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Research Article

A de Casteljau Algorithm for q-Bernstein-Stancu Polynomials

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This paper is concerned with a generalization of the q-Bernstein polynomials and Stancu operators, where the function is evaluated at intervals which are in geometric progression. It is shown that these polynomials can be generated by a de Casteljau algorithm, which is a generalization of that relating to the classical case and q-Bernstein case.

1. Introduction

Let $q > 0$. For any fixed real number $q > 0$ and for $n \in Z = \{0, \pm 1, \pm 2, \ldots\}$, the q-integers of the number $[n]$ are defined by

$$[n] = \frac{(1-q^n)}{(1-q)}, \quad \text{for } q \neq 1, \ [n] = n, \ \text{for } q = 1. \tag{1.1}$$

The q-factorial $[n]!$, for $n \in N_0 = \{0, 1, 2, \ldots\}$, is defined by

$$[n]! = [1][2]\cdots[n] \quad (n = 1, 2, \ldots), \ [0]! = 1. \tag{1.2}$$

For the integers n, k, $(n \geq k \geq 0)$, the q-binomial or the Gaussian coefficients are defined by (see [1, page 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \tag{1.3}$$

For $f \in C[0;1]$, $q > 0$, $\alpha \geq 0$ and each positive integer n, we introduce (see [2]) the following generalized q-Bernstein operators:

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]}{[n]}\right), \tag{1.4}$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{\prod_{i=0}^{k-1}(x+\alpha[i]) \prod_{s=0}^{n-1-k}(1-q^s x+\alpha[s])}{\prod_{i=0}^{n-1}(1+\alpha[i])}. \tag{1.5}$$

Note, that an empty product in (1.5) denotes 1. In the case where $\alpha = 0$, $B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips [3, 4] in 1997

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{i=0}^{n-k-1}\left(1-q^i x\right) f\left(\frac{[k]}{[n]}\right). \tag{1.6}$$

In the case where $q = 1$, $B_n^{q,\alpha}(f;x)$ reduces to Bernstein-Stancu polynomials, introduced by Stancu [5] in 1968

$$S_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+\alpha i) \prod_{s=0}^{n-k-1}(1-x+s\alpha)}{\prod_{i=0}^{n-1}(1+i\alpha)} f\left(\frac{k}{n}\right). \tag{1.7}$$

When $q = 1$ and $\alpha = 0$, we obtain the classical Bernstein polynomial defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.8}$$

Basic facts on Bernstein polynomials, their generalizations, and applications can be found for example in [6–8]. In recent years, the q-Bernstein polynomials have attracted much interest, and a great number of interesting results related to the $B_{n,q}(f)$ polynomials have been obtained (see [3, 4, 9–12]). Some approximation properties of the Stancu operators are presented in [5, 13–15].

Let $\Delta_q^0 f_j = f_j$, for $j = 0, 1, \ldots, n$, and recursively,

$$\Delta_q^{k+1} f_j = \Delta_q^k f_{j+1} - q^k \Delta_q^k f_j, \tag{1.9}$$

for $k = 0, 1, \ldots, n-j-1$ and $f_j = f([j]/[n])$. It is easily established by induction that q-differences satisfy the relation

$$\Delta_q^k f_j = \sum_{i=0}^{k} (-1)^k q^{i(i-1)/2} \begin{bmatrix} k \\ i \end{bmatrix} f_{j+k-i}. \tag{1.10}$$

In [2], we prove that the operators $B_n^{q,\alpha}(f;x)$ defined by (1.4) can be expressed in terms of q-differences

$$B_n^{q,\alpha} f(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Delta_q^k f_0 \prod_{i=0}^{k-1} \frac{x + \alpha[i]}{1 + \alpha[i]}, \tag{1.11}$$

which generalized the well-known result [3, 4] for the q-Bernstein polynomial. In this paper, we show that polynomials defined by (1.4) can be generated by a de Castljau algorithm, which is a generalization of that relating to the classical case [16] and q-Bernstein case [4, 11].

2. Auxiliary Results

We note that $B_n^{q,\alpha}(f;x)$ defined by (1.4), is a monotone linear operator for any $0 < q \leq 1$ and $\alpha \geq 0$. These operators reproduces linear functions [2], that is,

$$B_n^{q,\alpha}(ax + b; x) = ax + b, \quad a, b \in R. \tag{2.1}$$

They also satisfy the end point interpolation conditions $B_n^{q,\alpha}(f;0) = f(0)$ and $B_n^{q,\alpha}(f;1) = f(1)$. These properties are significant in designing curves and surfaces.

