Research Article

Initial Boundary Value Problem and Asymptotic Stabilization of the Two-Component Camassa-Holm Equation

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The nonhomogeneous initial boundary value problem for the two-component Camassa-Holm equation, which describes a generalized formulation for the shallow water wave equation, on an interval is investigated. A local in time existence theorem and a uniqueness result are achieved. Next by using the fixed-point technique, a result on the global asymptotic stabilization problem by means of a boundary feedback law is considered.

1. Introduction

In this paper, we are concerned with the initial boundary value problem and the asymptotic stabilization of the two-component Camassa-Holm equation on a compact interval by means of a stationary feedback law acting on the boundary. The two-component Camassa-Holm equation reads as follows:

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \overline{\rho}\,\overline{\rho}_x = 0,$$
$$\overline{\rho}_t + \overline{\rho}_x u + \overline{\rho}u_x = 0, \tag{1.1}$$

which was first derived as a bi-Hamiltonian models by Olver and Rosenau, see [1]. The system (1.1) shares many features with the Korteweg-De Vries equation Camassa-Holm equation and Degasperis-Procesi Equation; for instance, it has a Lax pair formulation, and it is integrable. In fact, the system (1.1) is related to the first negative flow of the AKNS hierarchy via a reciprocal transformation [2, 3]. In [4], Constantin and Ivanov deviated (1.1)

in the context of shallow water waves theory. As well as they showed that it has global strong solutions and also finite time blow-up solutions. Well-posedness and blow-up results are obtained in [5, 6].

For $\bar{\rho} \equiv 0$, the equation (1.1) becomes the Camassa-Holm equation, which is modeling the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [7–11]. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [12, 13]. It has a bi-Hamiltonian structure [3] and is completely integrable [7, 14]. Also there is a geometric interpretation of the equation (1.1) in terms of geodesic flow on the diffeomorphism group of the circle [15, 16]. Its solitary waves are peaked [17]. They are orbitally stable and interact like solitons [18, 19].

The Cauchy problem and initial-boundary value problem for the Camassa-Holm equation have been studied extensively in [20–26] and references within. It has been shown that this equation is locally well posed [20–23, 26] for some initial data. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [27, 28] (by wave breaking we understand that the wave remains bounded while its slope becomes unbounded in finite time [29]).

For $\bar{\rho} \neq 0$, the Cauchy problems of (1.1) have been discussed in [5, 30], respectively. Recently, a new global existence result and several new blow-up results of strong solutions for the Cauchy problem of (1.1) were obtained in [6]. And a new local existence result and several new blow-up results and blow-up rate of strong solutions for the Cauchy problem of (1.1) defined in a torus were obtained in [31]. Guan and Yin proved the existence of global week solutions to (1.1) provided the initial data satisfying some certain conditions, see [32].

As far as the initial boundary value problem and the asymptotic stabilization of the two-component Camassa-Holm equation on a compact interval are concerned, there are seldom results yet, to the authors' knowledge. Our aim of this paper is to prove the existence of the initial boundary value problem and the asymptotic stabilization of the two-component Camassa-Holm equation on a compact interval by acting on the boundary feedback law, precisely,

(1) the exact controllability problem: given two states $(u_0, \bar{\rho}_0)$ and $(u_1, \bar{\rho}_1)$ and a time $T > 0$, can one find a certain function $v(t)$ such that the solution to (1.1) satisfies $u(T) = u_1, \bar{\rho}(T) = \bar{\rho}_1$? and

(2) the stabilizability problem: can one find a stationary feedback law $v(x)$, such that for any state $(u_0, \bar{\rho}_0)$ a solution pair $(u(t), \bar{\rho}(t))$ to closed-loop system is global?

To explain our boundary formulation of (1.1), let us first introduce some transformation, precisely, $m = u - u_{xx}$ and $\rho = \bar{\rho} - 1$, which lead the system (1.1) to be equivalent to the system:

$$m_t + um_x + 2mu_x + \rho\rho_x + \rho_x = 0,$$

$$\rho_t + u\rho_x + \rho u_x + u_x = 0.$$

(1.2)

Let T be a positive number. In the following we take $\Omega_T = [0,T] \times [0,1]$. Let v_l and v_r be in $C^0([0,T], R)$ and $m_0 \in L^\infty(0,1)$, $\rho_0 \in W^{1,\infty}(0,1)$. We set

$$\Gamma_l = \{t \in [0,T] \mid v_l(t) > 0\}, \qquad \Gamma_r = \{t \in [0,T] \mid v_r(t) < 0\}. \tag{1.3}$$

In the following, we will always suppose that the sets

$$P_l = \{t \in [0,T] \mid v_l(t) = 0\}, \qquad P_r = \{t \in [0,T] \mid v_r(t) = 0\} \tag{1.4}$$

have a finite number of connected components. Finally, let $m_l, \rho_l \in L^\infty(\Gamma_l) \times W^{1,\infty}(\Gamma_l)$ and $m_r, \rho_r \in L^\infty(\Gamma_r) \times W^{1,\infty}(\Gamma_r)$. The given functions v_l, v_r, m_l, ρ_l, and m_r, ρ_r will be the boundary values for the equation; m_0, ρ_0 are the initial data. Let now $\mathfrak{A}(t,x)$ be the auxiliary function which lifts the boundary values v_l and v_r and is defined by

$$(1 - \partial_{xx})\mathfrak{A}(t,x) = 0, \quad \forall (t,x) \in \Omega_T,$$
$$\mathfrak{A}(t,0) = v_l(t), \quad \mathfrak{A}(t,1) = v_r(t), \quad \forall t \in [0,T]. \tag{1.5}$$

Setting $u = \theta + \mathfrak{A}$, we can further rewrite the system (1.1) as

$$m(t,x) = (1 - \partial_{xx})\theta(t,x),$$
$$\theta(t,0) = \theta(t,1) = 0, \tag{1.6}$$

$$m_t + (\theta + \mathfrak{A})m_x = -2m\partial_x(\theta + \mathfrak{A}) - \rho\rho_x - \rho_x,$$
$$\rho_t + (\theta + \mathfrak{A})\rho_x = -(\rho + 1)\partial_x(\theta + \mathfrak{A}),$$
$$m(0,\cdot) = m_0, \quad m(\cdot,0)|_{\Gamma_l} = m_l, \quad m(\cdot,1)|_{\Gamma_r} = m_r, \tag{1.7}$$
$$\rho(0,\cdot) = \rho_0, \quad \rho(\cdot,0)|_{\Gamma_l} = \rho_l, \quad \rho(\cdot,1)|_{\Gamma_r} = \rho_r.$$

Let $y = \binom{m}{\rho}$, $y_0 = \binom{m_0}{\rho_0}$, $b(t,x) = \left(\begin{smallmatrix} -2\partial_x(\theta+\mathfrak{A}) & 0 \\ 0 & -\partial_x(\theta+\mathfrak{A}) \end{smallmatrix}\right)$, $f(t,x) = \left(\begin{smallmatrix} -\rho\rho_x-\rho_x \\ -\partial_x(\theta+\mathfrak{A}) \end{smallmatrix}\right)$, and the system (1.7) can be written as

$$\partial_t y + (\theta + \mathfrak{A})\partial_x y = b(t,x)y + f(t,x),$$
$$y(0,\cdot) = y_0, \quad y(\cdot,0)|_{\Gamma_l} = y_l, \quad y(\cdot,1)|_{\Gamma_r} = y_r. \tag{1.8}$$

