag{2.37}$$

μ-a.e. on $\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cap \{t \in T : l\|v(t)\| - \sigma(t) \geq 0\}$. Therefore, by (2.34) and (2.37), we have

$$\begin{aligned}
\|u\|^0 + \|v\|^0 &= \frac{1}{k}[1 + \rho_M(ku)] + \frac{1}{l}[1 + \rho_M(lv)] \\
&\geq \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt\right] \\
&> \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t) - v(t)\|\right)dt\right] \\
&= \frac{k+l}{kl}\left[1 + \rho_M\left(\frac{kl}{k+l}(u-v)\right)\right] \\
&\geq \|u - v\|^0.
\end{aligned} \tag{2.38}$$

This implies $\|u\|^0 + \|v\|^0 > \|u - v\|^0$, a contradiction! Hence, $\mu(\{t \in T : k\|u(t)\| - \sigma(t) > 0\} \cup \{t \in T : l\|v(t)\| - \sigma(t) > 0\}) = 0$. Since three inequalities of (2.31) are equation, we deduce $kl/(k+l) \in K(u+v)$ and $\mu\{t \in T_1 : (kl/(k+l))(\|u(t)\| + \|v(t)\|) > e(t)\} = 0$. By (b), we have $kl/(k+l) \notin K(u-v)$ or $\mu\{t \in T_2 : (kl/(k+l))(\|u(t)\|+\|v(t)\|) > e(t)\} > 0$. If $kl/(k+l) \notin K(u-v)$, then

$$\begin{aligned}
\|u\|^0 + \|v\|^0 &= \frac{1}{k}[1 + \rho_M(ku)] + \frac{1}{l}[1 + \rho_M(lv)] \\
&\geq \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t)\| + \frac{kl}{k+l}\|v(t)\|\right)dt\right] \\
&\geq \frac{k+l}{kl}\left[1 + \int_T M\left(t, \frac{kl}{k+l}\|u(t) - v(t)\|\right)dt\right] \\
&= \frac{k+l}{kl}\left[1 + \rho_M\left(\frac{kl}{k+l}(u-v)\right)\right] \\
&> \|u - v\|^0.
\end{aligned} \tag{2.39}$$

This implies $\|u\|^0+\|v\|^0>\|u-v\|^0$, a contradiction! If $\mu\{t\in T_2:(kl/(k+l))(\|u(t)\|+\|v(t)\|)>e(t)\}>0$, then

$$\begin{aligned}
\|u\|^0+\|v\|^0 &= \frac{1}{k}\left[1+\rho_M(ku)\right]+\frac{1}{l}\left[1+\rho_M(lv)\right]\\
&\geq \frac{k+l}{kl}\left[1+\int_T M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt\right]\\
&> \frac{k+l}{kl}\left[1+\int_{T_2} M\left(t,\frac{kl}{k+l}\|u(t)-v(t)\|\right)dt\right]\\
&\quad+\frac{k+l}{kl}\left[1+\int_{T\setminus T_2} M\left(t,\frac{kl}{k+l}\|u(t)-(t)\|\right)dt\right]\\
&= \frac{k+l}{kl}\left[1+\rho_M\left(\frac{kl}{k+l}(u-v)\right)\right]\\
&\geq \|u-v\|^0.
\end{aligned} \tag{2.40}$$

This implies $\|u\|^0+\|v\|^0>\|u-v\|^0$, a contradiction!

Case 2. $\mu[(\{t\in T:\|u(t)\|\neq 0\}\cup\{t\in T:\|v(t)\|\neq 0\})\setminus\{t\in T:\sigma(t)>0\}]>0$. Let $E=(\{t\in T:\|u(t)\|\neq 0\}\cup\{t\in T:\|v(t)\|\neq 0\})\setminus\{t\in T:\sigma(t)>0\}$. Put

$$E_1=\{t\in E:\|u(t)\|\cdot\|v(t)\|=0\},\qquad E_2=\{t\in E:\|u(t)\|\cdot\|v(t)\|>0\}. \tag{2.41}$$

