# Research Article

# Two-Point Oscillation for a Class of Second-Order Damped Linear Differential Equations

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Using the comparison theorem, the two-point oscillation for linear differential equation with damping term  $y'' + (f(x)/(x-x^2)^{\alpha})y' + (g(x)/(x-x^2)^{\beta})y = 0$  is considered, where  $\alpha, \beta > 0$ ; f(x), g(x) > 0, and  $f(x), g(x) \in C(\overline{I})$ , I = (0,1). Results are obtained that  $0 < \alpha < 3/2, \beta > 3$  or  $\alpha > 3/2, \beta > 2\alpha$  imply the two-point oscillation of the equation.

#### 1. Introduction

Under the solution y(x) of a differential equation appearing in the paper, we mean a function y = y(x) such that  $y \in C^2(I)$ . Here, we allow that  $y \notin C(\overline{I})$ .

Recently, two-point oscillation of the differential equations caused the concern of many scholars ([1, 2]). In paper [1], Pašić and Wong construct the equation:

$$y'' + \left[\frac{1}{2}S(q')(x) + (q')^{2}(x)\right]y = 0, \quad x \in I,$$
(1.1)

where I = (0,1),  $q \in C^3(I)$ ,  $|q(0+)| = |q(1-)| = +\infty$ ,  $|q'(0+)| = |q'(1-)| = +\infty$ , q'(x) < 0,  $x \in I$ ,  $S(q') \in C(I)$ ,  $S(q')(x) = (q'''(x)/q'(x)) - (3/2)[q''(x)/q'(x)]^2$ , they study the following equation:

$$y'' + \frac{c(x)}{(x - x^2)^{\sigma}} y = 0, \quad x \in I,$$
(1.2)

by comparison theorem (where c(x) > 0,  $c(x) \in C(\overline{I})$ ), and they obtain that when  $\sigma > 2$ , (1.2) is two-point oscillatory.

In this paper, we construct the following equation with damping:

$$(p'(x)y')' - 2p''(x)y' + (p'(x))^{3}y = 0, \quad x \in I,$$
(1.3)

where  $p(x) \in C^2(I)$  and

$$|p(x)(0+)| = |p(x)(1-)| = \infty.$$
(1.4)

we study the two-point oscillation of the following damped equation by comparison theorem

$$y'' + \frac{f(x)}{(x - x^2)^{\alpha}}y' + \frac{g(x)}{(x - x^2)^{\beta}}y = 0,$$
(1.5)

where  $x \in I$ ,  $\alpha, \beta > 0$ ; f(x), g(x) > 0, f(x),  $g(x) \in C(\overline{I})$ , the result we obtained is new, and it continues the results obtained in [1].

#### 2. Two-Point Oscillation of (1.3)

*Definition* 2.1. A function y = y(x),  $y(x) \in C(I)$  is said to be two-point oscillation on the interval I, if there exist a decreasing sequence  $a_{(k)} \in I$  and an increasing sequence  $b_{(k)} \in I$  of consecutive zeros of y(x) such that  $a_{(k)} \setminus 0$  and  $b_{(k)} \nearrow 1$ .

Definition 2.2. A linear differential equation is said to be two-point oscillation on I if all its nontrivial solutions y = y(x),  $y(x) \in C^2(I)$  are two-point oscillatory on I.

By Sturm separation theorem, all nontrivial solutions of a linear differential equation are two-point oscillatory if there is a nontrivial solution is two-point oscillatory on *I*.

We know that  $y_1(x) = \cos p(x)$ ,  $y_2(x) = \sin p(x)$  are two linearly independent solutions of (1.3), so the general solution of (1.3) can be expressed as

$$y(x) = c_1 \cos p(x) + c_2 \sin p(x). \tag{2.1}$$

Because of  $|p(x)(0+)| = |p(x)(1-)| = \infty$ , the function y(x) is two-point oscillatory on I, then (1.3) is two-point oscillatory on I.

*Example 2.3.* Let  $p(x) = -\ln \ln(1/x)$ ,  $x \in I$ , then  $p'(x) = 1/(x \ln(1/x))$ ,  $p''(x) = (1 - \ln(1/x))/(x \ln(1/x))^2$ , p(x) satisfies the condition (1.4), so the following equation:

$$\left(\frac{1}{x\ln(1/x)}y'\right)' - 2\frac{1 - \ln(1/x)}{(x\ln(1/x))^2}y' + \left(\frac{1}{x\ln(1/x)}\right)^3 y = 0$$
 (2.2)

is two-point oscillatory on *I*.

