Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2011, Article ID 596971, 9 pages doi:10.1155/2011/596971

## Research Article

# **Approximate Best Proximity Pairs in Metric Space**

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Received 8 January 2011; Accepted 12 February 2011

Academic Editor: Norimichi Hirano

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Let A and B be nonempty subsets of a metric space X and also  $T:A\cup B\to A\cup B$  and  $T(A)\subseteq B$ ,  $T(B)\subseteq A$ . We are going to consider element  $x\in A$  such that  $d(x,Tx)\le d(A,B)+e$  for some e>0. We call pair (A,B) an approximate best proximity pair. In this paper, definitions of approximate best proximity pair for a map and two maps, their diameters, T-minimizing a sequence are given in a metric space.

#### 1. Introduction

Let X be a metric space and A and B nonempty subsets of X, and d(A, B) is distance of A and B. If  $d(x_0, y_0) = d(A, B)$ , then the pair  $(x_0, y_0)$  is called a best proximity pair for A and B and put

$$prox(A, B) := \{(x, y) \in A \times B : d(x, y) = d(A, B)\}$$
 (1.1)

as the set of all best proximity pair (A, B). Best proximity pair evolves as a generalization of the concept of best approximation. That reader can find some important result of it in [1-4].

Now, as in [5] (see also [4, 6–11]), we can find the best proximity points of the sets A and B, by considering a map  $T: A \cup B \to A \cup B$  such that  $T(A) \subseteq B$  and  $T(B) \subseteq A$ . Best proximity pair also evolves as a generalization of the concept of fixed point of mappings. Because if  $A \cap B \neq \emptyset$ , every best proximity point is a fixed point of T.

We say that the point  $x \in A \cup B$  is an approximate best proximity point of the pair (A, B), if  $d(x, Tx) \le d(A, B) + \epsilon$ , for some  $\epsilon > 0$ .

In the following, we introduce a concept of approximate proximity pair that is stronger than proximity pair.

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*Definition 1.1.* Let *A* and *B* be nonempty subsets of a metric space *X* and  $T: A \cup B \rightarrow A \cup B$  a map such that  $T(A) \subseteq B$ ,  $T(B) \subseteq A$ . put

$$P_T^a(A,B) = \{ x \in A \cup B : d(x,Tx) \le d(A,B) + \varepsilon \text{ for some } \varepsilon > 0 \}.$$
 (1.2)

We say that the pair (A, B) is an approximate best proximity pair if  $P_T^a(A, B) \neq \emptyset$ .

*Example 1.2.* Suppose that  $X = \mathbb{R}^2$ ,  $A = \{(x,y) \in X : (x-y)^2 + y^2 \le 1\}$ , and  $B = \{(x,y) \in X : (x+y)^2 + y^2 \le 1\}$  with T(x,y) = (-x,y) for  $(x,y) \in X$ . Then  $d((x,y),T(x,y)) \le d(A,B) + \epsilon$  for some  $\epsilon > 0$ . Hence  $P_T^a(A,B) \ne \emptyset$ .

### 2. Approximate Best Proximity

In this section, we will consider the existence of approximate best proximity points for the map  $T: A \cup B \to A \cup B$ , such that  $T(A) \subseteq B$ ,  $T(B) \subseteq A$ , and its diameter.

**Theorem 2.1.** Let A and B be nonempty subsets of a metric space X. Suppose that the mapping  $T: A \cup B \to A \cup B$  is satisfying  $T(A) \subseteq B$ ,  $T(B) \subseteq A$ , and

$$\lim_{n \to \infty} d\left(T^n x, T^{n+1} x\right) = d(A, B) \quad \text{for some } x \in A \cup B.$$
 (2.1)

Then the pair (A, B) is an approximate best proximity pair.

*Proof.* Let  $\epsilon > 0$  be given and  $x \in A \cup B$  such that  $\lim_{n \to \infty} d(T^n x, T^{n+1} x) = d(A, B)$ ; then there exists  $N_0 > 0$  such that

$$\forall n \ge N_0 : d\left(T^n x, T^{n+1} x\right) < d(A, B) + \epsilon. \tag{2.2}$$

If 
$$n = N_0$$
, then  $d(T^{N_0}(x), T(T^{N_0}(x))) < d(A, B) + \epsilon$ , and  $T^{N_0}(x) \in P_T^a(A, B)$  and  $P_T^a(A, B) \neq \emptyset$ .

