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Research Article

An Inequality of Meromorphic Vector Functions and Its Application

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Firstly, an inequality for vector-valued meromorphic functions is established which extend a corresponding inequality of Milloux for meromorphic scalar-valued function (1946). As an application, the relationship between the characteristic function of a vector-valued meromorphic function f and its derivative f' is studied, results are obtained to extend some related results for meromorphic scalar-valued function of Weitsman (1969) and Singh and Gopalakrishna (1971).

1. Introduction of Vector-Valued Meromorphic Function

In 1980s, Ziegler [1] established Nevanlinna's theory for the vector-valued meromorphic function in finite dimensional spaces. After Ziegler some works related to vector-valued meromorphic function were done in 1990s [2–4]. In this section, we shall introduce the following fundamental notations and results of vector-valued Nevanlinna theory which were quoted from Ziegler [1].

We denote by $\mathbb{C}^n$ the usual n dimensional complex Euclidean space with the coordinates $w = (w_1, w_2, \dots, w_n)$, the Hermitian scalar product

$$\langle v, w \rangle = v_1\overline{w}_1 + v_2\overline{w}_2 + \cdots + v_n\overline{w}_n, \quad (v, w \in \mathbb{C}^n), \tag{1.1}$$

and the distance

$$\|v - w\| = +\langle v - w, v - w \rangle^{1/2}. \tag{1.2}$$

Let

$$w_1 = f_1(z), \qquad w_2 = f_2(z), \dots, w_n = f_n(z) \tag{1.3}$$

be $n \geq 1$ complex valued functions of the complex variable z, which are meromorphic and not all constant in the Gaussian plane $\mathbb{C}^1 = \mathbb{C}$, or in a finite disc

$$\mathbb{C}_R = \{|z| < R\} \subset \mathbb{C}, \quad 0 < R < +\infty. \tag{1.4}$$

Thus in $\mathbb{C}_R,\ 0 < R \leq +\infty$ (we put $\mathbb{C}_{+\infty} = \mathbb{C}$), a vector-valued meromorphic function

$$f(z) = (f_1(z), f_2(z), \dots, f_n(z)) \tag{1.5}$$

is given, which does not reduce to the constant zero vector $0 = (0, 0, \dots, 0)$. The jth derivative $j = 1, 2, \dots$ of $f(z)$ are defined by

$$f^{(j)}(z) = \left(f_1^{(j)}(z), f_2^{(j)}(z), \dots, f_n^{(j)}(z)\right). \tag{1.6}$$

For such a function, the notations "zero," "pole," and "multiplicity" are defined as in the scalar case $n = 1$ of only one meromorphic function $f_1(z)$. More explicitly, in the punctured vicinity of each point $z_0 \in \mathbb{C}_R$, the vector function $w = f(z)$ can developed into a Laurent series

$$f(z) = c_{k_0}(z - z_0)^{k_0} + c_{k_0+1}(z - z_0)^{k_0+1} + \cdots, \tag{1.7}$$

where the coefficients are vectors

$$c_k = \left(c_k^1, c_k^2, \dots, c_k^n\right) \in \mathbb{C}^n, \quad c_{k_0} \neq (0, 0, \dots, 0). \tag{1.8}$$

In order to introduce the Nevanlinna theory of vector-valued meromorphic function, we will denote by "∞" the ideal element of the Aleksandrov one-point compactification of $\mathbb{C}^n$ (the two real infinities will be denoted by $+\infty$ and $-\infty$, resp.). Now, if $k_0 \leq 0$ in the above Laurent expansion, then z_0 will be called a pole or an ∞-point of $f(z)$ of multiplicity $-k_0$; in such a point z_0 at least one of the meromorphic component functions $f_j(z)$ has a pole of this multiplicity in the ordinary sense of function theory, so that in z_0 itself $f(z)$ is not defined. If $k_0 > 0$ in Laurent expansion, then z_0 is called a zero of $f(z)$ of multiplicity k_0; in such a point z_0, all component functions $f_j(z)$ vanish, each with at least this multiplicity.