Example 2.4. Let  $p(x) = -(1-2x)/(x-x^2)^{\varepsilon}$ ,  $x \in I$ , where  $\varepsilon > 0$ . Then,

$$p'(x) = \frac{2(x - x^2) + \varepsilon(1 - 2x)^2}{(x - x^2)^{(\varepsilon + 1)}},$$

$$p''(x) = -\frac{(\varepsilon + 1)(1 - 2x)(x - x^2)^{\varepsilon} \left[2(x - x^2) + \varepsilon(1 - 2x)^2\right] - (2 - 4\varepsilon)(1 - 2x)(x - x^2)^{\varepsilon + 1}}{(x - x^2)^{2\varepsilon + 2}},$$
(2.3)

when  $x \to 0$ ,  $p(x) \to -\infty$ ; when  $x \to 1$ ,  $p(x) \to -\infty$ , which satisfies the condition (1.4);  $p'(x) \in C(I)$ ,  $p''(x) \in C(I)$ . Substituting p'(x) and p''(x) into (1.3), we obtain that the following equation:

$$\left(\frac{2(x-x^{2})+\varepsilon(1-2x)^{2}}{(x-x^{2})^{(\varepsilon+1)}}y'\right)' - 2\frac{(2-4\varepsilon)(1-2x)(x-x^{2})^{\varepsilon+1}-(\varepsilon+1)(1-2x)(x-x^{2})^{\varepsilon}\left[2(x-x^{2})+\varepsilon(1-2x)^{2}\right]}{(x-x^{2})^{2\varepsilon+2}}y' + \left(\frac{2(x-x^{2})+\varepsilon(1-2x)^{2}}{(x-x^{2})^{(\varepsilon+1)}}\right)^{3}y = 0$$
(2.4)

is two-point oscillatory on I.

### 3. A New Comparison Theorem

**Theorem 3.1.** Suppose that the second order differential equations

$$(p_1(x)y'(x))' + r_1(x)y'(x) + q_1(x)y(x) = 0,$$
(3.1)

$$(p_2(x)z'(x))' + r_2(x)z'(x) + q_2(x)z(x) = 0, (3.2)$$

satisfy the existence and uniqueness theorem on I, and one of the following conditions holds:

(1) when  $0 < p_2 < p_1$  and  $r_1 \neq r_2$ ,

$$q_2 \ge q_1 + \frac{r_2^2}{4p_2} + \frac{(r_1 - r_2)^2}{4(p_1 - p_2)},$$
 (3.3)

(2) when  $0 < p_1 = p_2 = p$  and  $r_1 \neq r_2$ , there exists an  $n \in \mathbb{R}^+$ , which satisfies

$$q_2 \ge (1+n)q_1 + \frac{((1+n)r_1 - r_2)^2 + r_2^2}{4p},$$
 (3.4)

then (3.2) has at least one zero point between two consecutive zero point  $\alpha$ ,  $\beta$  ( $\alpha < \beta$ ) of any nontrivial solution y(x) of (3.1).

*Proof.* (1) We suppose that z(x) has no zero point on  $[\alpha, \beta]$  when  $0 < p_2 < p_1$ . Without loss of generality, let  $y(x) \ge 0$ , z(x) > 0,  $x \in [\alpha, \beta]$ , then we have