Moreover, the following holds.

Lemma 2.1. *Let $0 < q \leq 1$, $\alpha \geq 0$. Then,*

$$\prod_{u=0}^{m-1} \left(q^r - q^u x + \alpha([u] - [r]) \right) = \sum_{s=0}^{m} (-1)^s q^{s(s-1)/2 + (m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]), \tag{2.2}$$

for all $m \in N$, $r \in N_0 = N \cup \{0\}$ and $x \in [0;1]$.

Proof. We use induction on m. First, we see from equality $[-r] = -q^{-r}[r]$, $(r \in N)$, that (2.2) is evident for $m = 1$. Let us assume that (2.2) holds for a given $m \in N$. Then, using (2.2), we obtain

$$\begin{aligned}
&\prod_{u=0}^{m}\left(q^r - q^u x + \alpha([u] - [r])\right)\\
&\quad= \left(q^r - q^m x + \alpha([m] - [r])\right)\sum_{s=0}^{m}(-1)^s q^{s(s-1)/2+(m-s)r}\begin{bmatrix} m \\ s \end{bmatrix}\\
&\qquad\cdot\prod_{i=0}^{s-1}(x + \alpha[i])\prod_{j=s-r}^{m-r-1}(1 + \alpha[j])\\
&\quad= \sum_{s=0}^{m}(-1)^s q^{s(s-1)/2+(m-s)r}\left(q^r + \alpha[m] - \alpha[r] + \alpha q^m[s]\right)\begin{bmatrix} m \\ s \end{bmatrix}\\
&\qquad\cdot\prod_{i=0}^{s-1}(x + \alpha[i])\prod_{j=s-r}^{m-r-1}(1 + \alpha[j])\\
&\qquad+\sum_{s=1}^{m+1}(-1)^s q^{(s-1)(s-2)/2+(m-s+1)r+m}(1 + \alpha[s-r-1])\begin{bmatrix} m \\ s-1 \end{bmatrix}\\
&\qquad\cdot\prod_{i=0}^{s-1}(x + \alpha[i])\prod_{j=s-r}^{m-r-1}(1 + \alpha[j])\\
&\quad= q^{mr}\left(q^r + \alpha[m] - \alpha[r]\right)\prod_{j=-r}^{m-r-1}(1 + \alpha[j])\\
&\qquad+(-1)^{m+1}q^{m(m-1)/2+m}(1 + \alpha[m-r])\prod_{i=0}^{m}(x + \alpha[i])\\
&\qquad+\sum_{s=1}^{m}(-1)^s q^{s(s-1)/2+(m+1-s)r}U_s\prod_{i=0}^{s-1}(x + \alpha[i])\prod_{j=s-r}^{m-r-1}(1 + \alpha[j]),
\end{aligned} \tag{2.3}$$

where

$$U_s = \begin{bmatrix} m \\ s \end{bmatrix}\left(q^r + \alpha[m] - \alpha[r] + \alpha q^m[s]\right)q^{-r} + q^{m-s+1}\begin{bmatrix} m \\ s-1 \end{bmatrix}(1 + \alpha[s-r-1]). \tag{2.4}$$

Using the obvious equalities

$$\begin{gathered}\left(q^r+\alpha[m]-\alpha[r]\right)q^{-r}=1+\alpha[m-r],\\ \begin{bmatrix} m \\ s \end{bmatrix}[s]=\begin{bmatrix} m \\ s-1 \end{bmatrix}[m-s+1],\end{gathered} \tag{2.5}$$

we have

$$\begin{aligned}U_s=&\begin{bmatrix} m \\ s \end{bmatrix}(1+\alpha[m-r])\\ &+\begin{bmatrix} m \\ s-1 \end{bmatrix}q^{m-s+1}\Big(1+\alpha\Big([m-s+1]q^{s-r-1}+[s-r-1]\Big)\Big).\end{aligned} \tag{2.6}$$