Let $n(r,f)$ or $n(r,\infty)$ denote the number of poles of $f(z)$ in $|z| \le r$ and $n(r,a)$ denote the number of a-points of $f(z)$ in $|z| \le r$, counting with multiplicities. Define the volume function associated with vector-valued meromorphic function $f(z)$,

$$\begin{aligned} V(r,\infty) = V(r,f) &= \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log\|f(\xi)\| dx \wedge dy, \quad \xi = x+iy \\ V(r,a) = V\left(r, \frac{1}{f-a}\right) &= \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log\|f(\xi)-a\| dx \wedge dy, \quad \xi = x+iy \end{aligned} \tag{1.9}$$

and the counting function of finite or infinite a-points by

$$\begin{aligned} N(r,f) &= n(0,f)\log r + \int_0^r \frac{n(t,f)-n(0,f)}{t}dt, \\ N(r,\infty) &= n(0,\infty)\log r + \int_0^r \frac{n(t,\infty)-n(0,\infty)}{t}dt, \\ N(r,a) &= n(0,a)\log r + \int_0^r \frac{n(t,a)-n(0,a)}{t}dt, \end{aligned} \tag{1.10}$$

respectively. Next, we define

$$\begin{aligned} m(r,\infty) = m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left\|f\left(re^{i\theta}\right)\right\| d\theta, \\ m(r,a) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta})-a\|} d\theta, \\ T(r,f) &= m(r,f) + N(r,f). \end{aligned} \tag{1.11}$$

Let $\overline{n}(r,f)$ or $\overline{n}(r,\infty)$ denote the number of poles of $f(z)$ in $|z| \le r$ and $\overline{n}(r,a)$ denote the number of a-points of $f(z)$ in $|z| \le r$, ignoring multiplicities. Define the counting function of finite or infinite a-points by

$$\begin{aligned} \overline{N}(r,f) &= \overline{n}(0,f)\log r + \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt, \\ \overline{N}(r,\infty) &= \overline{n}(0,\infty)\log r + \int_0^r \frac{\overline{n}(t,\infty)-\overline{n}(0,\infty)}{t}dt, \\ \overline{N}(r,a) &= \overline{n}(0,a)\log r + \int_0^r \frac{\overline{n}(t,a)-\overline{n}(0,a)}{t}dt, \end{aligned} \tag{1.12}$$

respectively.

If $f(z)$ is a vector-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

$$\lambda(f) = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}. \tag{1.13}$$

We call the vector-valued meromorphic function f admissible if

$$\limsup_{r\to+\infty}\frac{T(r,f)}{\log r} = +\infty. \tag{1.14}$$

Definition 1.1. For a meromorphic function $f(z)$ (vector-valued or scalar-valued), we denote by $S(r,f)$ any quantity such that

$$S(r,f) = o(T(r,f)), \quad r \longrightarrow +\infty \tag{1.15}$$

without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure.

Definition 1.1 quoted from [2]. In [1], Ziegler established the following first main theorem, logarithmic derivative lemma, and deficient values theorem for meromorphic vector function.

Theorem A. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be a meromorphic vector function in $\mathbb{C}_R$. Then for $0 < r < R \leq +\infty$, $a \in \mathbb{C}^n$, $f(z) \not\equiv a$, then*

$$T(r,f) = V(r,a) + N(r,a) + m(r,a) + O(1). \tag{1.16}$$

Theorem B. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be a nonconstant meromorphic vector function in $\mathbb{C}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta}) - a\|}d\theta = S(r,f), \quad a \in \mathbb{C}^n. \tag{1.17}$$