We first define what we mean by a weak solution to (1.8). Our test functions will be in the space:

$$\mathrm{adm}(\Omega_T) = \Big\{\psi \in C^1(\Omega_T) \times C^1(\Omega_T) \mid \forall x \in [0,1], \psi(t,x) = 0; \ \forall t \in [0,T]/\Gamma_l,$$
$$\psi(t,0) = 0; \ \forall t \in [0,T]/\Gamma_r, \psi(t,1) = 0\}. \tag{1.9}$$

Definition 1.1. Given $y_0 = \begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix} \in L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)$, when $\theta \in L^\infty((0,T); \mathrm{Lip}([0,1]))$, a function pair $y = \begin{pmatrix} m \\ \rho \end{pmatrix} \in L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)$ is a weak solution to (1.8) if y satisfies

$$
\iint_{\Omega_T} \left[y \partial_t \psi + y(\theta + \mathfrak{A}) \partial_x \psi - y b(t,x) \psi - f(t,x) \psi \right] dt \, dx
$$
$$
= -\int_0^1 y_0 \psi(0,x) dx + \int_0^T \left[\psi(t,1) v_r(t) y_r(t) - \psi(t,0) v_l(t) y_l(t) \right] dt. \tag{1.10}
$$

It is obvious that $C_0^1(\Omega_T) \times C_0^1(\Omega_T) \subset \mathrm{adm}(\Omega_T)$; therefore, a weak solution to (1.8) is also a solution to (1.8) in the distribution sense.

Definition 1.2 (see [33]). For $(t,x) \in \Omega_T$, let $\varphi(\cdot, t, x)$ be the C^1 maximal solution to

$$
\partial_s \varphi(s,t,x) = a(s, \varphi(s,t,x)),
$$
$$
\varphi(t,t,x) = x, \tag{1.11}
$$

which is defined on a certain set $[e(t,x), h(t,x)]$ (which is closed because $[0,1]$ is compact) and with possibly $e(t,x)$ and/or $h(t,x) = t$.

We take into account the influence of the boundaries by introducing the sets:

$$
P = \left\{ (t,x) \in \Omega_T \mid \exists s \in [e(t,x), h(t,x)] \text{ such that } \varphi(s,t,x) \in \{0,1\}, a(s, \varphi(s,t,x)) = 0 \right\},
$$
$$
\cup \left\{ (s, \varphi(s,0,0)) \mid \forall s \in [0,T] \right\} \cup \left\{ (s, \varphi(s,0,1)) \mid \forall s \in [0,T] \right\},
$$
$$
I = \{ (t,x) \in \Omega_T / P \mid e(t,x) = 0 \},
$$
$$
L = \left\{ (t,x) \in \Omega_T / P \mid \varphi(e(t,x)t, x) = 0 \right\},
$$
$$
R = \left\{ (t,x) \in \Omega_T / P \mid \varphi(e(t,x)t, x) = 1 \right\},
$$
$$
\Gamma_l = \{ t \in [0,T] \mid a(t,0) > 0 \},
$$
$$
\Gamma_r = \{ t \in [0,T] \mid a(t,1) > 0 \}. \tag{1.12}
$$

The following lemma, see [33], will play an important role in proving the local time existence theorem and of a uniqueness result of the initial boundary value problem.

Lemma 1.3. *Let $a \in C^0([0,T]; C^1([0,1]))$, $b, f \in L^\infty(\Omega_T)$, $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$, and $y_r \in L^\infty(\Gamma_r)$. We will also suppose that the sets:*

$$
P_l = \{ t \in [0,T] \mid v_l(t) = 0 \}, \qquad P_r = \{ t \in [0,T] \mid v_r(t) = 0 \} \tag{1.13}
$$

have at most a countable number of connected components. Then the function y, defined by the formula

$$\text{for } (t, x) \in P, \quad y(t, x) = 0,$$

$$\text{for } (t, x) \in I, \quad y(t, x) = \exp\left(\int_0^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) y_0(\varphi(0, t, x))$$

$$+ \int_0^t \exp\left(\int_s^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) f(s, \varphi(s, t, x)) \mathrm{d}s,$$

$$\text{for } (t, x) \in L, \quad y(t, x) = \exp\left(\int_{e(t,x)}^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) y_l(e(t, x)) \tag{1.14}$$

$$+ \int_{e(t,x)}^t \exp\left(\int_s^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) f(s, \varphi(s, t, x)) \mathrm{d}s,$$

$$\text{for } (t, x) \in R, \quad y(t, x) = \exp\left(\int_{e(t,x)}^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) y_r(e(t, x))$$

$$+ \int_{e(t,x)}^t \exp\left(\int_s^t b(r, \varphi(r, t, x)) \mathrm{d}r \right) f(s, \varphi(s, t, x)) \mathrm{d}s,$$

is a weak solution of

$$\partial_t y + (\theta + \mathfrak{A}) \partial_x y = b(t, x) y + f(t, x) \tag{1.15}$$

and satisfies

$$\|y\|_{L^\infty(\Omega_T)} \le \left(\max\left\{ \|y_0\|_{L^\infty(\Omega_T)}, \|y_l\|_{L^\infty(\Gamma_l)}, \|y_r\|_{L^\infty(\Gamma_r)} \right\} + T \|f\|_{L^\infty(\Omega_T)} \right) e^{t \|b\|_{L^\infty(\Omega_T)}}. \tag{1.16}$$

However, if we let $m_l, \rho_l \in L^\infty(\Gamma_l) \times L^\infty(\Gamma_l)$ and $m_r, \rho_r \in L^\infty(\Gamma_r) \times L^\infty(\Gamma_r)$, note that f depends on the unknown ρ which is not a data; therefore Lemma 1.3 does not hold, or rather Theorem 6 from the Appendix of [33] can not be applied directly to (1.8) (or (1.1)). Indeed if θ and $\mathfrak{A}$ are given, one can solve the equation on ρ (the equation on the second component in (1.8) (or (1.1))), but this result only guarantee that ρ is in L^∞. Therefore the source term in the equation on m is not in L^∞ anymore but in $L^\infty((0, T); W^{-1,\infty}(0, 1))$, and then Lemma 1.3 cannot be used to solve the transport equation on m (the first component equation on (1.8)). One might try to get more regularity on ρ, but in this case more regularity is also needed on ρ_0, ρ_l, ρ_{r0} and even on θ and $\mathfrak{A}$ to get sufficient geometrical assumptions. Then, one might manage $\rho_0 \in W^{1,\infty}(0, 1)$, $\rho_l \in W^{1,\infty}(\Gamma_l)$, $\rho_r \in W^{1,\infty}(\Gamma_r)$ to obtain at least Lipschitz solution of the scalar transport equation on ρ and then get a weak solution on m.

The rest of this paper is organized as follows. In Section 2, the main results of the present paper are stated. Section 3 will be devoted to the proofs of a local time existence theorem and of a uniqueness result of the initial boundary value problem for the system (1.8) (or (1.1)). The problem of asymptotic stabilization for the system is analyzed, and a feedback control law will be investigated in Section 4.