It is easy to see that

$$\begin{gathered}[m-s+1]q^{s-r-1}+[s-r-1]=[m-r],\\ \begin{bmatrix} m \\ s \end{bmatrix}+\begin{bmatrix} m \\ s-1 \end{bmatrix}q^{m-s+1}=\begin{bmatrix} m+1 \\ s \end{bmatrix}.\end{gathered} \tag{2.7}$$

Therefore,

$$U_s=(1+\alpha[m-r])\begin{bmatrix} m+1 \\ s \end{bmatrix}. \tag{2.8}$$

From last equality and (2.3), we obtain

$$\begin{aligned}&\prod_{u=0}^{m}\left(q^r-q^u x+\alpha([u]-[r])\right)\\ &\quad=q^{mr}\left(q^r+\alpha[m]-\alpha[r]\right)\prod_{j=-r}^{m-r-1}\left(1+\alpha[j]\right)\\ &\qquad+(-1)^{m+1}q^{m(m-1)/2+m}(1+\alpha[m-r])\prod_{i=0}^{m}(x+\alpha[i])\\ &\qquad+\sum_{s=1}^{m}(-1)^s q^{s(s-1)/2+(m+1-s)r}\begin{bmatrix} m+1 \\ s \end{bmatrix}(1+\alpha[m-r])\prod_{i=0}^{s-1}(x+\alpha[i])\prod_{j=s-r}^{m-r-1}\left(1+\alpha[j]\right)\\ &\quad=\sum_{s=0}^{m+1}(-1)^s q^{s(s-1)/2+(m+1-s)r}\begin{bmatrix} m+1 \\ s \end{bmatrix}\prod_{i=0}^{s-1}(x+\alpha[i])\prod_{j=s-r}^{m-r}\left(1+\alpha[j]\right).\end{aligned} \tag{2.9}$$

This completes the proof of the lemma. □

input: $q; f([0]/[n]), f([1]/[n]), \ldots, f([n]/[n])$
for $r = 0$ to n

$$f_r^{[0]} := f\left(\frac{[r]}{[n]}\right)$$

next r
for $m = 1$ to n
for $r = 0$ to $n - m$

$$f_r^{[m]} := \frac{\{(q^r - q^{m-1}x + \alpha([m-1] - [r]))f_r^{[m-1]} + (x + \alpha[r])f_{r+1}^{[m-1]}\}}{1 + \alpha[m-1]}$$

next r
next m

Algorithm 1: De Casteljau type algorithm.

3. Main Result

The generalized q-Bernstein polynomials, defined by (1.4), may be evaluated by Algorithm 1.

In the case, where $\alpha = 0$, this is the de Casteljau algorithm for evaluating the q-Bernstein polynomial [3, 4]. Note that with $q = 1$ and $\alpha = 0$, we recover the original classical de Casteljau algorithm (see Hoschek and Lasser [16]). The algorithm is justifed by the following theorem.

Theorem 3.1. *Each intermediate point $f_r^{[m]}$ of the algorithm can be expressed as*

$$f_r^{[m]} = \left(\prod_{i=0}^{m-1}(1 + \alpha[i])\right)^{-1} \cdot \sum_{t=0}^{m} f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1}(x + \alpha[r+s]) \prod_{u=0}^{m-t-1}(q^r - q^u x + \alpha([u] - [r])), \tag{3.1}$$

and, in particular

$$f_0^{[n]} = B_n^{q,\alpha}(f; x). \tag{3.2}$$

Proof. We use induction on m. From the initial conditions in the algorithm, $f_r^{[0]} = f([r]/[n]) = f_r$, $0 \le r \le n$, it is clear that (3.1) holds for $m = 0$ and $0 \le r \le n$. Let us assume that (3.1) holds for some m such that $0 \le m < n$, and for all r such that $0 \le r \le n - m$. Then, for $0 \le r \le n - m - 1$, it follows from the algorithm that

$$f_r^{[m+1]} := \left\{(q^r - q^m x + \alpha([m] - [r]))f_r^{[m]} + (x + \alpha[r])f_{r+1}^{[m]}\right\}\frac{1}{1 + \alpha[m]}, \tag{3.3}$$