By the second main theorem, Ziegler [1] studies the following deficiency theorem for meromorphic vector function. For any vector $a \in \mathbb{C}^n$, we define the number $\delta(a) = \delta(a,f)$ by putting

$$\delta(a) = \delta(a,f) = \liminf_{r\to+\infty}\frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to+\infty}\frac{V(r,a)+N(r,a)}{T(r,f)},$$
$$\delta(\infty) = \delta(\infty,f) = \liminf_{r\to+\infty}\frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to+\infty}\frac{N(r,f)}{T(r,f)}, \tag{1.18}$$

and $\Theta(a) = \Theta(a, f)$ by putting

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r\to+\infty} \frac{V(r,a) + \overline{N}(r,a)}{T(r,f)},$$
$$\Theta(\infty) = \Theta(\infty,f) = 1 - \limsup_{r\to+\infty} \frac{\overline{N}(r,f)}{T(r,f)}, \tag{1.19}$$

Theorem C. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function in $\mathbb{C}$. Then the set $\{a \in \mathbb{C}^n \cup \{\infty\}, \delta(a) > 0\}$ is at most countable and summing over all such points we have*

$$\sum_a \delta(a) \le \sum_a \Theta(a) \le 2. \tag{1.20}$$

2. A Fundamental Inequality of Meromorphic Vector Function

For meromorphic scalar-valued function $f(z)$, Milloux [5] has proved the following theorem.

Theorem D. *If $f(z)$ is a nonconstant meromorphic scalar-valued function in Gaussian complex plane $\mathbb{C}$ and if a_i, $i = 1, 2, \ldots, q$, are distinct elements of $\mathbb{C}$ (where q is any positive integer), then*

$$qT(r,f) \le T(r,f') + \sum_{i=1}^{q} \overline{N}(r,a_i) + S(r,f). \tag{2.1}$$

For some alternative proofs of Theorem D, see [6] or [7]. It is natural to consider whether there exists a similar results if meromorphic scalar-valued function $f(z)$ is replaced by meromorphic vector-valued function $f(z)$. In this section, the main contribution is to extend Theorem D to vector-valued meromorphic function by referring the method of [1, 7].

Theorem 2.1. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function in $\mathbb{C}$ and if $a^{[j]}$, $j = 1, 2, \ldots, q$, are distinct elements of $\mathbb{C}^n$ (where q is any positive integer), then*

$$qT(r,f) \le T(r,f') + \sum_{j=1}^{q} \left(\overline{N}\left(r, a^{[j]}\right) + V\left(r, a^{[j]}\right)\right) + S(r,f). \tag{2.2}$$

Proof. Put

$$F(z) = \sum_{j=1}^{q} \frac{1}{\|f(z) - a^{[j]}\|}. \tag{2.3}$$

We can get

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ F\left(re^{i\theta}\right) d\theta \le m(r,0,f') + \frac{1}{2\pi}\int_0^{2\pi} \log^+\left\{F\left(re^{i\theta}\right)\left\|f'\left(re^{i\theta}\right)\right\|\right\} d\theta. \tag{2.4}$$

Put

$$\delta = \min_{i \neq j} \left\| a^{[i]} - a^{[j]} \right\|. \tag{2.5}$$

Let for the moment $\mu \in \{1, 2, \dots, q\}$ be fixed. Then we get in every point where

$$\left\| f(z) - a^{[\mu]} \right\| < \frac{\delta}{2q} \leq \frac{\delta}{4}, \tag{2.6}$$

the inequality

$$\left\| f(z) - a^{[\nu]} \right\| \geq \left\| a^{[\mu]} - a^{[\nu]} \right\| - \left\| f(z) - a^{[\mu]} \right\| \geq \frac{3\delta}{4}, \tag{2.7}$$

for $\mu \neq \nu$. Therefore, the set of points on $\partial \mathbb{C}_r$ which is determined by (2.6) is either empty or any two such sets for different μ have empty intersection. In any case,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log^+ F\left(re^{i\theta}\right) d\theta &\geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int_{\|f(z)-a^{[\mu]}\|<\delta/2q,\, |z|=r} \log^+ F\left(re^{i\theta}\right) d\theta \\ &\geq \frac{1}{2\pi} \sum_{\mu=1}^{q} \int_{\|f(z)-a^{[\mu]}\|<\delta/2q,\, |z|=r} \log^+ \frac{1}{\left\| f\left(re^{i\theta}\right) - a^{[\mu]} \right\|} d\theta. \end{aligned} \tag{2.8}$$