2. Main Results

Theorem 2.1. *For $T > 0$, we consider $v_l \in C^0(\Gamma_l)$, $v_r \in C^0(\Gamma_l)$ such that the sets P_l and P_r have only a finite number of connected components. Let $y_0 \in L^\infty(0,1) \times W^{1,\infty}(0,1)$, $y_l \in L^\infty(\Gamma_l) \times W^{1,\infty}(\Gamma_l)$, and $y_r \in L^\infty(\Gamma_r) \times W^{1,\infty}(\Gamma_r)$. There exist $T > 0$ and (θ, y) a weak solution of the system (1.8) (or (1.1)) with $\theta \in L^\infty((0,T); C^{1,1}([0,1]) \cap \mathrm{Lip}((0,T); H_0^1(0,1)$. Moreover any such solution θ is in fact in $C^0([0,T]; W^{2,p}(0,1)) \cap C^1([0,T]; W_0^{1,p}(0,1))$, for all $p < +\infty$. Furthermore the existence time of a maximal solution is larger than $\min(\widetilde{T}, T^*)$, with*

$$
\begin{aligned}
T^* &= \max_{\beta>0}\left(\frac{\ln(1+\beta/C)}{2[C_1 + (2 + \sinh(1))(\beta + C)]}\right), \\
C &= \max\{\|y_0\|_{L^\infty \times W^{1,\infty}}, \|y_l\|_{L^\infty \times W^{1,\infty}}, \|y_r\|_{L^\infty \times W^{1,\infty}}\}, \\
C_1 &= \frac{1}{\tanh(1)}\left(\|v_l\|_{L^\infty(0,T)}, \|v_r\|_{L^\infty(0,T)}\right).
\end{aligned}
\tag{2.1}
$$

In a second step, we will show a weak-strong uniqueness property.

Theorem 2.2. *Let $\theta \in L^\infty((0,T); C^{1,1}([0,1]) \cap \mathrm{Lip}((0,T); H_0^1(0,1)$, and let $y \in \mathrm{Lip}([0,1]) \times \mathrm{Lip}([0,1])$ be a weak solution of (1.8) (or (1.1)); then it is unique in the function space $[L^\infty((0,T); C^{1,1}([0,1]) \times L^\infty(\Omega_T))]^2$.*

Let $A_l > 2\sinh(1)$, $A_r > A_l \cosh(1) + \sinh(2)$, $T > 0$ and M a symmetric matrix, and assume that ρ_0, ρ_l, and ρ_r have compact supports in $(0,1)/\Gamma_l/\Gamma_r$, respectively. Our feedback law for (1.8) (or (1.1)) reads

$$
y \in C^0([0,1]) \times C^0([0,1]) \longmapsto
\begin{cases}
v_l(y) = A_l \|y\|_{C^0([0,1]) \times C^0([0,1])}, \\
v_r(y) = A_r \|y\|_{C^0([0,1]) \times C^0([0,1])}, \\
\dot{y}_l(t) = M y_l(t).
\end{cases}
\tag{2.2}
$$

Theorem 2.3. *For any $y_0 \in [C^0([0,1])]^2$ there exists $(y,v) \in [C^0(\Omega_T)]^2 \times C^2([0,1])$ a weak solution of (1.1) and (2.2) satisfying*

$$
\forall x \in [0,1], \quad y(0,x) = y_0(x).
\tag{2.3}
$$

Furthermore any maximal solution of (1.1), (2.2), and (2.3) is global, and if we let

$$
c = \min\left\{A_l - 2\sinh(1), \frac{A_r - A_l \cosh(1) + \sinh(2)}{\sinh(1)}\right\}, \qquad \tau = \frac{1}{\|M\|_2}\ln\left(\frac{c\|y\|_{C^0([0,1])}}{\|M\|_2}\right),
\tag{2.4}
$$

then we have

$$\|y\|_{C^0([0,1])} \leq \frac{\|M\|_2}{c[1 + \|M\|_2(t - \tau)]}. \tag{2.5}$$

Remark 2.4. For $\rho \equiv 0$, the system (1.1) becomes the classical Camassa-Holm equation, and the above theorems degenerate those of [33] with $k = 0$.

3. Proofs of the Main Theorems

3.1. Local Existence Theorem

This strategy is borrowed from [33]. We want to solve (1.6) (1.8) (or (1.1)). Equation (1.6) is a linear elliptic equation, and with θ fixed (1.8) is a linear transport equation in y, with boundary data.

Given $\theta \in L^\infty((0,T); C^{1,1}([0,1])) \cap \text{Lip}((0,T); H_0^1(0,1))$, we will define $y = \binom{m}{\rho}$ to be the solution to (1.8), and once we have m in $L^\infty(\Omega_T)$, we introduce $\widetilde{\theta}$ solution of

$$(1 - \partial_{xx})\widetilde{\theta} = m. \tag{3.1}$$

Then $\mathcal{F}$ is defined as the operator $\widetilde{\theta} = \mathcal{F}(\theta)$

Lemma 3.1. *The function $\mathfrak{A}$ defined by (1.5) satisfies*

$$\forall (t,x) \in \Omega_T, \quad \mathfrak{A}(t,x) \in C^0([0,T]; C^\infty([0,1])),$$

$$\mathfrak{A}(t,x) = \frac{1}{\sinh(1)}(\sinh(x)v_r(t) + \sinh(1-x)v_l(t)), \tag{3.2}$$

$$\|\mathfrak{A}(t,x)\|_{L^\infty((0,T); C^{1,1}([0,1]))} \leq \frac{\cosh(1)}{\sinh(1)}\left(\|v_l\|_{L^\infty(0,T)} + \|v_r\|_{L^\infty(0,T)}\right).$$

Then for a function $\theta \in L^\infty((0,T); C^{1,1}([0,1])) \cap \text{Lip}((0,T); H_0^1(0,1))$, we consider φ the flow of $\theta + \mathfrak{A}$.

Lemma 3.2. *The flow φ satisfies the following properties.*

(1) φ is C^1 with the following partial derivatives:

$$\partial_1\varphi(s,t,x) = (\theta + \mathfrak{A})(s, \varphi(s,t,x)),$$

$$\partial_2\varphi(s,t,x) = -(\theta + \mathfrak{A})\exp\left(\int_t^s \partial_x(\theta + \mathfrak{A})(r, \varphi(r,t,x))dr\right), \tag{3.3}$$

$$\partial_3\varphi(s,t,x) = \exp\left(\int_t^s \partial_x(\theta + \mathfrak{A})(r, \varphi(r,t,x))dr\right).$$

(2) *For all* $j = 1, 2, 3,$

$$\left\| \partial_j \varphi \right\|_{C^0} \leq \left(1 + \| \theta + \mathfrak{A} \|_{C^0(\Omega_T)} \right) e^{T \| \theta + \mathfrak{A} \|_{C^0(\Omega_T)}}. \tag{3.4}$$

For $(t, x) \in \Omega_T$, $\varphi(\cdot, t, x)$ *is defined on a set* $[e(t, x), h(t, x)]$, *here* $e(t, x)$ *is basically the entrance time in* Ω_T *of the characteristic curve going through* (t, x).

(3) *If* $e(t, x) > 0$, *then* $\varphi(e(t, x), t, x) \in \{0, 1\}$.

(4) *If* $h(t, x) < T$, *then* $\varphi(h(t, x), t, x) \in \{0, 1\}$.