and using (3.1), we obtain

$$\begin{aligned}
f_r^{[m+1]}\left(\prod_{i=0}^{m}(1+\alpha[i])\right) :=& \left(q^r - q^m x + \alpha([m]-[r])\right) \\
&\cdot \sum_{t=0}^{m} f_{r+t}\begin{bmatrix} m \\ t \end{bmatrix}\prod_{s=0}^{t-1}(x+\alpha[r+s]) \cdot \prod_{u=0}^{m-t-1}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
&+ (x+\alpha[r]) \cdot \sum_{t=0}^{m} f_{r+t+1}\begin{bmatrix} m \\ t \end{bmatrix}\prod_{s=0}^{t-1}(x+\alpha[r+s+1]) \\
&\cdot \prod_{u=0}^{m-t-1}\left(q^{r+1} - q^u x + \alpha([u]-[r+1])\right) \\
=& \left(q^r - q^m x + \alpha([m]-[r])\right) f_r \prod_{u=0}^{m-1}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
&+ \left(q^r - q^m x + \alpha([m]-[r])\right) \cdot \sum_{t=1}^{m} f_{r+t}\begin{bmatrix} m \\ t \end{bmatrix} \\
&\cdot \prod_{s=0}^{t-1}(x+\alpha[r+s]) \cdot \prod_{u=0}^{m-t-1}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
&+ (x+\alpha[r]) \cdot \sum_{t=1}^{m} f_{r+t}\begin{bmatrix} m \\ t-1 \end{bmatrix}\prod_{s=0}^{t-2}(x+\alpha[r+s+1]) \\
&\cdot \prod_{u=0}^{m-t}\left(q^{r+1} - q^u x + \alpha([u]-[r+1])\right) \\
&+ (x+\alpha[r]) f_{r+m+1}\prod_{s=0}^{m-1}(x+\alpha[r+s+1]) \\
=& f_r \prod_{u=0}^{m}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
&+ \sum_{t=1}^{m}\Bigg\{\left(q^r - q^m x + \alpha([m]-[r])\right)\begin{bmatrix} m \\ t \end{bmatrix} \\
&\quad \cdot \prod_{s=0}^{t-1}(x+\alpha[r+s]) \prod_{u=0}^{m-t-1}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
&\quad + (x+\alpha[r])\begin{bmatrix} m \\ t-1 \end{bmatrix}\prod_{s=0}^{t-2}(x+\alpha[r+s+1]) \\
&\quad \prod_{u=0}^{m-t}\left(q^{r+1} - q^u x + \alpha([u]-[r+1])\right)\Bigg\} f_{r+t} \\
&+ f_{r+m+1}\prod_{s=0}^{m}(x+\alpha[r+s]).
\end{aligned} \tag{3.4}$$

We see that

$$\begin{aligned}
&\prod_{u=0}^{m-t}\Big(q^{r+1}-q^{u}x+\alpha([u]-[r+1])\Big)\\
&\quad=\Big(q^{r+1}-x-\alpha[r+1]\Big)\prod_{u=0}^{m-t-1}\Big(q^{r+1}-q^{u+1}x+\alpha([u+1]-[r+1])\Big)\\
&\quad=\Big(q^{r+1}-x-\alpha[r+1]\Big)\prod_{u=0}^{m-t-1}\Big(q^{r+1}-q^{u+1}x+\alpha q([u]-[r])\Big)\\
&\quad=\Big(q^{r+1}-x-\alpha[r+1]\Big)q^{m-t}\prod_{u=0}^{m-t-1}\left(q^{r}-q^{u}x+\alpha([u]-[r])\right),
\end{aligned} \tag{3.5}$$

and hence,

$$\begin{aligned}
&f_r^{[m+1]}\left(\prod_{i=0}^{m}(1+\alpha[i])\right)\\
&\quad:= f_r\prod_{u=0}^{m}(q^{r}-q^{u}x+\alpha([u]-[r]))\\
&\qquad+\sum_{t=1}^{m}\left\{\begin{bmatrix} m\\ t\end{bmatrix}(q^{r}-q^{m}x+\alpha([m]-[r]))+\begin{bmatrix} m\\ t-1\end{bmatrix}\Big(q^{r+1}-x-\alpha[r+1]\Big)q^{m-t}\right\}\\
&\qquad\cdot f_{r+t}\prod_{s=0}^{t-1}(x+\alpha[r+s])\prod_{u=0}^{m-t-1}\left(q^{r}-q^{u}x+\alpha([u]-[r])\right)+f_{r+m+1}\prod_{s=0}^{m}(x+\alpha[r+s]).
\end{aligned} \tag{3.6}$$

