Research Article

Partial Sums of Generalized Class of Analytic Functions Involving Hurwitz-Lerch Zeta Function

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Let $f_n(z) = z + \sum_{k=2}^n a_k z^k$ be the sequence of partial sums of the analytic function $f(z) = z + \sum_{k=2}^\infty a_k z^k$. In this paper, we determine sharp lower bounds for $\Re\{f(z)/f_n(z)\}, \Re\{f_n(z)/f(z)\}, \Re\{f'(z)/f_n'(z)\}$, and $\Re\{f_n'(z)/f'(z)\}$. The usefulness of the main result not only provides the unification of the results discussed in the literature but also generates certain new results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^\infty a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. We also consider T a subclass of $\mathcal{A}$ introduced and studied by Silverman [1], consisting of functions of the form

$$f(z) = z - \sum_{k=2}^\infty a_k z^k, \quad z \in U. \tag{1.2}$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.3}$$

We recall here a general Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined in [2] by

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \tag{1.4}$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}$, when $|z| < 1$; $\Re(s) > 1$, when $|z| = 1)$, where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$, $(\mathbb{Z} := \{0, \pm1, \pm2, \pm3, \dots\})$; $\mathbb{N} := \{1, 2, 3, \dots\}$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [3], Ferreira and López [4], Garg et al. [5], Lin and Srivastava [6], Lin et al. [7], and others. Srivastava and Attiya [8] (see also Răducanu and Srivastava [9] and Prajapat and Goyal [10]) introduced and investigated the linear operator $\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$ defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \quad (z \in U; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mu \in \mathbb{C}; \ f \in \mathcal{A}), \tag{1.5}$$

where, for convenience,

$$G_{\mu,b}(z) := (1 + b)^{\mu} \left[\Phi(z, \mu, b) - b^{-\mu} \right] \quad (z \in U). \tag{1.6}$$

We recall here the following relationships (given earlier by [9, 10]) which follow easily by using (1.1), (1.5), and (1.6):

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + b}{k + b} \right)^{\mu} a_k z^k. \tag{1.7}$$

Motivated essentially by the Srivastava-Attiya operator [8], we introduce the generalized integral operator

$$\mathcal{J}_{\mu,b}^{m,\eta} f(z) = z + \sum_{k=2}^{\infty} C_k^m(b, \mu) a_k z^k = F(z), \tag{1.8}$$

where

$$C_k^m(b, \mu) = \left| \left(\frac{1 + b}{k + b} \right)^{\mu} \frac{m!(k + \eta - 2)!}{(\eta - 2)!(k + m - 1)!} \right|, \tag{1.9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mu \in \mathbb{C}$, $\eta \geq 2$ and $m > -1$. It is of interest to note that $J_{\mu,b}^{1,2}$ is the

Srivastava-Attiya operator [8] and $J_{0,b}^{m,\eta}$ is the well-known Choi-Saigo-Srivastava operator (see [11, 12]). Suitably specializing the parameters m, η, μ, and b in $\mathcal{J}_{\mu,b}^{m,\eta}f(z)$, we can get various integral operators introduced by Alexander [13] and Bernardi [14]. Further more, we get the Jung-Kim-Srivastava integral operator [15] closely related to some multiplier transformation studied by Flett [16].

Motivated by Murugusundaramoorthy [17–19] and making use of the generalized Srivastava-Attiya operator $\mathcal{J}_{\mu,b}^{m,\eta}$, we define the following new subclass of analytic functions with negative coefficients.

For $\lambda \geq 0$, $-1 \leq \gamma < 1$, and $\beta \geq 0$, let $P_\mu^\lambda(\gamma, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{z\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)' + \lambda z^2\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)''}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,\eta}f(z) + \lambda z\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)'} - \gamma\right\} > \beta\left|\frac{z\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)' + \lambda z^2\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)''}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,\eta}f(z) + \lambda z\left(\mathcal{J}_{\mu,b}^{m,\eta}f(z)\right)'} - 1\right|, \tag{1.10}$$

where $z \in U$. Shortly we can state this condition by

$$-\Re\left\{\frac{zG'(z)}{G(z)} - \gamma\right\} > \beta\left|\frac{zG'(z)}{G(z)} - 1\right|, \tag{1.11}$$

where

$$G(z) = (1-\lambda)F(z) + \lambda z F'(z) = z + \sum_{k=2}^{\infty}[1 + \lambda(k-1)]\left|C_k^m(b,\mu)\right|a_k z^k, \tag{1.12}$$

and $F(z) = \mathcal{J}_{\mu,b}^{m,\eta}f(z)$.

