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Research Article

Asymptotic Properties of Third-Order Delay Trinomial Differential Equations

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The aim of this paper is to study properties of the third-order delay trinomial differential equation $((1/r(t))y''(t))' + p(t)y'(t) + q(t)y(\sigma(t)) = 0$, by transforming this equation onto the second-/third-order binomial differential equation. Using suitable comparison theorems, we establish new results on asymptotic behavior of solutions of the studied equations. Obtained criteria improve and generalize earlier ones.

1. Introduction

In this paper, we will study oscillation and asymptotic behavior of solutions of third-order delay trinomial differential equations of the form

$$\left(\frac{1}{r(t)}\, y''(t)\right)' + p(t)y'(t) + q(t)y(\sigma(t)) = 0. \tag{E}$$

Throughout the paper, we assume that $r(t), p(t), q(t), \sigma(t) \in C([t_0, \infty))$ and

(i) $r(t) > 0, p(t) \geq 0, q(t) > 0, \sigma(t) > 0$,

(ii) $\sigma(t) \leq t$, $\lim_{t\to\infty}\sigma(t) = \infty$,

(iii) $R(t) = \int_{t_0}^{t} r(s)\, \mathrm{d}s \to \infty$ as $t \to \infty$.

By a solution of (E), we mean a function $y(t) \in C^2([T_x, \infty))$, $T_x \geq t_0$, that satisfies (E) on $[T_x, \infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and, otherwise, it is nonoscillatory. Equation (E) itself is said to be oscillatory if all its solutions are oscillatory.

Recently, increased attention has been devoted to the oscillatory and asymptotic properties of second- and third-order differential equations (see [1–22]). Various techniques appeared for the investigation of such differential equations. Our method is based on establishing new comparison theorems, so that we reduce the examination of the third-order trinomial differential equations to the problem of the observation of binomial equations.

In earlier papers [11, 13, 16, 20], a particular case of (E), namely, the ordinary differential equation (without delay)

$$y'''(t) + p(t)y'(t) + g(t)y(t) = 0, \tag{E_1}$$

has been investigated, and sufficient conditions for all its nonoscillatory solutions $y(t)$ to satisfy

$$y(t)y'(t) < 0 \tag{1.1}$$

or the stronger condition

$$\lim_{t\to\infty} y(t) = 0 \tag{1.2}$$

are presented. It is known that (E_1) has always a solution satisfying (1.1). Recently, various kinds of sufficient conditions for all nonoscillatory solutions to satisfy (1.1) or (1.2) appeared. We mention here [9, 11, 13, 16, 21]. But there are only few results for differential equations with deviating argument. Some attempts have been made in [8, 10, 18, 19]. In this paper we generalize these, results and we will study conditions under which all nonoscillatory solutions of (E) satisfy (1.1) and (1.2). For our further references we define as following.

Definition 1.1. We say that (E) has property (P_0) if its every nonoscillatory solution $y(t)$ satisfies (1.1).

In this paper, we have two purposes. In the first place, we establish comparison theorems for immediately obtaining results for third-order delay equation from that of third order equation without delay. This part extends and complements earlier papers [7, 8, 10, 18].

Secondly, we present a comparison principle for deducing the desired property of (E) from the oscillation of a second-order differential equation without delay. Here, we generalize results presented in [8, 9, 14, 15, 21].

Remark 1.2. All functional inequalities considered in this paper are assumed to hold eventually;0 that is, they are satisfied for all t large enough.

2. Main Results

It will be derived that properties of (E) are closely connected with the corresponding second-order differential equation

$$\left(\frac{1}{r(t)}\, v'(t)\right)' + p(t)v(t) = 0 \tag{E_v}$$

as the following theorem says.

Theorem 2.1. *Let $v(t)$ be a positive solution of (E_v). Then (E) can be written as*

$$\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' + q(t)v(t)y(\sigma(t)) = 0. \tag{E^c}$$

Proof. The proof follows from the fact that

$$\frac{1}{v(t)}\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' = \left(\frac{1}{r(t)}\, y''(t)\right)' + p(t)y'(t). \tag{2.1}$$

□

Now, in the sequel, instead of studying properties of the trinomial equation (E), we will study the behavior of the binomial equation (E^c). For our next considerations, it is desirable for (E^c) to be in a canonical form; that is,

$$\int^{\infty} v(t)\mathrm{d}t = \infty, \tag{2.2}$$

$$\int^{\infty} \frac{r(t)}{v^2(t)}\mathrm{d}t = \infty, \tag{2.3}$$

because properties of the canonical equations are nicely explored.