Because of

$$\begin{aligned} &\frac{1}{2\pi} \int_{\|f(z)-a^{[\mu]}\|<\delta/2q,\, |z|=r} \log^+ \frac{1}{\left\| f\left(re^{i\theta}\right) - a^{[\mu]} \right\|} d\theta \\ &\quad = m\left(r, a^{[\mu]}\right) - \frac{1}{2\pi} \int_{\|f(z)-a^{[\mu]}\|\geq\delta/2q,\, |z|=r} \log^+ \frac{1}{\left\| f\left(re^{i\theta}\right) - a^{[\mu]} \right\|} d\theta \\ &\quad \geq m\left(r, a^{[\mu]}\right) - \log^+ \frac{2q}{\delta}, \end{aligned} \tag{2.9}$$

it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ F\left(re^{i\theta}\right) d\theta \geq \sum_{\mu=1}^{q} m\left(r, a^{[\mu]}\right) - \log^+ \frac{2q}{\delta}, \tag{2.10}$$

so that by (2.4)

$$\sum_{\mu=1}^{q} m\left(r, a^{[\mu]}\right) \leq m(r, 0, f') + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\{ F\left(re^{i\theta}\right) \left\| f'\left(re^{i\theta}\right) \right\| \right\} d\theta + \log^+ \frac{2q}{\delta}. \tag{2.11}$$

Thus by Theorem B, we have

$$\sum_{\mu=1}^{q} m\left(r, a^{[\mu]}\right) \leq m(r, 0, f') + S(r, f). \tag{2.12}$$

It follows from Theorem A that

$$m(r, 0, f') + N(r, 0, f') + V(r, 0, f') = T(r, f') + O(1). \tag{2.13}$$

Thus from (2.12) and (2.13), we deduce

$$\sum_{\mu=1}^{q} m\left(r, a^{[\mu]}\right) \leq T(r, f') - N(r, 0, f') + S(r, f). \tag{2.14}$$

Adding $\sum_{\mu=1}^{q} N(r, a^{[\mu]})$ to both sides,

$$\begin{aligned}\sum_{\mu=1}^{q} T\left(r, \frac{1}{f - a^{[\mu]}}\right) &\leq T(r, f') + \sum_{\mu=1}^{q} N\left(r, a^{[\mu]}\right) - N(r, 0, f') + S(r, f) \\ &= T(r, f') + \sum_{\mu=1}^{q} \overline{N}\left(r, a^{[\mu]}\right) - N_0(r, 0, f') + S(r, f),\end{aligned} \tag{2.15}$$

where $N_0(r, 0, f')$ is formed with the zeros of f' which are not zeros of any of $f - a^{[\mu]}, (i = 1, 2, \ldots, q)$. Since $N_0(r, 0, f') \geq 0$, we have

$$\sum_{\mu=1}^{q} T\left(r, \frac{1}{f - a^{[\mu]}}\right) \leq T(r, f') + \sum_{\mu=1}^{q} \overline{N}\left(r, a^{[\mu]}\right) + S(r, f). \tag{2.16}$$

Since

$$T\left(r, \frac{1}{f - a^{[\mu]}}\right) + V\left(r, a^{[\mu]}\right) = T(r, f) + O(1), \tag{2.17}$$

it follows that

$$qT(r, f) \leq T(r, f') + \sum_{j=1}^{q} \left(\overline{N}\left(r, a^{[j]}\right) + V\left(r, a^{[j]}\right)\right) + S(r, f). \tag{2.18}$$

□

3. Characteristic Function of Derivative of Meromorphic Vector Function

Let $f(z)$ be a meromorphic scalar-valued function in $\mathbb{C}$. The characteristic function of derivative of $f(z)$ with $\sum_a \delta(a) = 2$ has been studied by Edrei [8], Shan and Singh [9], Singh and Gopalakrishna [7], Singh and Kulkarni [10] and Weitsman [11]. For example, Edrei [8] and Weitsman [11] have proved the following theorem.