Recently, Silverman [20] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. In the present paper and by following the earlier work by Silverman [20] (see [21–25]) on partial sums of analytic functions, we study the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k, \tag{1.13}$$

when the coefficients of $f(z)$ satisfy the condition (1.14). Also, we will determine sharp lower bounds for $\Re\{f(z)/f_n(z)\}$, $\Re\{f_n(z)/f(z)\}$, $\Re\{f'(z)/f_n'(z)\}$, and $\Re\{f_n'(z)/f'(z)\}$. It is seen that this study not only gives as a particular case, the results of Silverman [20], but also gives rise to several new results.

Before stating and proving our main results, we derive a sufficient condition giving the coefficient estimates for functions $f(z)$ to belong to this generalized function class.

Lemma 1.1. *A function $f(z)$ of the form (1.1) is in $P_\mu^\lambda(\gamma, \beta)$ if*

$$\sum_{k=2}^{\infty}(1 + \lambda(k-1))\left[k(1+\beta) - (\gamma+\beta)\right]|a_k|C_k^m(b,\mu) \leq 1 - \gamma, \tag{1.14}$$

where, for convenience,

$$\rho_k = \rho_k(\lambda, \gamma, \eta) = (1 + \lambda(k-1))[n(1+\beta) - (\gamma+\beta)]C_k^m(b,\mu), \tag{1.15}$$

$0 \leq \lambda \leq 1$, $-1 \leq \gamma < 1$, $\beta \geq 0$, *and* $C_k^m(b,\mu)$, *is given by* (1.9).

Proof. The proof of **Lemma** 1.1 is much akin to the proof of Theorem 1 obtained by Murugusundaramoorthy [17], hence we omit the details. □

2. Main Results

Theorem 2.1. *If f of the form* (1.1) *satisfies the condition* (1.14)*, then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda, \gamma, \eta) - 1 + \gamma}{\rho_{n+1}(\lambda, \gamma, \eta)} \quad (z \in U), \tag{2.1}$$

where

$$\rho_k = \rho_k(\lambda, \gamma, \eta) \geq \begin{cases} 1 - \gamma & \text{if } k = 2, 3, \ldots, n, \\ \rho_{n+1} & \text{if } k = n+1, n+2, \ldots. \end{cases} \tag{2.2}$$

The result (2.1) *is sharp with the function given by*

$$f(z) = z + \frac{1-\gamma}{\rho_{n+1}} z^{n+1}. \tag{2.3}$$

Proof. Define the function $w(z)$ by

$$\frac{1 + w(z)}{1 - w(z)} = \frac{\rho_{n+1}}{1 - \gamma}\left[\frac{f(z)}{f_n(z)} - \frac{\rho_{n+1} - 1 + \gamma}{\rho_{n+1}}\right]$$

$$= \frac{1 + \sum_{k=2}^n a_k z^{k-1} + (\rho_{n+1}/(1-\gamma)) \sum_{k=n+1}^\infty a_k z^{k-1}}{1 + \sum_{k=2}^n a_k z^{k-1}}. \tag{2.4}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (2.4) we can write

$$w(z) = \frac{(\rho_{n+1}/(1-\gamma)) \sum_{k=n+1}^\infty a_k z^{k-1}}{2 + 2\sum_{k=2}^n a_k z^{k-1} + (\rho_{n+1}/(1-\gamma)) \sum_{k=n+1}^\infty a_k z^{k-1}}. \tag{2.5}$$

Hence we obtain

$$|w(z)| \leq \frac{(\rho_{n+1}/(1-\gamma)) \sum_{k=n+1}^\infty |a_k|}{2 - 2\sum_{k=2}^n |a_k| - (\rho_{n+1}/(1-\gamma)) \sum_{k=n+1}^\infty |a_k|}. \tag{2.6}$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\rho_{n+1}}{1-\gamma}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 2 - 2\sum_{k=2}^{n}|a_k| \tag{2.7}$$

or, equivalently,

$$\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1-\gamma}|a_k| \leq 1. \tag{2.8}$$