Then, $\mu E_1>0$ or $\mu E_2>0$. If $\mu E_1>0$, then

$$\begin{aligned}
&\int_{E_1}\left(\frac{l}{k+l}M(t,k\|u(t)\|)dt+\frac{k}{k+l}M(t,l\|v(t)\|)\right)dt\\
&\qquad>\int_{E_1} M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt.
\end{aligned} \tag{2.42}$$

Therefore, by (2.42), we have

$$\begin{aligned}
\|u\|^0+\|v\|^0 &= \frac{1}{k}\left[1+\rho_M(ku)\right]+\frac{1}{l}\left[1+\rho_M(lv)\right]\\
&= \frac{k+l}{kl}\left[1+\int_T\left(\frac{l}{k+l}M(t,k\|u(t)\|)+\frac{k}{k+l}M(t,l\|v(t)\|)\right)dt\right]
\end{aligned}$$

$$
\begin{aligned}
&> \frac{k+l}{kl}\left[1+\int_{E_1} M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt\right] \\
&\quad + \frac{k+l}{kl}\left[1+\int_{T\setminus E_1} M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt\right] \\
&\geq \frac{k+l}{kl}\left[1+\int_{T} M\left(t,\frac{kl}{k+l}\|u(t)+v(t)\|\right)dt\right] \\
&= \frac{k+l}{kl}\left[1+\rho_M\left(\frac{kl}{k+l}(u+v)\right)\right] \\
&\geq \|u+v\|^0.
\end{aligned}
\tag{2.43}
$$

This implies $\|u\|^0 + \|v\|^0 > \|u+v\|^0$, a contradiction! If $\mu E_2 > 0$, then $\mu E_2^1 > 0$ or $\mu \mathrm{E}_2^2 > 0$ by Lemma 2.2, where

$$
\begin{aligned}
E_2^1 &= \{t \in E_2 : \|u(t)\| + \|v(t)\| > \|u(t)+v(t)\|\}, \\
E_2^2 &= \{t \in E_2 : \|u(t)\| + \|v(t)\| > \|u(t)-v(t)\|\}.
\end{aligned}
\tag{2.44}
$$

Without loss of generality, we may assume $\mu E_2^1 > 0$. Hence,

$$
\int_{E_2^1} M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt > \int_{E_2^1} M\left(t,\frac{kl}{k+l}\|u(t)+v(t)\|\right)dt. \tag{2.45}
$$

Therefore, by (2.45), we have

$$
\begin{aligned}
\|u\|^0 + \|v\|^0 &= \frac{1}{k}[1+\rho_M(ku)] + \frac{1}{l}[1+\rho_M(lv)] \\
&\geq \frac{k+l}{kl}\left[1+\int_{T} M\left(t,\frac{kl}{k+l}\|u(t)\|+\frac{kl}{k+l}\|v(t)\|\right)dt\right] \\
&> \frac{k+l}{kl}\left[1+\int_{E_2^1} M\left(t,\frac{kl}{k+l}\|u(t)+v(t)\|\right)dt\right] \\
&\quad + \frac{k+l}{kl}\left[1+\int_{T\setminus E_2^1} M\left(t,\frac{kl}{k+l}\|u(t)+v(t)\|\right)dt\right] \\
&= \frac{k+l}{kl}\left[1+\rho_M\left(\frac{kl}{k+l}(u+v)\right)\right] \\
&\geq \|u+v\|^0.
\end{aligned}
\tag{2.46}
$$

This implies $\|u\|^0 + \|v\|^0 > \|u+v\|^0$, a contradiction! This completes the proof. □

Corollary 2.6. $L_M^0(R)$ *is nonsquare space if and only if*

(a) *for any* $u \in L_M^0(R)$, *one has* $K(u) \neq \phi$,

(b) *at least one of the conditions*

(b1) $kl/(k+l) \notin K(u+v) \cap K(u-v)$,

(b2) $\mu(\{t \in T : k|u(t)| - \sigma(t) > 0\} \cup \{t \in T : l|v(t)| - \sigma(t) > 0\}) > 0$,

(b3) $\mu\{t \in \operatorname{supp} u \cap \operatorname{supp} v : (kl/(k+l))(|u(t)| + |v(t)|) > e(t)\} > 0$

is true, where $k \in K(u)$, $l \in K(v)$ *and* $u, v \in S(L_M^0(R))$.

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