For $\theta \in L^\infty((0, T); C^{1,1}([0, 1])) \cap \mathrm{Lip}((0, T); H_0^1(0, 1))$, we define the solution to (1.8) $y \in L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)$ by

$$\begin{aligned}
\text{for } (t, x) \in I, \quad y(t, x) &= \begin{pmatrix} \exp \displaystyle\int_0^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_0^t E_2 dr \end{pmatrix} y_0(\varphi(0, t, x)) \\[3ex]
&+ \int_0^t \begin{pmatrix} \exp \displaystyle\int_s^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho\rho_x - \rho_x \\[1ex] -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds, \\[4ex]
\text{for } (t, x) \in L, \quad y(t, x) &= \begin{pmatrix} \exp \displaystyle\int_{e(t,x)}^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_{e(t,x)}^t E_2 dr \end{pmatrix} y_l(e(t, x)) \\[3ex]
&+ \int_{e(t,x)}^t \begin{pmatrix} \exp \displaystyle\int_s^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho\rho_x - \rho_x \\[1ex] -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds, \\[4ex]
\text{for } (t, x) \in R, \quad y(t, x) &= \begin{pmatrix} \exp \displaystyle\int_{e(t,x)}^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_{e(t,x)}^t E_2 dr \end{pmatrix} y_r(e(t, x)) \\[3ex]
&+ \int_{e(t,x)}^t \begin{pmatrix} \exp \displaystyle\int_s^t E_1 dr & 0 \\[2ex] 0 & \exp \displaystyle\int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho\rho_x - \rho_x \\[1ex] -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds
\end{aligned} \tag{3.5}$$

with

$$E_1 = -2\partial_x(\theta + \mathfrak{A})(r, \varphi(r, t, x)), \qquad E_2 = -\partial_x(\theta + \mathfrak{A})(r, \varphi(r, t, x)). \tag{3.6}$$

And we have

(1) the function y is the unique weak solution of (1.8) in the sense of Definition 4.1; thanks to Lemma 1.3. y is in $L^\infty \cap W^{1,\infty}$.

Remark 3.3. y is the only weak solution of (1.8), and also y is in $L^\infty \cap W^{1,\infty}$ which is crucial for the stabilization problem because of the coupling between the two components of y. However, rather thanks to the regularity on the boundary data ρ is indeed Lipschitz inside the zones L, R, and I; it ensures that the transition between those zones should be continuous under the kind of compatibility conditions between ρ_0, ρ_l, and ρ_r; for example, all three have a compact support in $(0,1)/\Gamma_l/\Gamma_r$.

(2) Since $y \in L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)$, we immediately get $y \in W^{1,\infty}(0,T;H^{-1}(0,1)) \times W^{2,\infty}(0,T;H^{-1}(0,1))$ and satisfies (1.8). Also we can get the estimates:

$$\|y(t,x)\|_{L^\infty(\Omega_T) \times L^\infty(\Omega_T)} \le C_0 \exp\left[2T\left(\|\partial_x\theta\|_{L^\infty(\Omega_T)} + \|\partial_x\mathfrak{A}\|_{L^\infty(\Omega_T)}\right)\right],$$

$$\|\partial_t y(t,x)\|_{L^\infty(0,T;H^{-1}) \times L^\infty(0,T;H^{-1})} \le 3C_0 \exp\left[2T\left(\|\partial_x\theta\|_{L^\infty(\Omega_T)} + \|\partial_x\mathfrak{A}\|_{L^\infty(\Omega_T)}\right)\right], \tag{3.7}$$

$$\times \left(\|\theta\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + \|\mathfrak{A}\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}\right),$$

where

$$C_0 = \max\left\{\|y_0\|_{L^\infty \times W^{1,\infty}}, \|y_l\|_{L^\infty \times W^{1,\infty}}, \|y_r\|_{L^\infty \times W^{1,\infty}}\right\}$$

$$+ T\left[\left(\|\rho\|_{L^\infty} + 1\right)\|\rho_x\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + \|\partial_x\theta\|_{L^\infty(\Omega_T)} + \|\partial_x\mathfrak{A}\|_{L^\infty(\Omega_T)}\right]. \tag{3.8}$$

(3) If $(t,x) \in I \cup L \cup R$ and if $(s, s') \in [e(t,x), h(t,x)]^2$, one has the following property:

$$y(s, \varphi(s,t,x)) = \begin{pmatrix} \exp \displaystyle\int_0^t E_1 dr & 0 \\ 0 & \exp \displaystyle\int_0^t E_2 dr \end{pmatrix} y(s', \varphi(s', t, x)). \tag{3.9}$$

From the elliptic equation we can get

Lemma 3.4. *There exists a unique $\tilde{\theta} \in L^\infty((0,T); H_0^1(0,1))$ such that*

$$(1 - \partial_{xx})\tilde{\theta} = m \tag{3.10}$$

holds in $D'(0,1)$ for all $(t,x) \in \Omega_T$. Furthermore $\widetilde{\theta} \in L^\infty((0,T); C^{1,1}([0,1])) \cap \mathrm{Lip}([0,T]; H_0^1(0,1))$, since $m \in L^\infty(\Omega_T) \cap \mathrm{Lip}([0,T]; H^{-1}(0,1))$. Moreover we have the bounds:

$$
\begin{aligned}
\left\| \widetilde{\theta} \right\|_{L^\infty((0,T); C^{1,1}([0,1]))} &\le [1 + \sinh(1)] \|m\|_{L^\infty(\Omega_T)} \le [1 + \sinh(1)] \|y\|_{L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)}, \\
\left\| \partial_t \widetilde{\theta} \right\|_{L^\infty((0,T); H_0^1(0,1))} &\le \|\partial_t m\|_{L^\infty((0,T); H^{-1})} \le \|\partial_t y\|_{L^\infty((0,T); H^{-1}) \times W^{1,\infty}((0,T); H^{-1})}.
\end{aligned}
\tag{3.11}
$$

The proof can be found in [33] and omitted.

Thus, for $\theta \in L^\infty((0,T); C^{1,1}(0,1)) \cap \mathrm{Lip}([0,T]; H_0^1(0,1))$, the operator $\mathcal{F}$ can be defined as $\widetilde{\theta} = \mathcal{F}(\theta) \in L^\infty((0,T); C^{1,1}(0,1)) \cap \mathrm{Lip}([0,T]; H_0^1(0,1))$.

Let B_0 and B_1 be positive numbers, then we set

$$
\begin{aligned}
\mathcal{C}_{B_0,B_1,T} = \Big\{ \theta \in L^\infty\big((0,T); C^{1,1}(0,1)\big) \cap \mathrm{Lip}\big([0,T]; H_0^1(0,1)\big) \mid & \text{ both } \|\theta\|_{L^\infty((0,T); C^{1,1}(0,1))} \le B_0, \\
& \|\theta\|_{\mathrm{Lip}([0,T]; H^{-1}(0,1))} \le B_1 \Big\}.
\end{aligned}
\tag{3.12}
$$

Obviously $\mathcal{C}_{B_0,B_1,T}$ is convex. We will endow $\mathcal{C}_{B_0,B_1,T}$ with the norm $\|\theta\|_{L^\infty([0,T]; \mathrm{Lip}(0,1))}$.

Lemma 3.5. *There exist positive numbers B_0, B_1 and T, such that $\mathcal{F}$ maps $\mathcal{C}_{B_0,B_1,T}$ into itself.*

Proof. The proceeding of proof is similar to that of [33, Lemma 3], but the constant C_0 differs slightly from that of [33, Lemma 3]. Let us first introduce the two following constants:

$$
\begin{aligned}
C_0 &= \max\{ \|y_0\|_{L^\infty \times W^{1,\infty}}, \|y_l\|_{L^\infty \times W^{1,\infty}}, \|y_r\|_{L^\infty \times W^{1,\infty}} \} \\
&\quad + T\Big[(\|\rho\|_{L^\infty} + 1) \|\rho_x\|_{L^\infty((0,T); \mathrm{Lip}([0,1]))} + \|\partial_x \theta\|_{L^\infty(\Omega_T)} + \|\partial_x \mathfrak{A}\|_{L^\infty(\Omega_T)} \Big], \\
C_1 &= \frac{1}{\tanh(1)} \Big(\|v_l\|_{L^\infty(0,T)}, \|v_r\|_{L^\infty(0,T)} \Big).
\end{aligned}
\tag{3.13}
$$