From the condition (1.14), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1-\gamma}|a_k| \leq \sum_{k=2}^{\infty}\frac{\rho_k}{1-\gamma}|a_k| \tag{2.9}$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\rho_k - 1 + \gamma}{1-\gamma}\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\rho_k - \rho_{n+1}}{1-\gamma}\right)|a_k| \geq 0. \tag{2.10}$$

To see that the function given by (2.3) gives the sharp result, we observe that for $z = re^{i\pi/n}$,

$$\frac{f(z)}{f_n(z)} = 1 + \frac{1-\gamma}{\rho_{n+1}}z^n \longrightarrow 1 - \frac{1-\gamma}{\rho_{n+1}}$$

$$= \frac{\rho_{n+1} - 1 + \gamma}{\rho_{n+1}} \quad \text{when} \quad r \longrightarrow 1^-. \tag{2.11}$$

$\square$

We next determine bounds for $f_n(z)/f(z)$.

Theorem 2.2. *If f of the form* (1.1) *satisfies the condition* (1.14)*, then*

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\rho_{n+1}}{\rho_{n+1} + 1 - \gamma} \quad (z \in U), \tag{2.12}$$

where $\rho_{n+1} \geq 1 - \gamma$ and

$$\rho_k \geq \begin{cases} 1 - \gamma & \text{if } k = 2, 3, \ldots, n, \\ \rho_{n+1} & \text{if } k = n + 1, n + 2, \ldots. \end{cases} \tag{2.13}$$

The result (2.12) *is sharp with the function given by* (2.3)*.*

Proof. We write

$$
\frac{1 + w(z)}{1 - w(z)} = \frac{\rho_{n+1} + 1 - \gamma}{1 - \gamma}\left[\frac{f_n(z)}{f(z)} - \frac{\rho_{n+1}}{\rho_{n+1} + 1 - \gamma}\right]
$$

$$
= \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} - (\rho_{n+1}/(1 - \gamma))\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}},
$$

(2.14)

where

$$
|w(z)| \le \frac{((\rho_{n+1} + 1 - \gamma)/(1 - \gamma))\sum_{k=n+1}^{\infty}|a_k|}{2 - 2\sum_{k=2}^{n}|a_k| - ((\rho_{n+1} - 1 + \gamma)/(1 - \gamma))\sum_{k=n+1}^{\infty}|a_k|} \le 1.
$$

(2.15)

This last inequality is equivalent to

$$
\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1 - \gamma}|a_k| \le 1.
$$

(2.16)

We are making use of (1.14) to get (2.10). Finally, equality holds in (2.12) for the extremal function $f(z)$ given by (2.3).   □

We next turns to ratios involving derivatives.

Theorem 2.3. *If f of the form* (1.1) *satisfies the condition* (1.14), *then*

$$
\Re\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{\rho_{n+1} - (n+1)(1 - \gamma)}{\rho_{n+1}} \quad (z \in U),
$$

(2.17)

$$
\Re\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{\rho_{n+1}}{\rho_{n+1} + (n+1)(1 - \gamma)} \quad (z \in U),
$$

(2.18)

where $\rho_{n+1} \ge (n+1)(1 - \gamma)$ and

$$
\rho_k \ge \begin{cases} k(1 - \gamma) & \text{if } k = 2, 3, \ldots, n, \\ k\left(\dfrac{\rho_{n+1}}{n+1}\right) & \text{if } k = n+1, n+2, \ldots. \end{cases}
$$

(2.19)

The results are sharp with the function given by (2.3).

Proof. We write

$$
\frac{1 + w(z)}{1 - w(z)} = \frac{\rho_{n+1}}{(n+1)(1 - \gamma)}\left[\frac{f'(z)}{f_n'(z)} - \left(\frac{\rho_{n+1} - (n+1)(1 - \gamma)}{\rho_{n+1}}\right)\right],
$$

(2.20)

where

$$w(z) = \frac{(\rho_{n+1}/((n+1)(1-\gamma)))\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} k a_k z^{k-1} + (\rho_{n+1}/((n+1)(1-\gamma)))\sum_{k=n+1}^{\infty} k a_k z^{k-1}}. \tag{2.21}$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}}{(n+1)(1-\gamma)} \sum_{k=n+1}^{\infty} k|a_k| \leq 1. \tag{2.22}$$