Theorem E. *Let $f(z)$ be a transcendental meromorphic scalar-valued function of finite order and assume $\sum_{a\in\mathbb{C}} \delta(a) = \eta \geq 1$ and $\delta(\infty) = 2 - \eta$. Then*

$$T(r, f') \sim \eta T(r, f), \quad r \longrightarrow +\infty. \tag{3.1}$$

If $\sum_a \delta(a) = 2$ is replaced by $\sum_a \Theta(a) = 2$, Singh and Gopalakrishna [7] and Singh and Kulkarni [10] have proved the following theorem.

Theorem F. *Let $f(z)$ be a transcendental meromorphic scalar-valued function of finite order and assume $\sum_a \Theta(a) = 2$. Then*

$$\begin{aligned} &\lim_{r\to+\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty), \\ &\lim_{r\to+\infty} \frac{\overline{N}(r, a)}{T(r, f)} = 1 - \Theta(a) \end{aligned} \tag{3.2}$$

for every $a \in \mathbb{C} \cup \{\infty\}$.

It is natural to consider whether there exists a similar results if meromorphic scalar-valued function $f(z)$ is replaced by meromorphic vector-valued function $f(z)$. In this section, the main purpose is to extend the above theorems to vector-valued meromorphic function by referring the method of [1, 7].

Theorem 3.1. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_a \Theta(a) = 2$. Then*

$$\begin{gathered} \lim_{r\to+\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty), \\ \lim_{r\to+\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty), \qquad \lim_{r\to+\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a) \end{gathered} \tag{3.3}$$

for every $a \in \mathbb{C}^n$.

Proof. Now, basic estimates in vector-valued Nevanlinna theory [1] or [4] yields

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \\ &= m\left(r,\frac{ff'}{f}\right) + N(r,f') \\ &\le m\left(r,\frac{f'}{f}\right) + m(r,f) + N(r,f) + \overline{N}(r,f) \\ &\le T(r,f) + \overline{N}(r,f) + m\left(r,\frac{f'}{f}\right). \end{aligned} \tag{3.4}$$

By Theorem B and the above inequality, we have

$$\limsup_{r\to+\infty}\frac{T(r,f')}{T(r,f)} \le 2-\Theta(\infty). \tag{3.5}$$

Let $\{a^{[j]}\}$ be a sequence of distinct vector in $\mathbb{C}^n$ containing all the vector of $\delta(a^{[j]})>0$. From Theorem 2.1, for any positive integer q, we have

$$qT(r,f) \le T(r,f') + \sum_{j=1}^{q}\left(\overline{N}\left(r,a^{[j]}\right) + V\left(r,a^{[j]}\right)\right) + S(r,f). \tag{3.6}$$

Hence

$$\begin{aligned} q &\le \liminf_{r\to+\infty}\frac{T(r,f')}{T(r,f)} + \sum_{j=1}^{q}\limsup_{r\to+\infty}\frac{\overline{N}(r,a^{[j]}) + V(r,a^{[j]})}{T(r,f)} + \limsup_{r\to+\infty}\frac{S(r,f)}{T(r,f)} \\ &= \liminf_{r\to+\infty}\frac{T(r,f')}{T(r,f)} + \sum_{j=1}^{q}\left\{1-\Theta\left(a^{[j]}\right)\right\} + \limsup_{r\to+\infty}\frac{S(r,f)}{T(r,f)}. \end{aligned} \tag{3.7}$$