Estimates (3.7), and (3.11) on y and θ now read,

$$
\begin{aligned}
\|y(t,x)\|_{L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)} &\le C_0 \exp\Big[2T\big(\|\partial_x \theta\|_{L^\infty(\Omega_T)} + C_1 \big) \Big], \\
\|\partial_t y(t,x)\|_{L^\infty((0,T); H^{-1}) \times L^\infty((0,T); H^{-1})} &\le 3C_0 \exp\Big[2T\big(\|\partial_x \theta\|_{L^\infty(\Omega_T)} + C_1 \big) \Big] \\
&\quad \times \Big(\|\theta\|_{L^\infty((0,T); \mathrm{Lip}([0,1]))} + C_1 \Big), \\
\left\| \widetilde{\theta} \right\|_{L^\infty((0,T); C^{1,1}([0,1]))} &\le [1 + \sinh(1)] \|y(t,x)\|_{L^\infty(\Omega_T) \times W^{1,\infty}(\Omega_T)}, \\
\left\| \partial_t \widetilde{\theta} \right\|_{L^\infty((0,T); H_0^1(0,1))} &\le \|\partial_t y(t,x)\|_{L^\infty((0,T); H^{-1}) \times L^\infty((0,T); H^{-1})}.
\end{aligned}
\tag{3.14}
$$

Combining those estimates we get for all $\theta \in \mathcal{C}_{B_0,B_1,T}$:

$$\left\|\tilde{\theta}\right\|_{L^\infty((0,T);C^{1,1}([0,1]))} \leq C_0[1 + \sinh(1)] \exp\left[2T\left(\|\partial_x\theta\|_{L^\infty(\Omega_T)} + C_1\right)\right]$$

$$\leq C_0[1 + \sinh(1)] \exp[2T(B_0 + C_1)],$$

$$\left\|\partial_t\tilde{\theta}\right\|_{L^\infty((0,T);H_0^1(0,1))} \leq 3C_0 \exp\left[2T\left(\|\partial_x\theta\|_{L^\infty(\Omega_T)} + C_1\right)\right]\left(\|\theta\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + C_1\right)$$

$$\leq 3C_0 \exp[2T(B_0 + C_1)](B_0 + C_1).$$

$$(3.15)$$

To obtain $\tilde{\theta} \in \mathcal{C}_{B_0,B_1,T}$, it is sufficient that

$$C_0[1 + \sinh(1)] \exp[2T(B_0 + C_1)] \leq B_0,$$

$$B_0 + 3C_0 \exp[2T(B_0 + C_1)](B_0 + C_1) \leq B_1.$$

$$(3.16)$$

Once we have chosen T and B_0, it is easy to choose B_1 to satisfy the second inequality. For the first one we just choose B_0 sufficiently large and then T close to 0. More precisely,

$$C = \max\{\|y_0\|_{L^\infty \times W^{1,\infty}}, \|y_l\|_{L^\infty \times W^{1,\infty}}, \|y_r\|_{L^\infty \times W^{1,\infty}}\},$$

$$B_0 > C[1 + \sinh(1)], \qquad T \leq \frac{\ln B_0/(C[1 + \sinh(1)])}{2(B_0 + C_1)},$$

$$(3.17)$$

we complete the proof by taking $B_0/(C[1 + \sinh(1)]) = C + \beta, \beta > 0$. $\qquad\square$

Lemma 3.6. *$\mathcal{C}_{B_0,B_1,T}$ is compact with respect to the norm $\|\cdot\|_{L^\infty([0,T];\mathrm{Lip}(0,1))}$.*

The proof is very similar to that appeared in [33] and omitted.

Lemma 3.7. *For $y \in H^s \times H^{s-1}$, $s > 2$, $f(y)$ is bounded on bounded sets in $H^s \times H^{s-1}$. Therefore, $f(y)$ is bounded on bounded sets in $L^\infty \times L^\infty$ by the embedding theorem.*

The proof is very similar to that appeared in [33] and omitted.

Lemma 3.8. *The operator $\mathcal{F} : \mathcal{C}_{B_0,B_1,T} \to \mathcal{C}_{B_0,B_1,T}$ is continuous with respect to the norm $\|\theta\|_{L^\infty([0,T];\mathrm{Lip}(0,1))}$.*

Proof. Take a sequence $\{\theta_n\}$ which tends to θ with respect to $\|\cdot\|_{L^\infty([0,T];\mathrm{Lip}(0,1))}$, set $\tilde{\theta}_n = \mathcal{F}\theta_n$ and $\tilde{\theta} = \mathcal{F}\theta$, denote by φ_n the flow of $\theta_n + \mathfrak{A}$ and φ the flow of $\theta + \mathfrak{A}$, and we have that $\varphi_n \to \varphi$ locally in C^1 as $n \to \infty$, thanks to Proposition A.4 in [33]. What we will need to do is to show that $m_n \to m$ in $L^1(0,1)$ as $n \to \infty$ and $\rho_n \to \rho$ in $L^1(0,1)$ as $n \to \infty$.

Let $t \in [0,T]$, having supposed that P_l and P_r have only a finite number of connected components, we can assume, reducing t if necessary that v_l and v_r do not change sign on $[0,t]$. Since the characteristics of φ_n and φ may or may not cross before time t, we only consider the case that $\varphi(t,0,0) \leq \varphi_n(t,0,0) \leq \varphi(t,0,1) \leq \varphi_n(t,0,1)$, without loss of generality. The other

cases are proved in the same way. We first point out that since $\theta_n \in \mathcal{C}_{B_0, B_1, T}$ we have a bound for $\{y_n\}$ in $L^\infty(\Omega_T)$. Now

$$
\int_0^1 |y(t,x) - y_n(t,x)| \, dx
$$

$$
= \left(\int_0^{\varphi(t,0,0)} + \int_{\varphi(t,0,0)}^{\varphi_n(t,0,0)} + \int_{\varphi_n(t,0,0)}^{\varphi(t,0,1)} + \int_{\varphi(t,0,1)}^{\varphi_n(t,0,1)} + \int_{\varphi_n(t,0,1)}^1 \right) |y(t,x) - y_n(t,x)| \, dx \tag{3.18}
$$

$$
= I_1 + I_2 + I_3 + I_4 + I_5.
$$

Since $\varphi_n(t,0,0) \to \varphi(t,0,0)$ as $n \to \infty$ and $\varphi_n(t,0,1) \to \varphi(t,0,1)$ as $n \to \infty$ and thanks to the uniform bound on $\|y_n\|_{L^\infty}$, we see that both I_2 and I_4 tend to 0 when n goes to infinity.

For I_1 we have

$$
I_1 = \int_0^{\varphi(t,0,0)} |y(t,x) - y_n(t,x)| \, dx
$$

$$
= \int_0^{\varphi(t,0,0)} \left| \begin{pmatrix} \exp \int_{e(t,x)}^t E_1 dr & 0 \\ 0 & \exp \int_{e(t,x)}^t E_2 dr \end{pmatrix} y_l(e(t,x)) \right.
$$

$$
+ \int_{e(t,x)}^t \begin{pmatrix} \exp \int_s^t E_1 dr & 0 \\ 0 & \exp \int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho \rho_x - \rho_x \\ -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds \tag{3.19}
$$

$$
- \begin{pmatrix} \exp \int_{e_n(t,x)}^t E_{1n} dr & 0 \\ 0 & \exp \int_{e_n(t,x)}^t E_{2n} dr \end{pmatrix} y_l(e_n(t,x))
$$

$$
\left. - \int_{e_n(t,x)}^t \begin{pmatrix} \exp \int_s^t E_{1n} dr & 0 \\ 0 & \exp \int_s^t E_{2n} dr \end{pmatrix} \begin{pmatrix} -\rho \rho_x - \rho_x \\ -\partial_x(\theta_n + \mathfrak{A}) \end{pmatrix} ds \right| dx,
$$

where

$$
E_{1n} = -2\partial_x(\theta_n + \mathfrak{A})(r, \varphi_n(r,t,x)), \qquad E_2 = -\partial_x(\theta_n + \mathfrak{A})(r, \varphi_n(r,t,x)). \tag{3.20}
$$

Thanks to the boundedness on the $\|f\|_{L^\infty \times L^\infty}$ (Lemma 3.7) and Proposition A.2 of [33], if $(t,x) \overline{\in} P$ we have $\varphi_n(t,0,0) \to \varphi(t,0,0)$ as $n \to \infty$. This implies that if y_l was continuous, since we have a uniform bound on $\|\theta_n\|_{L^\infty((0,T);C^{1,1}(0,1))}$ the dominated convergence theorem would provide: $I_1 \to 0$. $I_3 \to 0$ and $I_5 \to 0$ which can be obtained by using the same method. Therefore, for y_0, y_l and y_r continuous we have $\|y(t,\cdot) - y_n(t,\cdot)\|_{L^1} \to 0$.