From the condition (1.14), it is sufficient to show that

$$\sum_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}}{(n+1)(1-\gamma)} \sum_{k=n+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty} \frac{\rho_k}{1-\gamma}|a_k| \tag{2.23}$$

which is equivalent to

$$\sum_{k=2}^{n} \left(\frac{\rho_k - (1-\gamma)k}{1-\gamma} \right)|a_k| + \sum_{k=n+1}^{\infty} \frac{(n+1)\rho_k - k\rho_{n+1}}{(n+1)(1-\gamma)}|a_k| \geq 0. \tag{2.24}$$

To prove the result (2.18), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{(n+1)(1-\gamma)+\rho_{n+1}}{(1-\gamma)(n+1)} \left[\frac{f_n'(z)}{f'(z)} - \frac{\rho_{n+1}}{(n+1)(1-\gamma)+\rho_{n+1}} \right], \tag{2.25}$$

where

$$w(z) = \frac{-(1+\rho_{n+1}/((n+1)(1-\gamma)))\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} k a_k z^{k-1} + (1-\rho_{n+1}/((n+1)(1-\gamma)))\sum_{k=n+1}^{\infty} k a_k z^{k-1}}. \tag{2.26}$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} k|a_k| + \left(\frac{\rho_{n+1}}{(n+1)(1-\gamma)} \right) \sum_{k=n+1}^{\infty} k|a_k| \leq 1. \tag{2.27}$$

It suffices to show that the left hand side of (2.27) is bounded previously by the condition

$$\sum_{k=2}^{\infty} \frac{\rho_k}{1-\gamma}|a_k|, \tag{2.28}$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\rho_k}{1-\gamma}-k\right)|a_k|+\sum_{k=n+1}^{\infty}\left(\frac{\rho_k}{1-\gamma}-\frac{\rho_{n+1}}{(n+1)(1-\gamma)}\right)k|a_k|\geq 0. \qquad (2.29)$$

$\square$

Remark 2.4. As a special case of the previous theorems, we can determine new sharp lower bounds for $\Re\{f(z)/f_n(z)\}$, $\Re\{f_n(z)/f(z)\}$, $\Re\{f'(z)/f'_n(z)\}$, and $\Re\{f'_n(z)/f'(z)\}$ for various function classes involving the Alexander integral operator [13] and Bernardi integral operators [14], Jung-Kim-Srivastava integral operator [15] and Choi-Saigo-Srivastava operator (see [11, 12]) on specializing the values of η, m, μ, and b.

Some other work related to partial sums and also related to zeta function can be seen in ([26–29]) for further views and ideas.
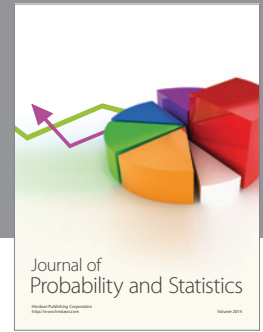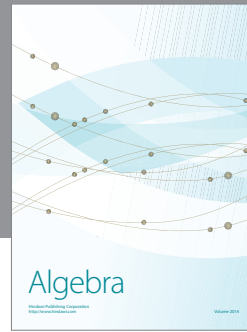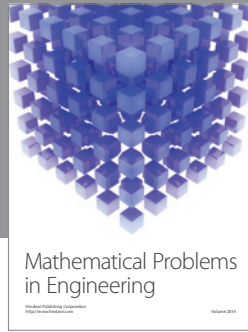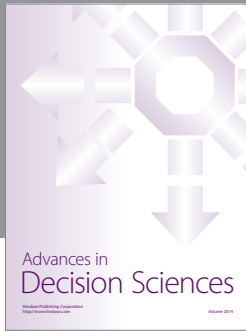
Acknowledgment

References

[1] H. Silverman, "Univalent functions with negative coefficients," *Proceedings of the American Mathematical Society*, vol. 51, pp. 109–116, 1975.

[2] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[3] J. Choi and H. M. Srivastava, "Certain families of series associated with the Hurwitz-Lerch zeta function," *Applied Mathematics and Computation*, vol. 170, no. 1, pp. 399–409, 2005.

[4] C. Ferreira and J. L. López, "Asymptotic expansions of the Hurwitz-Lerch zeta function," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 210–224, 2004.