**Theorem 2.2.** Let A and B be nonempty subsets of a metric space X. Suppose that the mapping  $T:A\cup B\to A\cup B$  is satisfying  $T(A)\subseteq B$ ,  $T(B)\subseteq A$  and

$$d(Tx,Ty) \le \alpha d(x,y) + \beta [d(x,Tx) + d(y,Ty)] + \gamma d(A,B)$$
(2.3)

for all  $x, y \in A \cup B$ , where  $\alpha, \beta, \gamma \ge 0$  and  $\alpha + 2\beta + \gamma < 1$ . Then the pair (A, B) is an approximate best proximity pair.

*Proof.* If  $x \in A \cup B$ , then

$$d(Tx, T^2x) \le \alpha d(x, Tx) + \beta \left[ d(x, Tx) + d(Tx, T^2x) \right] + \gamma d(A, B). \tag{2.4}$$

Therefore,

$$d\left(Tx, T^{2}x\right) \leq \frac{\alpha + \beta}{1 - \beta}d(x, Tx) + \frac{\gamma}{1 - \beta}d(A, B). \tag{2.5}$$

Now if  $k = (\alpha + \beta)/(1 - \beta)$ , then

$$d\left(Tx, T^2x\right) \le kd(x, Tx) + (1-k)d(A, B) \tag{2.6}$$

also

$$d(T^{2}x, T^{3}x) \le k^{2}d(x, Tx) + (1 - k^{2})d(A, B).$$
(2.7)

Therefore,

$$d(T^{n}x, T^{n+1}x) \le k^{n}d(x, Tx) + (1 - k^{n})d(A, B), \tag{2.8}$$

and so

$$d(T^n x, T^{n+1} x) \longrightarrow d(A, B), \text{ as } n \longrightarrow \infty.$$
 (2.9)

Therefore, by Theorem 2.1,  $P_T^a(A, B) \neq \emptyset$ ; then pair (A, B) is an approximate best proximity pair.

*Definition 2.3.* Let A and B be nonempty subsets of a metric space X. Suppose that the mapping  $T:A\cup B\to A\cup B$  is satisfying  $T(A)\subseteq B$ ,  $T(B)\subseteq A$ . We say that the sequence  $\{z_n\}\subseteq A\cup B$  is T-minimizing if

$$\lim_{n \to \infty} d(z_n, Tz_n) = d(A, B). \tag{2.10}$$

**Theorem 2.4.** Let A and B be nonempty subsets of a metric space X, suppose that the mapping T:  $A \cup B \to A \cup B$  is satisfying  $T(A) \subseteq B$ ,  $T(B) \subseteq A$ . If  $\{T^n x\}$  is a T-minimizing for some  $x \in A \cup B$ , then (A, B) is an approximate best pair proximity.

Proof. Since

$$\lim_{n \to \infty} d\left(T^n x, T^{n+1} x\right) = d(A, B) \quad \text{for some } x \in A \cup B,$$
 (2.11)

therefore, by Theorem 2.1,  $P_T^a(A, B) \neq \emptyset$ ; then pair (A, B) is an approximate best proximity pair.

**Theorem 2.5.** Let A and B be nonempty subsets of a normed space X such that  $A \cup B$  is compact. Suppose that the mapping  $T: A \cup B \to A \cup B$  is satisfying  $T(A) \subseteq B$ ,  $T(B) \subseteq A$ , T is continuous and

$$||Tx - Ty|| \le ||x - y||,$$
 (2.12)

where  $(x, y) \in A \times B$ . Then  $P_T^a(A, B)$  is nonempty and compact.

*Proof.* Since  $A \cup B$  compact, there exists a  $z_0 \in A \cup B$  such that

$$||z_0 - Tz_0|| = \inf_{z \in A \cup B} ||z - Tz||$$
 (\*)

If  $||z_0 - Tz_0|| > d(A, B)$ , then  $||Tz_0 - T^2z_0|| < ||z_0 - Tz_0||$  which contradict to the definition of  $z_0$ ,  $(Tz_0 \in A \cup B \text{ and by (*) } ||Tz_0 - T(Tz_0)|| \ge ||z_0 - Tz_0||)$ . Therefore,  $||z_0 - Tz_0|| = d(A, B) \le d(A, B) + \epsilon$  for some  $\epsilon > 0$  and  $z_0 \in P_T^a(A, B)$ . Therefore,  $P_T^a(A, B)$  is nonempty.