Now, we will study the properties of the positive solutions of (E_v) to recognize when (2.2)-(2.3) are satisfied. The following result (see, e.g., [7, 9] or [14]) is a consequence of Sturm's comparison theorem.

Lemma 2.2. *If*

$$\frac{R^2(t)}{r(t)}p(t) \le \frac{1}{4}, \tag{2.4}$$

then (E_v) possesses a positive solution $v(t)$.

To be sure that (E_v) possesses a positive solution, we will assume throughout the paper that (2.4) holds. The following result is obvious.

Lemma 2.3. *If $v(t)$ is a positive solution of (E_v), then $v'(t) > 0$, $((1/r(t))v'(t))' < 0$, and, what is more,* (2.2) *holds and there exists $c > 0$ such that $v(t) \le cR(t)$.*

Now, we will show that if (E_v) is nonoscillatory, then we always can choose a positive solution $v(t)$ of (E_v) for which (2.3) holds.

Lemma 2.4. *If $v_1(t)$ is a positive solution of (E_v) for which* (2.3) *is violated, then*

$$v_2(t) = v_1(t)\int_{t_0}^{\infty}\frac{r(s)}{v_1^2(s)}\mathrm{d}s \tag{2.5}$$

is another positive solution of (E_v) and, for $v_2(t)$, (2.3) *holds.*

Proof. First note that

$$v_2''(t) = v_1''(t)\int_{t_0}^{t}\frac{r(s)}{v_1^2(s)}\mathrm{d}s = -p(t)v_1(t)\int_{t_0}^{t}v_1^{-2}(s)\mathrm{d}s = -p(t)v_2(t). \tag{2.6}$$

Thus, $v_2(t)$ is a positive solution of (E_v). On the other hand, to insure that (2.3) holds for $v_2(t)$, let us denote $w(t) = \int_t^{\infty} r(s)/v_1^2(s)\,\mathrm{d}s$. Then $\lim_{t\to\infty} w(t) = 0$ and

$$\int_{t_1}^{\infty}\frac{r(s)}{v_2^2(s)}\mathrm{d}s = \int_{t_1}^{\infty}\frac{-w'(s)}{w(s)}\mathrm{d}s = \lim_{t\to\infty}\left(\frac{1}{w(t)} - \frac{1}{w(t_1)}\right) = \infty. \tag{2.7}$$

□

Combining Lemmas 2.2, 2.3, and 2.4, we obtain the following result.

Lemma 2.5. *Let* (2.4) *hold. Then trinomial (E) can be represented in its binomial canonical form (E^c).*

Now we can study properties of (E) with help of its canonical representation (E^c). For our reference, let us denote for (E^c)

$$L_0y = y, \quad L_1y = \frac{1}{v}\left(L_0y\right)', \quad L_2y = \frac{v^2}{r}\left(L_1y\right)', \quad L_3y = \left(L_2y\right)'. \tag{2.8}$$

Now, (E^c) can be written as $L_3y(t) + v(t)q(t)y(\sigma(t)) = 0$.

We present a structure of the nonoscillatory solutions of (E^c). Since (E^c) is in a canonical form, it follows from the well-known lemma of Kiguradze (see, e.g., [7, 9, 14]) that every nonoscillatory solution $y(t)$ of (E^c) is either of *degree 0*, that is,

$$yL_0y(t) > 0, \quad yL_1y(t) < 0, \quad yL_2y(t) > 0, \quad yL_3y(t) < 0, \tag{2.9}$$

or of *degree 2*, that is,

$$yL_0y(t) > 0, \quad yL_1y(t) > 0, \quad yL_2y(t) > 0, \quad yL_3y(t) < 0. \tag{2.10}$$

Definition 2.6. We say that (E^c) has property (A) if its every nonoscillatory solution $y(t)$ is of degree 0; that is, it satisfies (2.9).

Now we verify that property (P_0) of (E) and property (A) of (E^c) are equivalent in the sense that $y(t)$ satisfies (1.1) if and only if it obeys (2.9).