Thus

$$\liminf_{r\to+\infty}\frac{T(r,f')}{T(r,f)} \ge \sum_{j=1}^{q}\Theta\left(a^{[j]}\right). \tag{3.8}$$

Since q was arbitrary, we have

$$2-\Theta(\infty) = \sum_{a\in\mathbb{C}^n}\Theta(a) \le \liminf_{r\to+\infty}\frac{T(r,f')}{T(r,f)}. \tag{3.9}$$

This and (3.5) yield

$$\lim_{r\to+\infty}\frac{T(r,f')}{T(r,f)}=2-\Theta(\infty). \tag{3.10}$$

Let $a\in\mathbb{C}^n\cup\{\infty\}$ and $\{a^{[i]}\}_{i=1}^{+\infty}$ an infinite sequence of distinct elements of $\mathbb{C}^n\cup\{\infty\}$ which includes every $b\in\mathbb{C}^n\cup\{\infty\}$ satisfying $b\neq a$ and $\Theta(b)>0$. Then

$$\sum_{i=1}^{+\infty}\Theta\left(a^{[i]}\right)=\sum_{b\in\mathbb{C}^n\cup\{\infty\},\,b\neq a}\Theta(b)=2-\Theta(a). \tag{3.11}$$

Let q be any integer ≥ 3. From Generalized Second Main Theorem (see [1], Page 126), we have

$$(q-2)T(r,f)=\sum_{i=1}^{q-1}\left(\overline{N}\left(r,a^{[i]}\right)+V\left(r,a^{[i]}\right)\right)+\overline{N}(r,f)+S(r,f). \tag{3.12}$$

Hence

$$q-2\leq\sum_{i=1}^{q-1}\left\{1-\Theta\left(a^{[i]}\right)\right\}+\liminf_{r\to+\infty}\frac{\overline{N}(r,f)}{T(r,f)}. \tag{3.13}$$

Thus

$$\sum_{i=1}^{q-1}\Theta\left(a^{[i]}\right)-1\leq\liminf_{r\to+\infty}\frac{\overline{N}(r,f)}{T(r,f)}. \tag{3.14}$$

Since this holds for all $q\geq 3$, letting $q\to+\infty$ and combining (3.11), we get

$$1-\Theta(\infty)=\sum_{i=1}^{+\infty}\Theta\left(a^{[i]}\right)-1\leq\liminf_{r\to+\infty}\frac{\overline{N}(r,f)}{T(r,f)}\leq\limsup_{r\to+\infty}\frac{\overline{N}(r,f)}{T(r,f)}=1-\Theta(\infty). \tag{3.15}$$

So

$$\lim_{r\to+\infty}\frac{\overline{N}(r,f)}{T(r,f)}=1-\Theta(\infty). \tag{3.16}$$

For every $a\in\mathbb{C}^n$, Let q be any integer ≥ 3. From Generalized Second Main Theorem (see [1], Page 126), we have

$$(q-2)T(r,f)=\sum_{i=1}^{q-2}\left(\overline{N}\left(r,a^{[i]}\right)+V\left(r,a^{[i]}\right)\right)+\left(\overline{N}(r,a)+V(r,a)\right)+\overline{N}(r,f)+S(r,f). \tag{3.17}$$

Hence

$$q-2 \le \sum_{i=1}^{q-2}\left\{1-\Theta\left(a^{[i]}\right)\right\} + (1-\Theta(\infty)) + \liminf_{r\to+\infty}\frac{\overline{N}(r,a)+V(r,a)}{T(r,f)}. \tag{3.18}$$

Thus

$$\sum_{i=1}^{q-2}\Theta\left(a^{[i]}\right) + \Theta(\infty) - 1 \le \liminf_{r\to+\infty}\frac{\overline{N}(r,a)+V(r,a)}{T(r,f)}. \tag{3.19}$$