Also, if  $\{z_n\} \subseteq P_T^{\epsilon}(A,B)$ , then  $\|z_n - Tz_n\| < d(A,B) + \epsilon$ , for some  $\epsilon > 0$ , and by compactness of  $A \cup B$ , there exists a subsequence  $z_{n_k}$  and a  $z_0 \in A \cup B$  such that  $z_{n_k} \to z_0$  and so

$$||z_0 - Tz_0|| = \lim_{k \to \infty} ||z_{n_k} - Tz_{n_k}|| < d(A, B) + \epsilon$$
(2.13)

for some  $\epsilon > 0$ , hence  $P_T^a(A, B)$  is compact.

*Example 2.6.* If A = [-3, -1], B = [1, 3], and  $T : A \cup B \rightarrow A \cup B$  such that

$$T(x) = \begin{cases} \frac{1-x}{2}, & x \in A, \\ \frac{-1-x}{2}, & x \in B, \end{cases}$$
 (2.14)

then  $P_T^a(A, B)$  is compact, and we have

$$P_T^a(A,B) = \{x \in A \cup B : d(x,Tx) < d(A,B) + \epsilon \text{ for some } \epsilon > 0\}$$

$$= \{x \in A \cup B : d(x,Tx) < 2 + \epsilon \text{ for some } \epsilon > 0\}$$

$$= \{1,-1\}.$$
(2.15)

That is compact.

In the following, by diam( $P_T^a(A,B)$ ) for a set  $P_T^a(A,B) \neq \emptyset$ , we will understand the diameter of the set  $P_T^a(A,B)$ .

Definition 2.7. Let  $T: A \cup B \to A \cup B$  be a continuous map such that  $T(A) \subseteq B$ ,  $T(B) \subseteq A$  and  $\epsilon > 0$ . We define diameter  $P_T^a(A, B)$  by

$$\operatorname{diam}(P_T^a(A,B)) = \sup\{d(x,y) : x, y \in P_T^a(A,B)\}. \tag{2.16}$$

**Theorem 2.8.** Let  $T: A \cup B \to A \cup B$ , such that  $T(A) \subseteq B$ ,  $T(B) \subseteq A$  and  $\epsilon > 0$ . If there exists an  $\alpha \in [0,1]$  such that for all  $(x,y) \in A \times B$ 

$$d(Tx, Ty) \le \alpha d(x, y), \tag{2.17}$$

then

$$\operatorname{diam}(P_T^a(A,B)) \le \frac{2\epsilon}{1-\alpha} + \frac{2d(A,B)}{1-\alpha}.$$
 (2.18)

*Proof.* If  $x, y \in P_T^a(A, B)$ , then

$$d(x,y) \le d(x,Tx) + d(Tx,Ty) + d(Ty,y)$$
  

$$\le \epsilon_1 + \alpha d(x,y) + 2d(A,B) + \epsilon_2.$$
(2.19)

Put  $\epsilon = \text{Max}\{\epsilon_1, \epsilon_2\}$ , therefore,  $d(x, y) \le 2\epsilon/(1-\alpha) + (2d(A, B))/(1-\alpha)$ . Hence  $\text{diam}(P_T^a(A, B)) \le 2\epsilon/(1-\alpha) + (2d(A, B))/(1-\alpha)$ .

## 3. Approximate Best Proximity for Two Maps

In this section, we will consider the existence of approximate best proximity points for two maps  $T: A \cup B \to A \cup B$  and  $S: A \cup B \to A \cup B$ , and its diameter.