Theorem 2.7. *Let* (2.4) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *Then (E^c) has property (A) if and only if (E) has property (P_0).*

Proof. $\rightarrow$ We suppose that $y(t)$ is a positive solution of (E). We need to verify that $y'(t) < 0$. Since $y(t)$ is also a solution of (E^c), then it satisfies (2.9). Therefore, $0 > L_1 y(t) = y'(t)/v(t)$.

$\leftarrow$ Assume that $y(t)$ is a positive solution of (E^c). We will verify that (2.9) holds. Since $y(t)$ is also a solution of (E), we see that $y'(t) < 0$; that is, $L_1 y(t) < 0$. It follows from (E^c) that $L_3 y(t) = -v(t)q(t)y(\sigma(t)) < 0$. Thus, $L_2 y(t)$ is decreasing. If we admit $L_2 y(t) < 0$ eventually, then $L_1 y(t)$ is decreasing, and integrating the inequality $L_1 y(t) < L_1 y(t_1)$, we get $y(t) < y(t_1) + L_1 y(t_1) \int_{t_1}^{t} v(s)\, ds \rightarrow -\infty$ as $t \rightarrow \infty$. Therefore, $L_2 y(t) > 0$ and (2.9) holds. □

The following result which can be found in [9, 14] presents the relationship between property (A) of delay equation and that of equation without delay.

Theorem 2.8. *Let* (2.4) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *Let*

$$\sigma(t) \in C^1([t_0, \infty)), \quad \sigma'(t) > 0. \tag{2.11}$$

If

$$\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' + \frac{v(\sigma^{-1}(t))q(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))}\, y(t) = 0 \tag{E_2}$$

has property (A), then so does (E^c).

Combining Theorems 2.7 and 2.8, we get a criterion that reduces property (P_0) of (E) to the property (A) of (E_2).

Corollary 2.9. *Let* (2.4) *and* (2.11) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *If (E_2) has property (A) then (E) has property (P_0).*

Employing any known or future result for property (A) of (E_2), then in view of Corollary 2.9, we immediately obtain that property (P_0) holds for (E).

Example 2.10. We consider the third-order delay trinomial differential equation

$$\left(\frac{1}{t} y''(t)\right)' + \frac{\alpha(2-\alpha)}{t^3} y'(t) + q(t)y(\sigma(t)) = 0, \tag{2.12}$$

where $0 < \alpha < 1$ and $\sigma(t)$ satisfies (2.11). The corresponding equation (E_v) takes the form

$$\left(\frac{1}{t}v'(t)\right)' + \frac{\alpha(2-\alpha)}{t^3}\, v(t) = 0, \tag{2.13}$$

and it has the pair of the solutions $v(t) = t^\alpha$ and $\widehat{v}(t) = t^{2-\alpha}$. Thus, $v(t) = t^\alpha$ is our desirable solution, which permits to rewrite (2.12) in its canonical form. Then, by Corollary 2.9, (2.12) has property (P_0) if the equation

$$\left(t^{2\alpha-1}\left(t^{-\alpha}\, y'(t)\right)'\right)' + \frac{\left(\sigma^{-1}(t)\right)^{\alpha} q\left(\sigma^{-1}(t)\right)}{\sigma'\left(\sigma^{-1}(t)\right)} y(t) = 0 \tag{2.14}$$

has property (A).

Now, we enhance our results to guarantee stronger asymptotic behavior of the nonoscillatory solutions of (E). We impose an additional condition on the coefficients of (E) to achieve that every nonoscillatory solution of (E) tends to zero as $t \to \infty$.

Corollary 2.11. *Let* (2.4) *and* (2.11) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *If (E_2) has property (A) and*

$$\int_{t_0}^{\infty} v(s_3) \int_{s_3}^{\infty} \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^{\infty} v(s_1) q(s_1) ds_1\, ds_2\, ds_3 = \infty, \tag{2.15}$$

then every nonoscillatory solution $y(t)$ of (E) satisfies (1.2).