$$\frac{d}{dx} \left[ \frac{y}{z} (zp_1 y' - yp_2 z') \right] \\
= \frac{y}{z} \left[ z(p_1 y')' + p_1 z' y' - y(p_2 z')' - p_2 y' z' \right] + \frac{y'z - z'y}{z^2} (p_1 y'z - p_2 yz') \\
= \frac{y}{z} \left[ z(-q_1 y - r_1 y') - y(-q_2 z - r_2 z') + (p_1 - p_2) y' z' \right] + \frac{y'z - z'y}{z^2} (p_1 y'z - p_2 yz') \\
= (q_2 - q_1) y^2 + r_2 \frac{y^2 z'}{z} - r_1 yy' + (p_1 - p_2) \frac{yy'z'}{z} + p_1 (y')^2 - p_1 \frac{yy'z'}{z} \\
- p_2 \frac{yy'zz'}{z^2} + p_2 \frac{(yz')^2}{z^2} \\
= (q_2 - q_1) y^2 + r_2 \frac{y^2 z'}{z} - r_1 yy' + (p_1 - p_2) (y')^2 + p_2 \left( y' - \frac{yz'}{z} \right)^2 \\
= (q_2 - q_1) y^2 + (\sqrt{p_1 - p_2} y')^2 - (r_1 - r_2) yy' - r_2 y \left( y' - \frac{yz'}{z} \right) + \left[ \sqrt{p_2} \left( y' - \frac{yz'}{z} \right) \right]^2 \\
+ \left( \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}} y \right)^2 + \left( \frac{r_2 y}{2\sqrt{p_2}} \right)^2 - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)} y^2 - \frac{r_2^2}{4p_2} y^2 \\
= \left[ \sqrt{p_1 - p_2} y' - \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}} y \right]^2 + \left[ \frac{r_2 y}{2\sqrt{p_2}} - \sqrt{p_2} \left( y' - \frac{yz'}{z} \right) \right]^2 \\
+ \left[ q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)} \right] y^2. \tag{3.5}$$

Integrating the above equation from  $\alpha$  to  $\beta$ , we obtain

$$0 = \int_{\alpha}^{\beta} \left[ \sqrt{p_1 - p_2} y' - \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}} y \right]^2 dx$$

$$+ \int_{\alpha}^{\beta} \left[ \frac{r_2 y}{2\sqrt{p_2}} - \sqrt{p_2} \left( y' - \frac{yz'}{z} \right) \right]^2 dx$$

$$+ \int_{\alpha}^{\beta} \left[ q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)} \right] y^2 dx,$$
(3.6)

that is,

$$0 \ge \int_{\alpha}^{\beta} \left[ q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)} \right] y^2 dx. \tag{3.7}$$

From previous equality and assumption (3.3), we obtain the next equalities:

$$\sqrt{p_1 - p_2} y' = \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}} y, \tag{3.8}$$

$$\frac{r_2 y}{2\sqrt{p_2}} = \sqrt{p_2} \left( y' - \frac{y z'}{z} \right). \tag{3.9}$$

By (3.8), we obtain  $y'/y = (r_1 - r_2)/2(p_1 - p_2)$ ,  $y = e^{\int ((r_1 - r_2)/2(p_1 - p_2))dx}$ . By (3.9), we obtain  $(y'/y) - (z'/z) = (r_2/2p_2)$ . In summary, we obtain  $z = ye^{-\int (r_2/2p_2)dx}$ , that is,  $z(\alpha) = z(\beta) = 0$ , which contradicts with the assumption.

(2) We suppose that z(x) has no zero point on  $[\alpha, \beta]$  when  $0 < p_1 = p_2 = p$ . Without loss of generality, let  $y(x) \ge 0$ , z(x) > 0,  $x \in [\alpha, \beta]$ , for all  $n \in \mathbb{R}^+$ , then

$$\frac{d}{dx} \left[ \frac{y}{z} (zpy' - ypz') + nypy' \right] \\
= \frac{y}{z} \left( z(py')' - y(pz')' \right) + p \frac{y'z - z'y}{z^2} (y'z - yz') + p(y')^2 + ny(-q_1y - r_1y') \\
= \frac{y}{z} \left[ z(-q_1y - r_1y') - y(-q_2z - r_2z') \right] + p \frac{(y'z - z'y)^2}{z^2} + p(y')^2 + ny(-q_1y - r_1y') \\
= \left[ (q_2 - (1+n)q_1)y^2 + r_2 \frac{y^2z'}{z} - (1+n)r_1yy' \right] + p \frac{(y'z - z'y)^2}{z^2} + p(y')^2 \\
= \left[ (q_2 - (1+n)q_1)y^2 - ((1+n)r_1 - r_2)yy' - r_2y \left( y' - \frac{yz'}{z} \right) \right] + p \frac{(y'z - z'y)^2}{z^2} + p(y')^2 \\
= (q_2 - (1+n)q_1)y^2 - ((1+n)r_1 - r_2)yy' - r_2y \left( y' - \frac{yz'}{z} \right) \right] \\
+ p \left( y' - \frac{yz'}{z} \right)^2 + p(y')^2 + \frac{r_2^2}{4p}y^2 + \frac{(2r_1 - r_2)^2}{4p}y^2 - \frac{((1+n)r_1 - r_2)^2}{4p}y^2 - \frac{r_2^2}{4p}y^2 \right] \\
= \left[ \frac{r_2y}{2\sqrt{p}} - \sqrt{p} \left( y' - \frac{yz'}{z} \right) \right]^2 + \left[ \frac{((1+n)r_1 - r_2)y}{2\sqrt{p}} - y'\sqrt{p} \right]^2 \\
+ \left[ q_2 - (1+n)q_1 - \frac{(2r_1 - r_2)^2 + r_2^2}{4p} \right] y^2. \tag{3.10}$$