It is easy to verify that

$$\begin{aligned}
\begin{bmatrix} m\\ t\end{bmatrix}+q^{m-t+1}\begin{bmatrix} m\\ t-1\end{bmatrix}&=\begin{bmatrix} m+1\\ t\end{bmatrix},\\
\begin{bmatrix} m\\ t-1\end{bmatrix}+q^{t}\begin{bmatrix} m\\ t\end{bmatrix}&=\begin{bmatrix} m+1\\ t\end{bmatrix}.
\end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned}
&\begin{bmatrix} m \\ t \end{bmatrix}(q^r - q^m x + \alpha([m]-[r])) + \begin{bmatrix} m \\ t-1 \end{bmatrix}\left(q^{r+1} - x - \alpha[r+1]\right)q^{m-t} \\
&= q^r\left(\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1}\begin{bmatrix} m \\ t-1 \end{bmatrix}\right) - xq^{m-t}\left(\begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t\begin{bmatrix} m \\ t \end{bmatrix}\right) \\
&\quad + \frac{\alpha}{1-q}\left\{q^r\left(\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1}\begin{bmatrix} m \\ t-1 \end{bmatrix}\right) - q^{m-t}\left(\begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t\begin{bmatrix} m \\ t \end{bmatrix}\right)\right\} \\
&= \begin{bmatrix} m+1 \\ t \end{bmatrix}\left\{\left(q^r - xq^{m-t}\right) + \alpha([m-t]-[r])\right\}.
\end{aligned} \tag{3.8}$$

Consequently,

$$\begin{aligned}
f_r^{[m+1]}\left(\prod_{i=0}^{m}(1+\alpha[i])\right) := {} & f_r\prod_{u=0}^{m}(q^r - q^u x + \alpha([u]-[r])) \\
& + \sum_{t=1}^{m}\begin{bmatrix} m+1 \\ t \end{bmatrix}\left\{\left(q^r - xq^{m-t}\right) + \alpha([m-t]-[r])\right\} \\
& \cdot f_{r+t}\prod_{s=0}^{t-1}(x+\alpha[r+s])\prod_{u=0}^{m-t-1}\left(q^r - q^u x + \alpha([u]-[r])\right) \\
& + f_{r+m+1}\prod_{s=0}^{m}(x+\alpha[r+s]) \\
= {} & \sum_{t=0}^{m+1}\begin{bmatrix} m+1 \\ t \end{bmatrix} \cdot f_{r+t}\prod_{s=0}^{t-1}(x+\alpha[r+s])\prod_{u=0}^{m-t}(q^r - q^u x + \alpha([u]-[r])).
\end{aligned} \tag{3.9}$$

Thus, one has the desired result. □

Theorem 3.2. *For $0 \le m \le n$ and $0 \le r \le n-m$, we have*

$$f_r^{[m]} = \sum_{s=0}^{m} q^{(m-s)r}\begin{bmatrix} m \\ s \end{bmatrix}\Delta_q^s f_r \frac{\prod_{i=r}^{s+r-1}(x+\alpha[i])}{\prod_{j=0}^{s-1}(1+\alpha[j])}, \tag{3.10}$$

for all $x \in [0;1]$.

Proof. Using (2.2) and (3.1), we have

$$f_r^{[m]}\prod_{i=0}^{m-1}(1+\alpha[i]) = \sum_{t=0}^{m}\begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} S_t(m), \tag{3.11}$$

where

$$S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \times \prod_{s=r}^{t+r-1} (x+\alpha[s]) \prod_{i=0}^{u-1} (x+\alpha[i]) \prod_{j=u-r}^{m-t-r-1} (1+\alpha[j]) \quad (0 \le t \le m). \tag{3.12}$$

First, we prove that

$$S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \cdot \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t}^{m-1} (1+\alpha[j]) \tag{3.13}$$

for all $m \in N_0 = \{0, 1, 2, \ldots\}$, $t \in N_0$, and $x \in [0; 1]$. Note that an empty sum denotes 0.