Proof. Assume that $y(t)$ is a positive solution of (E). Then, it follows from Corollary 2.9 that $y'(t) < 0$. Therefore, $\lim_{t\to\infty} y(t) = \ell \geq 0$. Assume $\ell > 0$. On the other hand, $y(t)$ is also a solution of (E^c), and, in view of Theorem 2.7, it has to be of *degree 0*; that is, (2.9) is fulfilled. Then, integrating (E^c) from t to ∞, we get

$$L_2 y(t) \geq \int_t^{\infty} v(s) q(s) y(\sigma(s)) \mathrm{d}s \geq \ell \int_t^{\infty} v(s) q(s) \mathrm{d}s. \tag{2.16}$$

Multiplying this inequality by $r(t)/v^2(t)$ and then integrating from t to ∞, we have

$$-L_1 y(t) \geq \ell \int_t^{\infty} \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^{\infty} v(s_1) q(s_1) \mathrm{d}s_1\, \mathrm{d}s_2. \tag{2.17}$$

Multiplying this by $v(t)$ and then integrating from t_1 to t, we obtain

$$y(t_1) \geq \ell \int_{t_1}^{t} v(s_3) \int_{s_3}^{\infty} \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^{\infty} v(s_1) q(s_1) \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3 \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{2.18}$$

This is a contradiction, and we deduce that $\ell = 0$. The proof is complete. $\square$

Example 2.12. We consider once more the third-order equation (2.12). It is easy to see that (2.15) takes the form

$$\int_{t_0}^{\infty} s_3^{\alpha} \int_{s_3}^{\infty} s_2^{1-2\alpha} \int_{s_2}^{\infty} s_1^{\alpha} q(s_1) \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}s_3 = \infty. \tag{2.19}$$

Then, by Corollary 2.11, every nonoscillatory solution of (2.12) tends to zero as $t \to \infty$ provided that (2.19) holds and (2.14) has property (A).

In the second part of this paper, we derive criteria that enable us to deduce property (P_0) of (E) from the oscillation of a suitable second-order differential equation. The following theorem is a modification of Tanaka's result [21].

Theorem 2.13. *Let* (2.4) *and* (2.11) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *Let*

$$\int^{\infty} v(s) q(s) ds < \infty. \tag{2.20}$$

If the second-order equation

$$\left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) q(s) ds\right) z(\sigma(t)) = 0 \tag{E_3}$$

is oscillatory, then (E^c) has property (A).

Proof. Assume that $y(t)$ is a positive solution of (E^c), then $y(t)$ is either of *degree 0* or of *degree 2*. Assume that $y(t)$ is of *degree 2*; that is, (2.10) holds. An integration of (E^c) yields

$$L_2 y(t) \geq \int_t^{\infty} v(s) q(s) y(\sigma(s)) \mathrm{d}s. \tag{2.21}$$

On the other hand,

$$y(t) \geq \int_{t_1}^{t} v(x) L_1 y(x) \mathrm{d}x. \tag{2.22}$$

Combining the last two inequalities, we get

$$\begin{aligned} L_2 y(t) &\geq \int_t^{\infty} v(s) q(s) \int_{t_1}^{\sigma(s)} v(x) L_1 y(x) \mathrm{d}x \, \mathrm{d}s \\ &\geq \int_t^{\infty} v(s) q(s) \int_{\sigma(t)}^{\sigma(s)} v(x) L_1 y(x) \mathrm{d}x \, \mathrm{d}s \\ &= \int_{\sigma(t)}^{\infty} L_1 y(x) v(x) \int_{\sigma^{-1}(x)}^{\infty} v(s) q(s) \mathrm{d}s \, \mathrm{d}x. \end{aligned} \tag{2.23}$$

Integrating the previous inequality from t_1 to t, we see that $w(t) \equiv L_1 y(t)$ satisfies

$$w(t) \geq w(t_1) + \int_{t_1}^{t} \frac{r(s)}{v^2(s)} \int_{\sigma(s)}^{\infty} L_1 y(x) v(x) \int_{\sigma^{-1}(x)}^{\infty} v(\delta) q(\delta) \mathrm{d}\delta \, \mathrm{d}x \, \mathrm{d}s. \tag{2.24}$$

Denoting the right-hand side of (2.24) by $z(t)$, it is easy to see that $z(t) > 0$ and

$$\begin{aligned} 0 &= \left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) g(s) ds\right) w(\sigma(t)) = 0 \\ &\geq \left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) g(s) ds\right) z(\sigma(t)) = 0. \end{aligned} \tag{2.25}$$

By Theorem 2 in [14], the corresponding equation (E_3) also has a positive solution. This is a contradiction. We conclude that $y(t)$ is of *degree 0*; that is, (E^c) has property (A). □

If (2.20) does not hold, then we can use the following result.

