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Research Article

Analogue of Lebesgue-Radon-Nikodym Theorem with respect to p-adic q-Measure on $\mathbb{Z}_p$

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Recently, Lebesgue-Radon-Nikodym theorem with respect to fermionic p-adic invariant measure on $\mathbb{Z}_p$ was studied in Kim. In this paper we will give the analogue of the Lebesgue-Radon-Nikodym theorem with respect to p-adic q-measure on $\mathbb{Z}_p$. In special case, $q = 1$, we can derive the same results in Kim.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p| = p^{-\nu_p(p)} = 1/p$ and $\nu_p(0) = \infty$.

When one speaks of q-extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q| < 1$, and we use the notations of q-numbers as follows:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \qquad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \tag{1.1}$$

For any positive integer N, let

$$a + p^N\mathbb{Z}_p = \left\{x \in \mathbb{Z}_p \mid x \equiv a\left(\bmod\ p^N\right)\right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < p^N$ (see [1–8]).

It is known that the fermionic p-adic q-measure on $\mathbb{Z}_p$ is given by Kim as follows:

$$\mu_{-q}(a+p^n\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}} = \frac{1+q}{1+q^{p^N}}(-q)^a, \tag{1.3}$$

(see [6, 9–12]). Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. From (1.3), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.4}$$

where $f \in C(\mathbb{Z}_p)$ (see [1, 6, 9–12]).

Various proofs of the Radon-Nikodym theorem can be found in many books on measure theory, analysis, or probability theory. Usually they use the Hahn decomposition theorem for signed measure, the Riesz representation theorem for functionals on Hilbert space, or a martingale theory (see [13, 14]). In the previous paper [3], the author has studied the analogue of the Lebesgue-Radon-Nikodym theorem with respect to fermionic p-adic invariant measure on $\mathbb{Z}_p$. The purpose of this paper is to derive the analogue of the Lebesgue-Radon-Nikodym theorem with respect to p-adic q-measure on $\mathbb{Z}_p$ in the sense of fermionic.

2. Lebesgue-Radon-Nikodym's Type Theorem with respect to p-Adic q-Measure on $\mathbb{Z}_p$

For any positive integer a and n with $a < p^n$ and $f \in C(\mathbb{Z}_p)$, let us define

$$\mu_{f,-q}(a+p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} f(x)d\mu_{-q}(x), \tag{2.1}$$

where the integral is the fermionic p-adic q-integral on $\mathbb{Z}_p$.

From (1.3), (1.4), and (2.1), we note that

$$\begin{aligned}
\mu_{f,-q}(a+p^n\mathbb{Z}_p) &= \lim_{m\to\infty}\frac{1}{[p^{m+n}]_{-q}}\sum_{x=0}^{p^m-1} f(a+p^n x)(-q)^{a+p^n x} \\
&= \lim_{m\to\infty}\frac{1}{[p^m]_{-q}}\sum_{x=0}^{p^{m-n}-1} f(a+p^n x)(-q)^a q^{p^n x}(-1)^x \\
&= \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \lim_{m\to\infty}\frac{1}{[p^{m-n}]_{-q^{p^n}}}\sum_{x=0}^{p^{m-n}-1} f(a+p^n x)\left(-q^{p^n}\right)^x \\
&= \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \int_{\mathbb{Z}_p} f(a+p^n x)d\mu_{-q^{p^n}}(x).
\end{aligned} \tag{2.2}$$

By (2.2), we get

$$\mu_{f,-q}(a+p^n\mathbb{Z}_p) = \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \int_{\mathbb{Z}_p} f(a+p^n x)\, d\mu_{-q^{p^n}}(x). \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. *For $f, g \in C(\mathbb{Z}_p)$, one has*

$$\mu_{\alpha f+\beta g,-q} = \alpha\mu_{f,-q} + \beta\mu_{g,-q}, \tag{2.4}$$

where α, β are constants.

From (2.2) and (2.4), we note that

$$\left|\mu_{f,-q}(a+p^n\mathbb{Z}_p)\right| \leq M\|f\|_\infty, \tag{2.5}$$

where $\|f\|_\infty = \sup_{x\in\mathbb{Z}_p}|f(x)|$ and M is some positive constant.

Now, we recall the definition of the strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. If μ_{-q} is satisfied the following equation:

$$\left|\mu_{-q}(a+p^n\mathbb{Z}_p) - \mu_{-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right| \leq \delta_{n,q}, \tag{2.6}$$

where $\delta_{n,q} \to 0$ and $n \to \infty$ and $\delta_{n,q}$ is independent of a, then μ_{-q} is called the weakly fermionic p-adic q-measure on $\mathbb{Z}_p$.

If $\delta_{n,q}$ is replaced by $Cp^{-\nu_p(1-q^n)}$ (C is some constant), then μ_{-q} is called strongly fermionic p-adic q-measure on $\mathbb{Z}_p$.

Let $P(x) \in C_p[[x]]_q$ be an arbitrary q-polynomial with $\sum a_i[x]_q^i$. Then we see that $\mu_{P,-q}$ is strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = [x]_q^k$.