*Definition 3.1.* Let *A* and *B* be nonempty subsets of a metric space (*X*, *d*) and let *T* : *A* ∪ *B* →  $A \cup BS : A \cup B \to A \cup B$  two maps such that  $T(A) \subseteq B$ ,  $S(B) \subseteq A$ . A point (*x*, *y*) in  $A \times B$  is said to be an approximate-pair fixed point for (*T*, *S*) in *X* if there exists  $\epsilon > 0$ 

$$d(Tx, Sy) \le d(A, B) + \epsilon. \tag{3.1}$$

We say that the pair (T, S) has the approximate-pair fixed property in X if  $P^a_{(T,S)}(A, B) \neq \emptyset$ , where

$$P_{(T,S)}^{a}(A,B) = \{(x,y) \in A \times B : d(Tx,Sy) \le d(A,B) + \epsilon \text{ for some } \epsilon > 0\}.$$
 (3.2)

**Theorem 3.2.** Let A and B be nonempty subsets of a metric space (X, d) and let  $T : A \cup B \to A \cup B$  and  $S : A \cup B \to A \cup B$  be two maps such that  $T(A) \subseteq B$ ,  $S(B) \subseteq A$ . If, for every  $(x, y) \in A \times B$ ,

$$d(T^{n}(x), S^{n}(y)) \longrightarrow d(A, B), \tag{3.3}$$

then (T, S) has the approximate-pair fixed property.

*Proof.* For  $\epsilon > 0$ , Suppose  $(x, y) \in A \times B$ . Since

$$d(T^{n}(x), S^{n}(y)) \longrightarrow d(A, B),$$

$$\exists n_{0} > 0 \quad \text{s.t. } \forall n \geq n_{0} : d(T^{n}(x), S^{n}(y)) < d(A, B) + \epsilon,$$

$$(3.4)$$

then  $d(T(T^{n-1}(x), S(S^{n-1}(y)) < d(A, B) + \epsilon \text{ for every } n \ge n_0. \text{ Put } x_0 = T^{n_0-1}(x) \text{ and } y_0 = S^{n_0-1}(y)). \text{ Hence } d(T(x_0), S(y_0)) \le d(A, B) + \epsilon \text{ and } P^a_{(T,S)}(A, B) \ne \emptyset.$ 

**Theorem 3.3.** Let A and B be nonempty subsets of a metric space (X, d) and let  $T : A \cup B \to A \cup B$  and  $S : A \cup B \to A \cup B$  be two maps such that  $T(A) \subseteq B$ ,  $S(B) \subseteq A$  and, for every  $(x, y) \in A \times B$ ,

$$d(Tx, Sy) \le \alpha d(x, y) + \beta [d(x, Tx) + d(y, Sy)] + \gamma d(A, B), \tag{3.5}$$

where  $\alpha, \beta, \gamma \ge 0$  and  $\alpha + 2\beta + \gamma < 1$ . Then if x is an approximate fixed point for T, or y is an approximate fixed point for S, then  $P^a_{(TS)}(A, B) \ne \emptyset$ .

*Proof.* If  $(x, y) \in A \times B$ , then

$$d(Tx, S(Tx)) \le \alpha d(x, Tx) + \beta [d(x, Tx) + d(Tx, S(Tx))] + \gamma d(A, B). \tag{3.6}$$

Therefore,

$$d(Tx, S(Tx)) \le \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B). \tag{3.7}$$

Now if  $k = (\alpha + \beta)/(1 - \beta)$ , then

$$d(Tx, S(Tx)) \le kd(x, Tx) + (1 - k)d(A, B) \tag{*}$$

also

$$d(Sy, T(Sy)) \le kd(y, Sy) + (1 - k)d(A, B).$$
 (\*\*)

If x is an approximate fixed point for T, then there exists a  $\epsilon > 0$  and by (\*)

$$d(Tx, S(Tx)) \le kd(x, Tx) + (1 - k)d(A, B)$$

$$\le k(d(A, B) + \epsilon) + (1 - k)d(A, B)$$

$$= d(A, B) + k\epsilon$$

$$< d(A, B) + \epsilon.$$
(3.8)

And  $(x,Tx) \in P^a_{(T,S)}(A,B)$ ; also if y is an approximate fixed point for S, then there exists a  $\epsilon > 0$  and by (\*\*)

$$d(Sy,T(Sy)) \le kd(y,Sy) + (1-k)d(A,B)$$

$$\le k(d(A,B) + \epsilon) + (1-k)d(A,B)$$

$$= d(A,B) + k\epsilon$$

$$< d(A,B) + \epsilon.$$
(3.9)

And  $(y, Sy) \in P^a_{(T,S)}(A, B)$ . Therefore,  $P^a_{(T,S)}(A, B) \neq \emptyset$ .