We use the induction on m. First, we see that (3.13) holds for $m = 0$ and all $t \in N_0$. Let us assume that (3.13) holds for a given m, and for all $t \in N_0$. Then, from (3.12) and (3.13), we obtain

$$\begin{aligned} S_t(m+1) &= (x+\alpha[t+r-1]) \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot \prod_{i=r}^{t+u-2+r} (x+\alpha[i]) \prod_{j=u+t-1}^{m-1} (1+\alpha[j]) \\ &= \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t-1}^{m-1} (1+\alpha[j]) \\ &\quad + \alpha \sum_{u=1}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot ([t+r+1]-[t+u+r+1]) \prod_{i=r}^{t+u-2+r} (x+\alpha[i]) \prod_{j=u+t-1}^{m-1} (1+\alpha[j]). \end{aligned} \tag{3.14}$$

We see that

$$\begin{bmatrix} m-t+1 \\ u \end{bmatrix} ([t+r-1]-[t+u+r-1]) = -q^{t+r-1}[m-t-u+2] \begin{bmatrix} m-t+1 \\ u-1 \end{bmatrix}, \tag{3.15}$$

and hence,

$$\begin{aligned} S_t(m+1) &= \sum_{u=0}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t-1}^{m-1} (1+\alpha[j]) \\ &\quad + \alpha \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} q^{u+t-1} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot [m-t-u+1] \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t}^{m-1} (1+\alpha[j]) \\ &= (-1)^{m-t+1} q^{(m-t+1)(m-t)/2} \prod_{i=r}^{m+r} (x+\alpha[i]) \\ &\quad + \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \\ &\quad \cdot \left(1+\alpha[u+t-1] + \alpha q^{u+t-1}[m-t-u+1]\right) \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\ &\quad \cdot \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t}^{m-1} (1+\alpha[j]). \end{aligned} \tag{3.16}$$

Next, in view of the equality

$$\left(1+\alpha[u+t-1] + \alpha q^{u+t-1}[m-t-u+1]\right) = 1+\alpha[m], \tag{3.17}$$

we obtain (3.13). Consequently, in view of (3.11) and (3.13), we get

$$\begin{aligned} f_r^{[m]} \prod_{i=0}^{m-1} (1+\alpha[i]) &= \sum_{t=0}^{m} \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \\ &\quad \cdot \begin{bmatrix} m-t \\ u \end{bmatrix} \prod_{i=r}^{t+u+r-1} (x+\alpha[i]) \prod_{j=u+t}^{m-1} (1+\alpha[j]) \\ &= \sum_{t=0}^{m} \sum_{u=t}^{m} \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} (-1)^{u-t} q^{(u-t)(u-t-1)/2+(m-u)r} \\ &\quad \cdot \begin{bmatrix} m-t \\ u-t \end{bmatrix} \prod_{i=r}^{u+r-1} (x+\alpha[i]) \prod_{j=u}^{m-1} (1+\alpha[j]). \end{aligned} \tag{3.18}$$

Next, in view of the equality

$$\begin{bmatrix} m \\ t \end{bmatrix}\begin{bmatrix} m-t \\ u-t \end{bmatrix} = \begin{bmatrix} m \\ u \end{bmatrix}\begin{bmatrix} u \\ t \end{bmatrix}, \tag{3.19}$$

we obtain

$$\begin{aligned} f_r^{[m]}\prod_{i=0}^{m-1}(1+\alpha[i]) &= \sum_{u=0}^{m}\sum_{t=0}^{u}\begin{bmatrix} m \\ u \end{bmatrix} f_{r+t}(-1)^{u-t}q^{(u-t)(u-t-1)/2+(m-u)r} \\ &\quad \cdot \begin{bmatrix} u \\ t \end{bmatrix}\prod_{i=r}^{u+r-1}(x+\alpha[i])\prod_{j=u}^{m-1}(1+\alpha[j]) \\ &= \sum_{u=0}^{m}\begin{bmatrix} m \\ u \end{bmatrix} q^{(m-u)r}\prod_{i=r}^{u+r-1}(x+\alpha[i])\prod_{j=u}^{m-1}(1+\alpha[j]) \\ &\quad \cdot \sum_{t=0}^{u}\begin{bmatrix} u \\ t \end{bmatrix}(-1)^{u-t}q^{(u-t)(u-t-1)/2}f_{r+t}. \end{aligned} \tag{3.20}$$

The condition (1.10) completes the proof. □

Theorems 3.1 and 3.2 are generalizations of Theorems 2.1 and 2.3 in [11].

Note that when $m = n$ and $r = 0$, (3.10) does indeed reduce to (1.11)

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