Let a be an integer with $0 \leq a < p^n$. Then we get

$$\begin{aligned}
\mu_{P,-q}(a+p^n\mathbb{Z}_p) &= \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \lim_{m\to\infty}\frac{1}{[p^{m-n}]_{-q^{p^n}}}\sum_{i=0}^{p^{m-n}}[a+ip^n]_q^k(-1)^i q^{p^n i}, \\
q^{p^n i} &= \sum_{l=0}^{i}\binom{i}{l}[p^n]_q^l(q-1)^l.
\end{aligned} \tag{2.7}$$

By (2.7), we easily get

$$\begin{aligned}
\mu_{P,-q}(a+p^n\mathbb{Z}_p) &\equiv \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a[a]_q^k \quad \left(\operatorname{mod}[p^n]_q\right) \\
&\equiv \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a P(a) \quad \left(\operatorname{mod}[p^n]_q\right).
\end{aligned} \tag{2.8}$$

Let x be an arbitrary in $\mathbb{Z}_p$ with $x \equiv x_n (\operatorname{mod} p^n)$ and $x \equiv x_{n+1} (\operatorname{mod} p^{n+1})$, where x_n and x_{n+1} are positive integers such that $0 \leq x_n < p^n$ and $0 \leq x_{n+1} < p^{n+1}$. Thus, by (2.8), we have

$$\left| \mu_{P,-q}\left(a + p^n \mathbb{Z}_p\right) - \mu_{P,-q}\left(a + p^{n+1} \mathbb{Z}_p\right) \right| \leq C p^{-\nu_p\left(1-q^{p^n}\right)}, \tag{2.9}$$

where C is a positive some constant and $n \gg 0$.

Let

$$f_{\mu_{P,-q}}(a) = \lim_{n \to \infty} \mu_{P,-q}\left(a + p^n \mathbb{Z}_p\right). \tag{2.10}$$

Then, (2.5), (2.7), and (2.8), we get

$$f_{\mu_{P,-q}}(a) = \frac{[2]_q}{2} (-q)^a [a]_q^k = \frac{[2]_q}{2} (-q)^a P(a). \tag{2.11}$$

Since $f_{\mu_{P,-q}}(x)$ is continuous on $\mathbb{Z}_p$, it follows for all $x \in \mathbb{Z}_p$

$$f_{\mu_{P,-q}}(x) = \frac{[2]_q}{2} (-q)^x P(x). \tag{2.12}$$

Let $g \in C(\mathbb{Z}_p)$. By (2.10), (2.11), and (2.12), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} g(x) d\mu_{P,-q}(x) &= \lim_{m \to \infty} \sum_{i=0}^{p^n-1} g(i) \mu_{P,-q}\left(i + p^n \mathbb{Z}_p\right) \\ &= \frac{[2]_q}{2} \sum_{i=0}^{p^n-1} g(i) (-q)^i [i]_q^k \\ &= \int_{\mathbb{Z}_p} g(x) [x]_q^k d\mu_{-q}(x). \end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.2. *Let $P(x) \in C_p[[x]]_q$ be an arbitrary q-polynomial with $\sum a_i [x]_q^i$. Then $\mu_{P,-q}$ is a strongly fermionic p-adic q-measure on $\mathbb{Z}_p$ and for all $x \in \mathbb{Z}_p$*

$$f_{\mu_{P,-q}} = (-1)^x \frac{[2]_q}{2} q^x P(x). \tag{2.14}$$

Furthermore, for all $g \in C(\mathbb{Z}_p)$, one has

$$\int_{\mathbb{Z}_p} g(x) d\mu_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) d\mu_{-q}(x), \tag{2.15}$$

where the second integral is fermionic p-adic q-integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_{n,q} \binom{x}{n}_q$ be the q-Mahler expansion of continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!} \tag{2.16}$$

(see [4]). Then we note that $\lim_{m\to\infty} |a_{n,q}| = 0$.

Let

$$f_m(x) = \sum_{i=0}^{m} a_{i,q} \binom{x}{i}_q \in C_p[[x]]_q. \tag{2.17}$$

Then

$$\|f - f_m\|_\infty \leq \sup_{m \leq n} |a_{n,q}|. \tag{2.18}$$

Writing $f = f_m + f - f_m$, we easily get

$$\begin{aligned}
&\left|\mu_{f,-q}(a + p^n \mathbb{Z}_p) - \mu_{f,-q}\left(a + p^{n+1}\mathbb{Z}_p\right)\right| \\
&\quad \leq \max\left\{\left|\mu_{f_m,-q}(a + p^n \mathbb{Z}_p) - \mu_{f_m,-q}\left(a + p^{n+1}\mathbb{Z}_p\right)\right|,\right. \\
&\qquad \left.\left|\mu_{f-f_m,-q}(a + p^n \mathbb{Z}_p) - \mu_{f-f_m,-q}\left(a + p^{n+1}\mathbb{Z}_p\right)\right|\right\}.
\end{aligned} \tag{2.19}$$

From Theorem 2.2, we note that

$$\left|\mu_{f-f_m,-q}(a + p^n \mathbb{Z}_p)\right| \leq \|f - f_m\|_\infty \leq C_1 p^{-\nu_p(1-q^{p^n})}, \tag{2.20}$$

where C_1 is some positive constant.