**Theorem 3.4.** Let A and B be nonempty subsets of a metric space (X, d) and let  $T : A \cup B \to A \cup B$  and  $S : A \cup B \to A \cup B$  be two continuous maps such that  $T(A) \subseteq B$ ,  $S(B) \subseteq A$ . If, for every  $(x, y) \in A \times B$ ,

$$d(Tx, Sy) \le \alpha d(x, y) + \gamma d(A, B), \tag{3.10}$$

where  $\alpha, \gamma \geq 0$  and  $\alpha + \gamma = 1$ , also let  $\{x_n\}$  and  $\{y_n\}$  be as follows:

$$x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n \quad \text{for some } (x_1, y_1) \in A \times B, \ n \in \mathbb{N}.$$
 (3.11)

If  $\{x_n\}$  has a convergent subsequence in A, then there exists a  $x_0 \in A$  such that  $d(x_0, Tx_0) = d(A, B)$ . Proof. We have

$$d(x_{n+1}, y_{n+1}) = d(Tx_n, Sy_n)$$

$$\leq \alpha d(x_n, y_n) + \gamma (d(A, B))$$

$$\leq \cdots$$

$$\leq \alpha^{n+1} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^n) \gamma d(A, B).$$
(3.12)

If  $\{x_{n_k}\}_{k\geq 1}$  converges to  $x_1\in A$ , that is,  $x_{n_k}\to x_1$ , then

$$d(x_{n_{K+1}}, y_{n_{k+1}}) \le \alpha^{n_{k+1}} d(x_0, y_0) + (1 + \alpha + \dots + \alpha_k^n) \gamma d(A, B).$$
(3.13)

Since *T* is continuous, then

$$d(x_{n_{k+1}}, Tx_{n_k}) \longrightarrow \frac{\gamma}{1-\alpha} d(A, B) = d(A, B). \tag{3.14}$$

Therefore, 
$$d(x_1, Tx_1) = d(A, B)$$
.

Definition 3.5. Let  $T:A\cup B\to A\cup B$  and  $S:A\cup B\to A\cup B$  be continues maps such that  $T(A)\subseteq B$  and  $S(B)\subseteq A$ . We define diameter  $P^a_{(T,S)}(A,B)$  by

$$\operatorname{diam}\left(P_{(T,S)}^{a}(A,B)\right) = \sup\left\{d(x,y): d(Tx,Ty) \le \epsilon + d(A,B) \text{ for some } \epsilon > 0\right\}. \tag{3.15}$$

Example 3.6. Suppose  $A = \{(x,0) : 0 \le x \le 1\}$ ,  $B = \{(x,1) : 0 \le x \le 1\}$ , T(x,0) = T(x,1) = (1/2,1), and S(x,1) = S(x,0) = (1/2,0). Then d(T(x,0),S(y,1)) = 1 and  $diam(P^a_{(T,S)}(A,B)) = diam(A \times B) = \sqrt{2}$ .

**Theorem 3.7.** Let  $T: A \cup B \to A \cup B$  and  $S: A \cup B \to A \cup B$  be continuous maps such that  $T(A) \subseteq B$ ,  $S(B) \subseteq A$ . If there exists a  $k \in [0,1]$ ,

$$d(x,Tx) + d(Sy,y) \le kd(x,y), \tag{3.16}$$

then

$$\operatorname{diam}\left(P_{(T,S)}^{a}(A,B)\right) \leq \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k} \quad \text{for some } \epsilon > 0. \tag{3.17}$$

*Proof.* If  $(x, y) \in P^a_{(T,S)}(A, B)$ , then

$$d(x,y) \le d(x,Tx) + d(Tx,Sy) + d(Sy,y)$$

$$\le \epsilon + kd(x,y) + d(A,B).$$
(3.18)

Therefore,  $d(x,y) \le \epsilon/(1-k) + (d(A,B))/(1-k)$ . Then  $diam(P^a_{(T,S)}(A,B)) \le \epsilon/(1-k) + (d(A,B))/(1-k)$ .

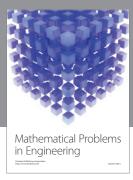
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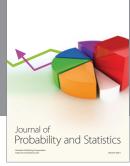
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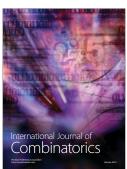








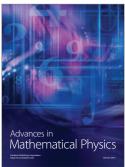




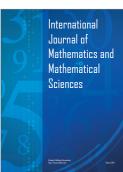


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