Theorem 2.14. *Let* (2.4) *and* (2.11) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If*

$$\int^{\infty} v(s) q(s) ds = \infty, \tag{2.26}$$

then (E^c) *has property (A).*

Proof. Assume that $y(t)$ is a positive solution of (E^c) and $y(t)$ is of *degree 2*. An integration of (E^c) yields

$$\begin{aligned} L_2 y(t_1) &\geq \int_{t_1}^{t} v(s) q(s) y(\sigma(s)) \mathrm{d}s \\ &\geq y(\sigma(t_1)) \int_{t_1}^{t} v(s) q(s) \mathrm{d}s \longrightarrow \infty \quad \text{as } t \longrightarrow \infty, \end{aligned} \tag{2.27}$$

which is a contradiction. Thus, $y(t)$ is of *degree 0*. The proof is complete now. □

Taking Theorem 2.13 and Corollary 2.9 into account, we get the following criterion for property (P_0) of (E).

Corollary 2.15. *Let* (2.4), (2.11), *and* (2.20) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If* (E_3) *is oscillatory, then* (E) *has property* (P_0).

Applying any criterion for oscillation of (E_3), Corollary 2.15 yields a sufficient condition property (P_0) of (E).

Corollary 2.16. *Let* (2.4), (2.11), *and* (2.20) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If*

$$\liminf_{t\to\infty}\left(\int_{t_0}^{\sigma(t)} \frac{r(s)}{v^2(s)}ds\right)\left(\int_t^{\infty} v(\sigma(x))\sigma'(x)\int_x^{\infty} v(s)g(s)ds\,dx\right) > \frac{1}{4}, \tag{2.28}$$

then (E) *has property* (P_0).

Proof. It follows from Theorem 11 in [9] that condition (2.28) guarantees the oscillation of (E_3). The proof arises from Corollary 2.16. □

Imposing an additional condition on the coefficients of (E), we can obtain that every nonoscillatory solution of (E) tends to zero as $t \to \infty$.

Corollary 2.17. *Let* (2.4) *and* (2.11) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If* (2.28) *and* (2.15) *hold, then every nonoscillatory solution* $y(t)$ *of* (E) *satisfies* (1.2).

Example 2.18. We consider again (2.12). By Corollary 2.17, every nonoscillatory solution of (2.12) tends to zero as $t \to \infty$ provided that (2.19) holds and

$$\liminf_{t\to\infty}\sigma^{2-2\alpha}(t)\left(\int_t^{\infty} \sigma^{\alpha}(x)\sigma'(x)\int_x^{\infty} s^{\alpha}q(s)\mathrm{d}s\,\mathrm{d}x\right) > \frac{2-2\alpha}{4}. \tag{2.29}$$

For a special case of (2.12), namely, for

$$\left(\frac{1}{t}\,y''(t)\right)' + \frac{\alpha(2-\alpha)}{t^3}y'(t) + \frac{a}{t^4}\,y(\lambda t)) = 0, \tag{2.30}$$

with $0 < \alpha < 1$, $0 < \lambda < 1$, and $a > 0$, we get that every nonoscillatory solution of (2.30) tends to zero as $t \to \infty$ provided that

$$\frac{a\lambda^{3-\alpha}}{(3-\alpha)(1-\alpha)^2} > 1. \tag{2.31}$$

If we set $a = \beta[(\beta+1)(\beta+3)+\alpha(2-\alpha)]\lambda^{\beta}$, where $\beta > 0$, then one such solution of (2.12) is $y(t) = t^{-\beta}$.

On the other hand, if for some $\gamma \in (1+\alpha, 3-\alpha)$ we have $a = \gamma[(\gamma-1)(3-\gamma)+\alpha(\alpha-2)]\lambda^{-\gamma} > 0$, then (2.31) is violated and (2.12) has a nonoscillatory solution $y(t) = t^{\gamma}$ which is of *degree 2*.

3. Summary

In this paper, we have introduced new comparison theorems for the investigation of properties of third-order delay trinomial equations. The comparison principle established in Corollaries 2.9 and 2.11 enables us to deduce properties of the trinomial third-order equations from that of binomial third-order equations. Moreover, the comparison theorems presented in Corollaries 2.15–2.17 permit to derive properties of the trinomial third-order equations from the oscillation of suitable second-order equations. The results obtained are of high generality, are easily applicable, and are illustrated on suitable examples.

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