Integrating the above equation from  $\alpha$  to  $\beta$ , we obtain

$$0 \ge \int_{\alpha}^{\beta} \left[ q_2 - (1+n)q_1 - \frac{((1+n)r_1 - r_2)^2 + r_2^2}{4p} \right] y^2 dx. \tag{3.11}$$

We can find the contradiction similarly; here, we delete the details. This completes the proof.

When z(x) = w(z(x)) is a nonlinear term, where  $w(z) : R \to R$  is a continuous function and zw(z) > 0 for  $z \ne 0$ , Zhuang and Wu established some comparison theorems if  $w'(z) \ge K > 0$  holds in [3]. The condition of Corollary 2.2 in [3] is identical with (3.3) when w'(z) is smooth and K = 1, but there's no condition about the situation of  $p_1 = p_2 = p$ . We put "nyp'y" added to Picone identity, which solve the problem of the vacuousness of (3.3) when  $p_1 = p_2 = p$ . Then, we obtain (3.4) and establish the integrated comparison theorem of second order damped linear differential equations.

We can easily obtain the following corollaries by Theorem 3.1.

**Corollary 3.2.** Suppose (3.1), (3.2) satisfy the existence and uniqueness theorem on I. If (3.1) is two-point oscillatory on I,  $r_1 \neq r_2$  and satisfies one of the following conditions:

(1) when  $0 < p_2 < p_1$ , the following condition is satisfied on I,

$$q_2 \ge q_1 + \frac{r_2^2}{4p_2} + \frac{(r_2 - r_1)^2}{4(p_1 - p_2)},$$
 (3.12)

(2) when  $0 < p_1 = p_2 = p$ , the following condition is satisfied on I,

$$q_2 \ge (1+n)q_1 + \frac{((1+n)r_1 - r_2)^2 + r_2^2}{4p},$$
 (3.13)

then (3.2) is two-point oscillatory on I.

**Corollary 3.3.** *Consider the second order equation* (1.3) *and the following equation:* 

$$(A(x)y')' + B(x)y' + C(x)y = 0, (3.14)$$

where  $x \in I$ , p(x) satisfies condition (1.4),  $B(x) \neq -2p''(x)$ . Suppose they satisfy the existence and uniqueness theorem on I. When  $x \to 0+$  and  $x \to 1-$ , if 0 < A(x) < p'(x) is satisfied on I, and

$$C(x) \ge (p'(x))^3 + \frac{B(x)^2}{4A(x)} + \frac{(B(x) + 2p''(x))^2}{4(p'(x) - A(x))},$$
(3.15)

then (3.14) is two-point oscillatory on I.

Remark 3.4. The two-point oscillation of (1.2) is studied by comparison theorem and two-point oscillatory equation in [1]. When  $p_1(x) = p_2(x) = 1$  and  $r_1(x) = r_2(x) = 0$ , Theorem 3.1 reduces to Theorem 2.1 in [1].

As an application of Corollary 3.3, we discuss the two-point oscillation of (1.5). Since Example 2.4 is the known two-point oscillatory equation, that is  $p(x) = -(1-2x)/(x-x^2)^{\varepsilon}$ ,  $x \in I$ , where  $\varepsilon > 0$ ,  $p'(x) \sim (x-x^2)^{-(\varepsilon+1)}$  as  $x \to 0+$  or  $x \to 1-$ ;  $p''(x) \sim (x-x^2)^{-(\varepsilon+2)}$  as  $x \to 0+$  or  $x \to 1-$ . For (1.5), A(x) = 1,  $B(x) = f(x)/(x-x^2)^{\alpha}$ ,  $C(x) = g(x)/(x-x^2)^{\beta}$ . Because of f(x) > 0,  $f(x) \in C(\overline{I})$ , there exists M > 0 such that f(x) < M for all  $x \in I$ . Therefore,