From the inequality (56) in [33], we obtain

$$
\begin{aligned}
\left\|y(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \leq \Big[& \left\|y_0\right\|_{L^1(0,1)\times L^1(0,1)} + \left\|y_l\right\|_{L^1(0,t)\cap\Gamma_l \times L^1(0,t)\cap\Gamma_l} \\
& + \left\|y_r\right\|_{L^1(0,t)\cap\Gamma_r \times L^1(0,t)\cap\Gamma_r} + \left(\left\|\rho\right\|_{L^1(0,1)\times L^1(0,1)} + 1\right)\left\|\rho_x\right\|_{L^1(0,1)\times L^1(0,1)} \\
& + \left\|\partial_x(\theta + \mathfrak{A})\right\|_{L^1(0,1)} \Big]\left\|\partial_x(\theta + \mathfrak{A})\right\|_{L^\infty(\Omega_T)} \exp\left(3t\|\partial_x(\theta + \mathfrak{A})\|_{L^\infty(\Omega_T)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\|y_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \leq \Big[& \left\|y_0\right\|_{L^1(0,1)\times L^1(0,1)} + \left\|y_l\right\|_{L^1(0,t)\cap\Gamma_l \times L^1(0,t)\cap\Gamma_l} \\
& + \left\|y_r\right\|_{L^1(0,t)\cap\Gamma_r \times L^1(0,t)\cap\Gamma_r} + \left(\left\|\rho_n\right\|_{L^1(0,1)\times L^1(0,1)} + 1\right)\left\|\rho_{n,x}\right\|_{L^1(0,1)\times L^1(0,1)} \\
& + \left\|\partial_x(\theta_n + \mathfrak{A})\right\|_{L^1(0,1)} \Big]\left\|\partial_x(\theta_n + \mathfrak{A})\right\|_{L^\infty(\Omega_T)} \exp\left(3t\|\partial_x(\theta_n + \mathfrak{A})\|_{L^\infty(\Omega_T)}\right).
\end{aligned}
\tag{3.21}
$$

So the general case of convergence $\left\|y(t,\cdot) - y_n(t,\cdot)\right\|_{L^1} \to 0$ follows from the density of C^0 in L^1 and the uniform bound on $\|\theta_n\|_{L^\infty((0,T);\mathrm{Lip}(0,1))}$.

Now only the restriction on t remains; we recall that until now we supposed that v_l and v_r did not change sign on $[0,t]$. If v_l and v_r do not change sign on $[0,t_1]$ and then on $[t_1,t]$, we have

$$
\left\|y(t_1,\cdot) - y_n(t_1,\cdot)\right\|_{L^1} \longrightarrow 0.
\tag{3.22}
$$

Let $\widetilde{y}_n$ the solution of

$$
\begin{aligned}
& \partial_t \widetilde{y}_n + (\theta_n + \mathfrak{A})\partial_x \widetilde{y}_n = b_n(t,x)\widetilde{y}_n + f_n(t,x), \\
& \widetilde{y}_n(t_1,\cdot) = y_{t_1}, \qquad \widetilde{y}_n(\cdot,0)|_{\Gamma_l} = y_l, \qquad \widetilde{y}_n(\cdot,1)|_{\Gamma_r} = y_r.
\end{aligned}
\tag{3.23}
$$

We can conclude that as $n \to \infty$,

$$
\begin{aligned}
& \left\|y(t,\cdot) - y_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \\
& \quad \leq \left\|y(t,\cdot) - \widetilde{y}_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} + \left\|\widetilde{y}_n(t,\cdot) - y_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \\
& \quad \leq \left\|y(t_1,\cdot) - \widetilde{y}_n(t_1,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \exp\left(3(t - t_1)\|\partial_x(\theta_n + \mathfrak{A})\|_{L^\infty(\Omega_T)}\right) \\
& \qquad + \left\|y(t,\cdot) - \widetilde{y}_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \\
& \quad \leq \left\|y(t_1,\cdot) - y_n(t_1,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \exp\left(3(t - t_1)\|\partial_x(\theta_n + \mathfrak{A})\|_{L^\infty(\Omega_T)}\right) \\
& \qquad + \left\|y(t,\cdot) - \widetilde{y}_n(t,\cdot)\right\|_{L^1(0,1)\times L^1(0,1)} \longrightarrow 0.
\end{aligned}
\tag{3.24}
$$

Thus the convergence in $L^1(0,1)$ propagates on each interval where v_l and v_r do not change sign; thanks to the hypothesis on P_r and P_l we have that for all $t \in [0,T]$, $\|y(t,\cdot) - y_n(t,\cdot)\|_{L^1(0,1)} \to 0$, as $n \to \infty$. Combining this first convergence result with the uniform bound and using the dominated convergence theorem in the time variable, we obtain $\|y - y_n\|_{L^1(\Omega_T)} \to 0$ which implies that

$$\left\| \widetilde{\theta} - \widetilde{\theta}_n \right\|_{L^1(0,T;W^{2,1}(0,1))} \longrightarrow 0. \tag{3.25}$$

From the compactness of $\mathcal{C}_{B_0,B_1,T}$, we get that $\widetilde{\theta}_n \to \widetilde{\theta}$ holds in $\mathcal{C}_{B_0,B_1,T}$. □

All the above lemmas result in the application of Schauders fixed point theorem to $\mathcal{F}$ and we get a solution

$$\theta \in L^\infty\left((0,T); C^{1,1}(0,1)\right) \cap \text{Lip}\left([0,T]; H_0^1(0,1)\right). \tag{3.26}$$

From the construction of F and from Proposition A.8 in [33] the additional regularity properties of any solution θ,

$$\theta \in C^0\left([0,T]; W^{2,p}(0,1)\right) \cap C^1\left([0,T]; W_0^{1,p}(0,1)\right), \quad \forall p < +\infty. \tag{3.27}$$

3.2. Uniqueness

In this subsection, we will show that the solution to the system (1.6) and (1.8) is unique; that to say, given (y,θ) and $(\widetilde{y},\widetilde{\theta})$ be two solutions of (1.6) and (1.8) for the same initial and boundary data, we will get $y = \widetilde{y}$ and $\theta = \widetilde{\theta}$.