Since this holds for all $q \ge 3$, letting $q \to +\infty$ and combining (3.11), we get

$$\begin{aligned} 1-\Theta(a) = \sum_{i=1}^{+\infty}\Theta\left(a^{[i]}\right) - 1 &\le \liminf_{r\to+\infty}\frac{\overline{N}(r,a)+V(r,a)}{T(r,f)} \\ &\le \limsup_{r\to+\infty}\frac{\overline{N}(r,a)+V(r,a)}{T(r,f)} = 1-\Theta(a). \end{aligned} \tag{3.20}$$

So

$$\lim_{r\to+\infty}\frac{\overline{N}(r,a)+V(r,a)}{T(r,f)} = 1-\Theta(a). \tag{3.21}$$

□

From Theorem 3.1, we have the following corollary

Corollary 3.2. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a\in\mathbb{C}^n}\Theta(a) = 2$. Then*

$$T(r,f') \sim 2T(r,f), \quad r \longrightarrow +\infty. \tag{3.22}$$

Corollary 3.3. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a\in\mathbb{C}^n}\Theta(a) = \eta \ge 1$ and $\delta(\infty) = 2-\eta$. Then*

$$T(r,f') \sim \eta T(r,f), \quad r \longrightarrow +\infty. \tag{3.23}$$

Corollary 3.4. *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_a\delta(a) = 2$. Then*

$$\begin{gathered} \lim_{r\to+\infty}\frac{T(r,f')}{T(r,f)} = 2-\delta(\infty), \\ \lim_{r\to+\infty}\frac{N(r,f)}{T(r,f)} = 1-\delta(\infty), \qquad \lim_{r\to+\infty}\frac{N(r,a)+V(r,a)}{T(r,f)} = 1-\delta(a) \end{gathered} \tag{3.24}$$

for every $a \in \mathbb{C}^n$.

Proof. Since $\delta(a) \leq \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$ and Theorem C, it follows that, if $\sum_a \delta(a) = 2$, then $\sum_a \Theta(a) = 2$ and $\delta(a) = \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$. Hence

$$\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty) \tag{3.25}$$

follows by Theorem 3.1.

Now, for every $a \in \mathbb{C}^n$,

$$\lim_{r \to +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a) = 1 - \delta(a). \tag{3.26}$$

Further

$$\overline{N}(r, a) \leq N(r, a). \tag{3.27}$$

Hence

$$\begin{aligned} 1 - \delta(a) &\leq \lim_{r \to +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} \\ &\leq \liminf_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} \\ &\leq \limsup_{r \to +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} \\ &= 1 - \delta(a). \end{aligned} \tag{3.28}$$

Similarly,

$$\lim_{r \to +\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty) = 1 - \delta(\infty). \tag{3.29}$$

Further

$$\overline{N}(r, \infty) \leq N(r, \infty). \tag{3.30}$$

Hence

$$1 - \delta(\infty) \leq \lim_{r \to +\infty} \frac{\overline{N}(r, \infty)}{T(r, f)} \leq \liminf_{r \to +\infty} \frac{N(r, \infty)}{T(r, f)} \leq \limsup_{r \to +\infty} \frac{N(r, \infty)}{T(r, f)} = 1 - \delta(\infty). \tag{3.31}$$

□

From Corollary 3.4, we have the following corollary.

Corollary 3.5. *Let* $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ *be an admissible meromorphic vector function of finite order in* $\mathbb{C}$ *and assume* $\sum_{a\in\mathbb{C}^n}\delta(a) = 2$. *Then*

$$T(r, f') \sim 2T(r, f), \quad r \longrightarrow +\infty. \tag{3.32}$$

Corollary 3.6. *Let* $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ *be an admissible meromorphic vector function of finite order in* $\mathbb{C}$ *and assume* $\sum_{a\in\mathbb{C}^n}\delta(a) = \eta \geq 1$ *and* $\delta(\infty) = 2 - \eta$. *Then*