[5] M. Garg, K. Jain, and H. M. Srivastava, "Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions," *Integral Transforms and Special Functions*, vol. 17, no. 11, pp. 803–815, 2006.

[6] S.-D. Lin and H. M. Srivastava, "Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 725–733, 2004.

[7] S.-D. Lin, H. M. Srivastava, and P.-Y. Wang, "Some expansion formulas for a class of generalized Hurwitz-Lerch zeta functions," *Integral Transforms and Special Functions*, vol. 17, no. 11, pp. 817–827, 2006.

[8] H. M. Srivastava and A. A. Attiya, "An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination," *Integral Transforms and Special Functions*, vol. 18, no. 3-4, pp. 207–216, 2007.

[9] D. Răducanu and H. M. Srivastava, "A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function," *Integral Transforms and Special Functions*, vol. 18, no. 11-12, pp. 933–943, 2007.

[10] J. K. Prajapat and S. P. Goyal, "Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions," *Journal of Mathematical Inequalities*, vol. 3, no. 1, pp. 129–137, 2009.

[11] J. H. Choi, M. Saigo, and H. M. Srivastava, "Some inclusion properties of a certain family of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 432–445, 2002.

[12] Y. Ling and F. Liu, "The Choi-Saigo-Srivastava integral operator and a class of analytic functions," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 613–621, 2005.

[13] J. W. Alexander, "Functions which map the interior of the unit circle upon simple regions," *Annals of Mathematics*, vol. 17, no. 1, pp. 12–22, 1915.

[14] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.

[15] I. B. Jung, Y. C. Kim, and H. M. Srivastava, "The Hardy space of analytic functions associated with certain one-parameter families of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 176, no. 1, pp. 138–147, 1993.

[16] T. M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities," *Journal of Mathematical Analysis and Applications*, vol. 38, pp. 746–765, 1972.

[17] G. Murugusundaramoorthy, "A unified class of analytic functions with negative coefficients involving the Hurwitz-Lerch zeta function," *Bulletin of Mathematical Analysis and Applications*, vol. 1, no. 3, pp. 71–84, 2009.

[18] G. Murugusundaramoorthy, "Subordination results and integral means inequalities for k-uniformly starlike functions defined by convolution involving the Hurwitz-Lerch Zeta function," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. LV, no. 3, pp. 69–82, 2010.

[19] G. Murugusundaramoorthy, "A subclass of analytic functions associated with the Hurwitz-Lerch zeta function," *Hacettepe Journal of Mathematics and Statistics*, vol. 39, no. 2, pp. 265–272, 2010.

[20] H. Silverman, "Partial sums of starlike and convex functions," *Journal of Mathematical Analysis and Applications*, vol. 209, no. 1, pp. 221–227, 1997.

[21] B. A. Frasin, "Partial sums of certain analytic and univalent functions," *Acta Mathematica. Academiae Paedagogicae Nyíregyháziensis*, vol. 21, no. 2, pp. 135–145, 2005.

[22] B. A. Frasin, "Generalization of partial sums of certain analytic and univalent functions," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 735–741, 2008.

[23] T. Rosy, K. G. Subramanian, and G. Murugusundaramoorthy, "Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, article 64, 8 pages, 2003.

[24] T. Sheil-Small, "A note on the partial sums of convex schlicht functions," *The Bulletin of the London Mathematical Society*, vol. 2, pp. 165–168, 1970.

[25] E. M. Silvia, "On partial sums of convex functions of order α," *Houston Journal of Mathematics*, vol. 11, no. 3, pp. 397–404, 1985.

[26] R. W. Ibrahim and M. Darus, "Partial sums for certain classes of meromorphic functions," *Tamkang Journal of Mathematics*, vol. 41, no. 1, pp. 39–49, 2010.

[27] R. W. Ibrahim and M. Darus, "Operator defined by convolution with zeta functions," *Far East Journal of Mathematical Sciences*, vol. 40, no. 1, pp. 93–105, 2010.

[28] M. Darus and R. W. Ibrahim, "On partial sums of generalized differential operator," *Proceedings of the Pakistan Academy of Sciences*, vol. 46, no. 4, pp. 209–215, 2009.

[29] R. W. Ibrahim and M. Darus, "Partial sums of analytic functions of bounded turning with applications," *Computational & Applied Mathematics*, vol. 29, no. 1, pp. 81–88, 2010.