For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$.

So

$$\left|\mu_{f_m,-q}(a + p^n \mathbb{Z}_p) - \mu_{f_m,-q}\left(a + p^{n+1}\mathbb{Z}_p\right)\right| \leq C_2 p^{-\nu_p(1-q^{p^n})}, \tag{2.21}$$

where C_2 is also some positive constant.

By (2.20) and (2.21), we see that

$$\begin{aligned}
&\left|f(a) - \mu_{f,-q}(a + p^n \mathbb{Z}_p)\right| \\
&\quad \leq \max\left\{\left|f(a) - f_m(a)\right|, \left|f_m(a) - \mu_{f_m,-q}(a + p^n \mathbb{Z}_p)\right|, \left|\mu_{f-f_m,-q}(a + p^n \mathbb{Z}_p)\right|\right\} \\
&\quad \leq \max\left\{\left|f(a) - f_m(a)\right|, \left|f_m(a) - \mu_{f_m,-q}(a + p^n \mathbb{Z}_p)\right|, \leq \|f - f_m\|_\infty\right\}.
\end{aligned} \tag{2.22}$$

If we fix $\epsilon > 0$ and fix m such that $\|f - f_m\| \le \epsilon$, then, for $n \gg 0$, we have

$$\left|f(a) - \mu_{f,-q}\left(a + p^n\mathbb{Z}_p\right)\right| \le \epsilon. \tag{2.23}$$

Hence, we have

$$f_{\mu_{f,-q}}(a) = \lim_{n\to\infty}\mu_{f,-q}\left(a + p^n\mathbb{Z}_p\right) = \frac{[2]_q}{2}(-q)^a f(a). \tag{2.24}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \le p^{-n}$.

Then we get

$$\begin{aligned}\mu_{f,-q}\left(a + p^n\mathbb{Z}_p\right) &= \mu_{f_m,-q}\left(a + p^n\mathbb{Z}_p\right) + \mu_{f-f_m,-q}\left(a + p^n\mathbb{Z}_p\right)\\ &= \mu_{f_m,-q}\left(a + p^n\mathbb{Z}_p\right)\\ &= (-1)^a q^a \frac{[2]_q}{2} f(a)\left(\bmod\, [p^n]_q\right).\end{aligned} \tag{2.25}$$

For all $g \in C(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)d\mu_{f,-q}(x) = \int_{\mathbb{Z}_p} f(x)g(x)d\mu_{-q}(x). \tag{2.26}$$

Assume that f is the function from $C(\mathbb{Z}_p, C_p)$ to $Lip(\mathbb{Z}_p, C_p)$. By the definition of μ_{-q}, we easily see that μ_{-q} is a strongly p-adic q-measure on $\mathbb{Z}_p$ and for $n \gg 0$

$$\left|f_{\mu_{-q}}(a) - \mu_{-q}\left(a + p^n\mathbb{Z}_p\right)\right| \le C_3 p^{-\nu_p(1-q^{p^n})}, \tag{2.27}$$

where C_3 is some positive constant.

If $\mu_{1,-q}$ is associated strongly fermionic p-adic q-measure on $\mathbb{Z}_p$, then we have

$$\left|\mu_{1,-q}\left(a + p^n\mathbb{Z}_p\right) - f_{\mu_{-q}}(a)\right| \le C_4 p^{-\nu_p(1-q^{p^n})}, \tag{2.28}$$

where $n \gg 0$ and C_4 is some positive constant.

From (2.28), we get

$$\left|\mu_{-q}\left(a+p^n\mathbb{Z}_p\right)-\mu_{1,-q}\left(a+p^n\mathbb{Z}_p\right)\right| \le \left|\mu_{-q}\left(a+p^n\mathbb{Z}_p\right)-f_{\mu_{-q}}(a)\right| + \left|f_{\mu_{-q}}(a)-\mu_{1,-q}\left(a+p^n\mathbb{Z}_p\right)\right| \le K, \tag{2.29}$$

where K is some positive constant.

Therefore, $\mu_{-q} - \mu_{1,-q}$ is a q-measure on $\mathbb{Z}_p$. Hence, we obtain the following theorem.

Theorem 2.3. *Let μ_{-q} be a strongly fermionic p-adic q-measure on $\mathbb{Z}_p$, and assume that the fermionic Radon-Nikodym derivative $f_{\mu_{-q}}$ on $\mathbb{Z}_p$ is continuous function on $\mathbb{Z}_p$. Suppose that $\mu_{1,-q}$ is the strongly fermionic p-adic q-measure associated to $f_{\mu_{-q}}$. Then there exists a q-measure $\mu_{2,-q}$ on $\mathbb{Z}_p$ such that*

$$\mu_{-q} = \mu_{1,-q} + \mu_{2,-q}. \tag{2.30}$$

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