$$B(x) = \frac{f(x)}{(x - x^2)^{\alpha}} \sim \left(x - x^2\right)^{-\alpha}, \quad \text{as } x \longrightarrow 0 + \text{or } x \longrightarrow 1 -,$$

$$C(x) = \frac{g(x)}{(x - x^2)^{\beta}} \sim \left(x - x^2\right)^{-\beta}, \quad \text{as } x \longrightarrow 0 + \text{or } x \longrightarrow 1 -,$$

$$(3.16)$$

Thus, when  $x \to 0+$  and  $x \to 1-$ ,  $p'(x) \to +\infty$ ,  $A(x) \equiv 1$ ,  $-2p''(x) \neq B(x)$ ,

$$\frac{(B(x) + 2p''(x))^2}{4(p'(x) - A(x))} \sim \frac{\left[ (x - x^2)^{-\alpha} + (x - x^2)^{-\varepsilon - 2} \right]^2}{(x - x^2)^{-(\varepsilon + 1)}}$$

$$\sim (x - x^2)^{-(3+\varepsilon)} + (x - x^2)^{-(2\alpha - \varepsilon - 1)} + (x - x^2)^{-(\alpha + 1)},$$
as  $x \to 0 + \text{or } x \to 1-,$ 

$$\frac{B(x)^2}{4A(x)} \sim \left(x - x^2\right)^{-2\alpha}, \quad \text{as } x \longrightarrow 0 + \text{or } x \longrightarrow 1 -,$$

$$\left(p'\right)^3 = \left[\frac{2(x - x^2) + \varepsilon(1 - 2x)^2}{(x - x^2)^{(\varepsilon + 1)}}\right]^3 \sim \left(x - x^2\right)^{-3(\varepsilon + 1)}, \quad \text{as } x \longrightarrow 0 + \text{or } x \longrightarrow 1 -.$$
(3.17)

By (3.17), the following condition need to be satisfied if (3.15) holds,

$$\left(x - x^{2}\right)^{-\beta} \ge \left(x - x^{2}\right)^{-3(\varepsilon + 1)} + \left(x - x^{2}\right)^{-2\alpha} + \left(x - x^{2}\right)^{-(3+\varepsilon)} + \left(x - x^{2}\right)^{-(2\alpha - \varepsilon - 1)} + \left(x - x^{2}\right)^{-(\alpha + 1)},$$
as  $x \longrightarrow 0 + \text{or } x \longrightarrow 1-,$ 
(3.18)

that is,

$$\beta > \max\{3(\varepsilon+1), 2\alpha, \alpha+1\}. \tag{3.19}$$

In summary, let  $\varepsilon \to 0$ , then,

when  $0 < \alpha < 3/2, 3 + 3\varepsilon > \max\{2\alpha, \alpha + 1\}$ , condition (3.19) holds with  $\beta > 3$ , (1.5) is two-point oscillatory on I in this case,

when  $\alpha > 3/2$ ,  $2\alpha > \max\{3 + 3\varepsilon, \alpha + 1\}$ , condition (3.19) holds with  $\beta > 2\alpha$ , (1.5) is two-point oscillatory on I in this case.

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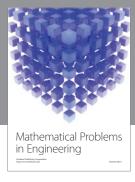
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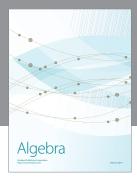
#### References

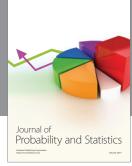
- [1] M. Pašić and J. S. W. Wong, "Two-point oscillations in second-order linear differential equations," *Differential Equations & Applications*, vol. 1, no. 1, pp. 85–122, 2009.
- [2] M. K. Kwong, M. Pašić, and J. S. W. Wong, "Rectifiable oscillations in second-order linear differential equations," *Journal of Differential Equations*, vol. 245, no. 8, pp. 2333–2351, 2008.
- [3] R.-K. Zhuang and H.-W. Wu, "Sturm comparison theorem of solution for second order nonlinear differential equations," *Applied Mathematics and Computation*, vol. 162, no. 3, pp. 1227–1235, 2005.













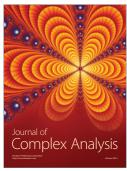




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