$$T(r, f') \sim \eta T(r, f), \quad r \longrightarrow +\infty. \tag{3.33}$$

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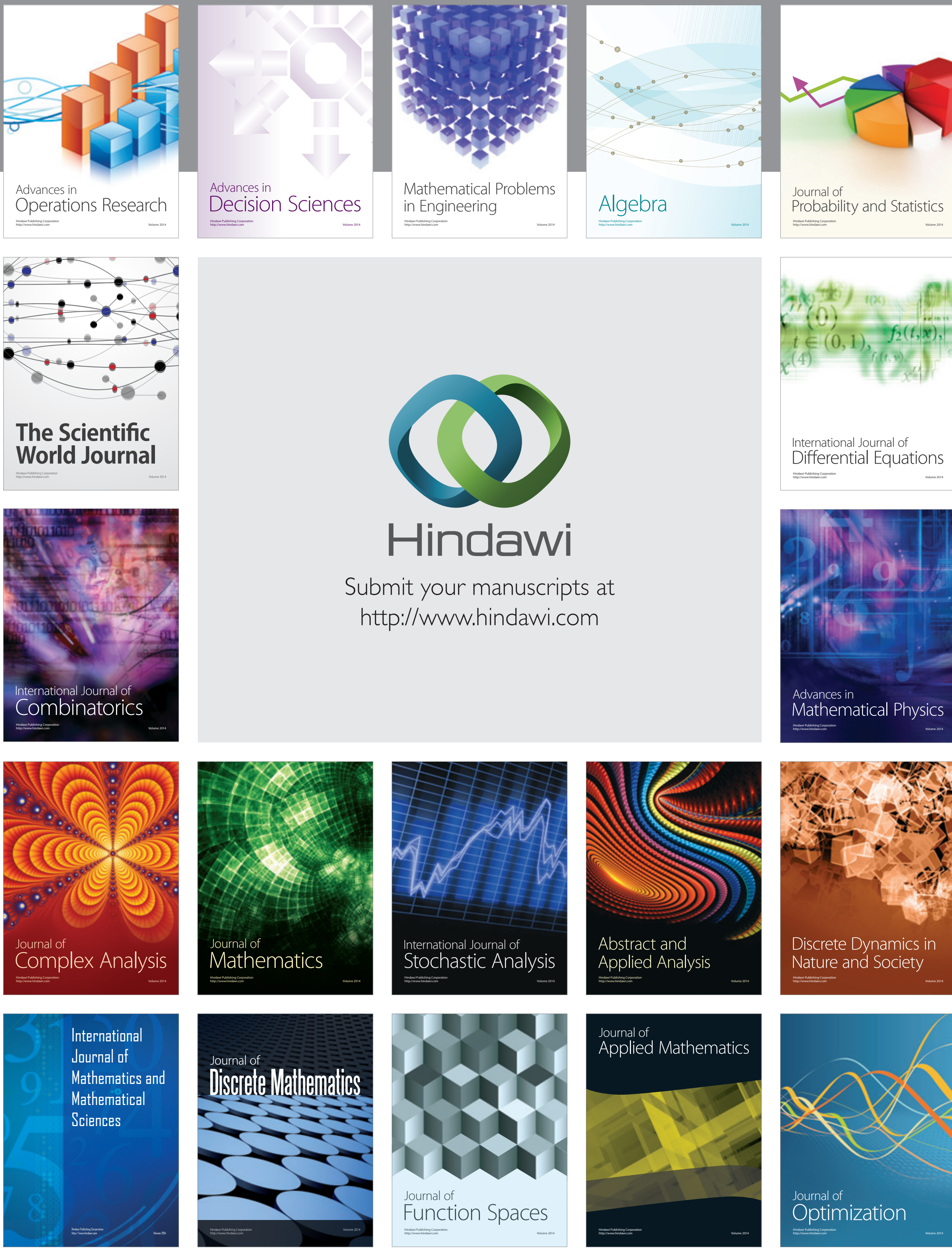
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