Let $Y = y - \widetilde{y}$, $\Theta = \theta - \widetilde{\theta}$, $B(t,x) = b(t,x) - \widetilde{b}(t,x)$, $F(t,x) = f(t,x) - \widetilde{f}(t,x)$, then $\Theta \in \text{Lip}([0,T]; H_0^1(0,1))$. And we have $M = m - \widetilde{m} \in L^\infty(\Omega_T)$ and $Y \in L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ is the solution to

$$\partial_t Y + (\theta + \mathfrak{A})\partial_x Y = bY + B(t,x)\widetilde{y} - \Theta\partial_x\widetilde{y} + F(t,x),$$

$$Y(0,\cdot) = 0, \qquad Y(\cdot,0)|_{\Gamma_l} = 0, \qquad Y(\cdot,1)|_{\Gamma_r} = 0. \tag{3.28}$$

Using the lemma again with $b(t,x) = \begin{pmatrix} -2\partial_x(\theta+\mathfrak{A}) & 0 \\ 0 & -\partial_x(\theta+\mathfrak{A}) \end{pmatrix}$ and $\overline{f} = B(t,x)\widetilde{y} - U\partial_x\widetilde{y} + F(t,x)$, we get

for $(t,x) \in P$, $Y(t,x) = 0$,

for $(t,x) \in I$, $Y(t,x) = \int_0^t \exp\left(\int_s^t b(r,\varphi(r,t,x))dr\right)\overline{f}(s,\varphi(s,t,x))ds,$

$$\text{for } (t,x) \in L, \quad Y(t,x) = \int_{e(t,x)}^{t} \exp\left(\int_{s}^{t} b(r,\varphi(r,t,x))\mathrm{d}r\right)\overline{f}(s,\varphi(s,t,x))\mathrm{d}s,$$

$$\text{for } (t,x) \in R, \quad Y(t,x) = \int_{e(t,x)}^{t} \exp\left(\int_{s}^{t} b(r,\varphi(r,t,x))\mathrm{d}r\right)\overline{f}(s,\varphi(s,t,x))\mathrm{d}s.$$

$$(3.29)$$

Since $\|U(t,\cdot)\|_{L^{\infty}(0,1)} \leq C'\|Y(t,\cdot)\|_{L^{\infty}\times L^{\infty}}$ for some positive constant C', and y, $\partial_x y$ bounded, we see that for some $C'' > 0$,

$$\left\|\overline{f}(t,\cdot)\right\|_{L^{\infty}\times L^{\infty}} \leq C''\|Y(t,\cdot)\|_{L^{\infty}\times L^{\infty}}.$$

$$(3.30)$$

And since $b(t,x)$ is bounded, we get that

$$\|Y(t,\cdot)\|_{L^{\infty}\times L^{\infty}} \leq C'' \int_{0}^{s} \|Y(s,\cdot)\|_{L^{\infty}\times L^{\infty}}\mathrm{d}s.$$

$$(3.31)$$

Then we complete the proof of the uniqueness by using Gronwall's lemma.

4. Stabilization

4.1. Preliminary Results

The equilibrium state that we want to stabilize is $y = 0$, $\theta = \mathfrak{A} = 0$. A natural idea is using Lyapunov indirection method to investigate whether the linearized system around the equilibrium state is stabilizable or not. Its stabilization would provide a local stabilization result on the nonlinear system. Unfortunately, the linearized system is not stabilizable, for the state of the linearized system around the equilibrium state is constant. We see that the sign of $\theta + \mathfrak{A}$ controls the geometry of the characteristics, and the sign of $\partial_x(\theta + \mathfrak{A})$ controls the ingredient information of y along the characteristics. We will use the return method that Coron introduced in [34]. We would like our feedback law, $v_l(y) = A_l\|y\|_{C^0([0,1])\times C^0([0,1])}$, $v_r(y) = A_r\|y\|_{C^0([0,1])\times C^0([0,1])}$, to provide $\theta + \mathfrak{A} \geq 0$ and $\partial_x(\theta + \mathfrak{A}) \geq 0$. However, there is a difficulty in the stabilization problem. It needs not to be true that the transition between those zones is continuous rather thanks to the regularity on the boundary data ρ is indeed Lipschitz inside the zones L, R, and I. To achieve this target, we have to prescribe y_l, and we just need to make a continuous transition at $(t,x) = (0,0)$ and let y_l asymptotically converge in time; we assume that compatibility conditions hold; precisely, ρ_0, ρ_l and ρ_r have compact supports in $(0,1)/\Gamma_l/\Gamma_r$, respectively. To achieve this target, we have to prescribe y_l, and we just need to make a continuous transition at $(t,x) = (0,0)$ and let y_l asymptotically converge in time. This is guaranteed by

$$\partial_t y_l = M y_l, \qquad (4.1)$$

where M, symmetric matrix, is the unique matrix solution to the matrix function:

$$PM + M^T P = -Q, \qquad (4.2)$$

for some symmetric positive-definite matrices, P and Q. Indeed, let $V(t, y_l) = y_l^T P y_l$ be the Lyapunov candidate, and that y_l asymptotically converges in time is equivalent to that the time derivative of the V, $\dot{V} = y_l^T (PM + M^T P) y_l$ is strictly negative. A fixed-point strategy will be used again to prove the existence of a solution to the closed-loop system. We begin by defining the domain of the operator.

Definition 4.1. Let X be the space of $(g, N) \in C^0([0, T]; [0, 1])^2 \times C^0([0, T])$ satisfying

(1) $g(0, x) = y_0(x)$, $g(t, 0) = y_0(0)e^{Mt}$,

(2) $\|g(t, \cdot)\|_{C^0([0,1])} \leq N(t)$,

(3) $N(t)$ is nonincreasing and $N(0) \leq \|y_0\|_{C^0([0,1]) \times C^0([0,1])}$.

Lemma 4.2. *The domain X is nonempty, convex, bounded, and closed with respect to the uniform topology.*

Taking $M = \begin{pmatrix} \lambda, 0 \\ 0, \lambda \end{pmatrix}$, $\lambda < 0$, satisfying (4.1) and (4.2), and $(y_0(x)e^{Mt}, \|y_0\|_{C^0([0,1])^2}e^{Mt}) \in X$, so X is nonempty.

Now for $(y, N(t)) \in X$ we define $\breve{\theta}$ and $\breve{\mathfrak{A}}$ as the solutions of

$$m(t, x) = (1 - \partial_{xx})\breve{\theta}(t, x),$$

$$\breve{u}(t, 0) = \breve{\theta}(t, 1) = 0, \quad \forall(t, x) \in \Omega_T,$$

$$(1 - \partial_{xx})\breve{\mathfrak{A}}(t, x) = 0,$$

$$\breve{\mathfrak{A}}(t, 0) = A_l N(t), \qquad \breve{\mathfrak{A}}(t, 1) = A_r N(t), \quad \forall(t, x) \in \Omega_T.$$

(4.3)

That is,

$$\forall(t, x) \in \Omega_T, \quad \breve{u}(t, x) = -\int_0^x \sinh(x - \xi)m(t, \xi)d\xi,$$

$$\breve{\mathfrak{A}}(t, x) = \frac{N(t)}{\sinh(1)}[\sinh(x)A_r + \sinh(1 - x)A_l].$$

(4.4)

Thus, we have the estimates:

$$\left|\breve{\theta}(t, x)\right| \leq 2\sinh(1)\|y\|_{C^0([0,1]) \times C^0([0,1])},$$

$$\left|\partial_x \breve{\theta}(t, x)\right| \leq 2\cosh(1)\|y\|_{C^0([0,1]) \times C^0([0,1])},$$

$$\left|\partial_{xx}\breve{\theta}(t, x)\right| \leq [1 + 2\sinh(1)]\|y\|_{C^0([0,1]) \times C^0([0,1])},$$

$$\left|\partial_x \breve{\mathfrak{A}}(t, x)\right| \geq \frac{A_r - 2\cosh(1)}{\sinh(1)}N(t),$$

$$\left|\breve{\mathfrak{A}}(t, x)\right| \geq A_l N(t).$$

(4.5)

And in turn,

$$\left(\breve{\theta} + \breve{\mathfrak{A}}\right)(t, x) \geq [A_l - 2\sinh(1)] \|y\|_{C^0([0,1]) \times C^0([0,1])}, \tag{4.6}$$

$$\partial_x \left(\breve{\theta} + \breve{\mathfrak{A}}\right)(t, x) \geq \frac{A_r - 2\cosh(1)A_l - \sinh(2)}{\sinh(1)} \|y\|_{C^0([0,1]) \times C^0([0,1])}. \tag{4.7}$$

Now, if φ is the flow of $\breve{\theta} + \breve{\mathfrak{A}}$, e is C^1 and since $\breve{u} + \breve{\mathfrak{A}} \geq 0$, $\varphi(\cdot, t, x)$ is nondecreasing. Thus we can define the entrance time and then the operator $\mathcal{S}$ as follows.

Let $e(t, x) = \min\{s \in [0, T] \mid \varphi(s, t, x) = 0\}$, for $(t, x) \in [0, T] \times [0, 1]$, $\mathcal{S}(y, N) = (\widetilde{y}, \widetilde{N})$ with

(1) if $x \geq \varphi(t, 0, 0)$,

$$\begin{aligned}
\widetilde{y}(t, x) = &\begin{pmatrix} \exp\displaystyle\int_0^t E_1 dr & 0 \\ 0 & \exp\displaystyle\int_0^t E_2 dr \end{pmatrix} y_0(\varphi(0, t, x)) \\
&+ \int_0^t \begin{pmatrix} \exp\displaystyle\int_s^t E_1 dr & 0 \\ 0 & \exp\displaystyle\int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho\rho_x - \rho_x \\ -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds,
\end{aligned} \tag{4.8}$$

(2) if $x \leq \varphi(t, 0, 0)$,

$$\begin{aligned}
\widetilde{y}(t, x) = &\begin{pmatrix} \exp\displaystyle\int_0^t E_1 dr & 0 \\ 0 & \exp\displaystyle\int_0^t E_2 dr \end{pmatrix} e^{Me(t,x)} y_0(0) \\
&+ \int_{e(t,x)}^t \begin{pmatrix} \exp\displaystyle\int_s^t E_1 dr & 0 \\ 0 & \exp\displaystyle\int_s^t E_2 dr \end{pmatrix} \begin{pmatrix} -\rho\rho_x - \rho_x \\ -\partial_x(\theta + \mathfrak{A}) \end{pmatrix} ds,
\end{aligned} \tag{4.9}$$

(3) $N(t) = \|y(t, \cdot)\|_{C^0([0,1]) \times C^0([0,1])}$.

From Lemma 1.3 we know that $\widetilde{y}$ is the weak solution of

$$\partial_t \widetilde{y} + (\theta + \mathfrak{A})\partial_x \widetilde{y} = b(t, x)\widetilde{y} + f(t, x),$$
$$\widetilde{y}(0, \cdot) = 0, \qquad \widetilde{y}(t, 0) = e^{Mt} y_0(0). \tag{4.10}$$

Lemma 4.3. *(1) The operator $\mathcal{S}$ maps X to X.*

(2) The family $\mathcal{S}(X)$ is uniformly bounded and equicontinuous.

(3) S is continuous with respect to the uniform topology.

The proof is very similar to [33] except for here the state y is a two-component vector and the proof is omitted.

We can apply Schauder's fixed-point theorem to S and get (y, N) fixed point of S.

4.2. Stabilization and Global Existence

From (4.6) and (4.7), for all $(t, x) \in \Omega_T$,

$$
\left(\breve{\theta} + \breve{\mathfrak{A}} \right)(t, x) \geq c \| y(r, \cdot) \|_{C^0([0,1]) \times C^0([0,1])},
$$

$$
\partial_x \left(\breve{\theta} + \breve{\mathfrak{A}} \right)(t, x) \geq c \| y(r, \cdot) \|_{C^0([0,1]) \times C^0([0,1])}, \tag{4.11}
$$

y is the solution of the transport equation (1.8) and it satisfies

$$
y(t, x) = \begin{pmatrix} \exp \displaystyle\int_s^t E_1 \mathrm{d}r' & 0 \\ 0 & \exp \displaystyle\int_s^t E_2 \mathrm{d}r \end{pmatrix} y\big(s, \varphi(s, t, x)\big)
$$

$$
+ \int_s^t \begin{pmatrix} \exp \displaystyle\int_r^t E_1 \mathrm{d}r & 0 \\ 0 & \exp \displaystyle\int_r^t E_2 \mathrm{d}r' \end{pmatrix} f(r, x) \mathrm{d}r. \tag{4.12}
$$

We get for $t \geq s$

$$
|y(t, x)| \leq \exp\left(- \int_s^t c \| y(r, \cdot) \|_{C^0([0,1]) \times C^0([0,1])} \mathrm{d}r \right) y\big|s, \varphi(s, t, x)\big|. \tag{4.13}
$$

This implies that $|y(t, x)| = \sqrt{|m(t, x)|^2 + |\rho(t, x)|^2}$ decreases along the characteristics (strictly for the times where $y(t, \cdot) \neq 0$). But we have also imposed $y(t, 0) = y(s, 0)e^{M(t-s)}$; therefore $|y(t, x)|$ also decreases along $x = 0$. This already shows, thanks to the existence theorem, that a maximal solution of the closed loop system is global. To get a more precise statement, we consider all the characteristics between time t and s, and we obtain

$$
\| y(t, \cdot) \|_{C^0([0,1]) \times C^0([0,1])} \leq \max_{r \in [s,t]} \left\{ e^{\|M\|_2 (s-r)} \exp\left(- \int_s^t c \| y(r, \cdot) \|_{C^0([0,1]) \times C^0([0,1])} \mathrm{d}r \right) \right\}
$$

$$
\times \| y(t, \cdot) \|_{C^0([0,1]) \times C^0([0,1])}. \tag{4.14}
$$

Now we define

$$g(r) = e^{\|M\|_2(s-r)} \exp\left(-\int_s^t c\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])} dr\right).$$
(4.15)

Then

$$g'(r) = \left[c\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])} - \|M\|_2\right]g(r),$$
(4.16)

and we know that as long as the quantity $\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])}$ is not equal to zero, it strictly decreases. So if $\|y_0\|_{C^0([0,1])\times C^0([0,1])} > \|M\|_2/c$, for t small enough $\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])} \geq \|M\|_2/c$, and we have

$$\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])} \leq e^{-\|M\|_2 t}\|y_0\|_{C^0([0,1])\times C^0([0,1])},$$
(4.17)

which implies $\|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])} \leq \|M\|_2/c$. This provides for $\tau \leq s \leq t$, the inequality

$$\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])} \leq e^{-\int_s^t c\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])} dr}\|y(s,\cdot)\|_{C^0([0,1])\times C^0([0,1])}.$$
(4.18)

And we conclude with a classical comparison principle for ODES.

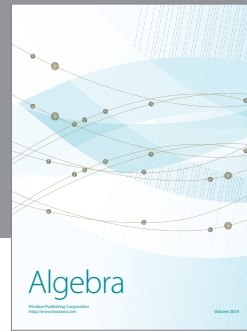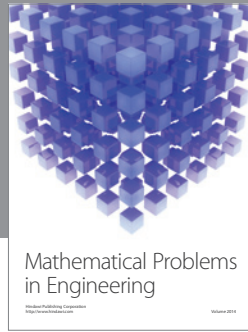## Acknowledgments

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