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*Research Article*

# Positive Solution of Fourth-Order Integral Boundary Value Problem with Two Parameters

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The author investigates the fourth-order integral boundary value problem with two parameters $u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t,u), t \in (0,1)$, $u(0) = u(1) = 0$, $u''(0) = \int_0^1 u(s)\phi_1(s)ds, u''(1) = \int_0^1 u(s)\phi_2(s)ds$, where nonlinear term function $f$ is allowed to change sign. Applying the fixed point index theorem on cone together with the operator spectrum theorem, some results on the existence of positive solution are obtained.

## 1. Introduction

The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can all be reduced to nonlocal problems with integral boundary conditions (see, e.g., [1–3]). For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [4], Karakostas and Tsamatos [5], and Lomtatidze and Malaguti [6] and the references therein. For more information about the general theory of integral equations and their relation to boundary value problems, we refer to the books of Corduneanu [7] and Agarwal and O'Regan [8].

Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoints and nonlocal boundary value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the reader to [9–15] and the references therein.

In the recent literature, several sorts of boundary value problems with integral boundary conditions have been studied further, see [16–20]. Especially, Ruyun Ma and Yulian

An [18] investigated the global structure of positive solutions for nonlocal boundary value problems

$$u''(t)+\lambda h(t)f(u(t))=0,\quad 0<t<1,$$
$$u(0)=0,\qquad u(1)=\int_0^1 u(s)dA(s),\tag{1.1}$$

by using global bifurcation techniques, where $f\in C([0,\infty),[0,\infty)), h\in C((0,1),[0,\infty))$. In [19], Jiqiang Jiang et al. investigated the existence of positive solution for second-order singular Sturm-Liouville integral boundary value problems

$$-u''(t)=\lambda h(t)f(t,u(t)),\quad 0<t<1,$$
$$\alpha u(0)-\beta u'(0)=\int_0^1 a(s)u(s)ds,\tag{1.2}$$
$$\gamma u(1)+\delta u'(1)=\int_0^1 b(s)u(s)ds,$$

by using the fixed point theory in cones, where $f\in C([0,1]\times(0,\infty),[0,\infty))$.

On the other hand, the fourth-order boundary value problem describe the deformations of an elastic beam in equilibrium state. Owing to its importance in physics, the existence of solutions to this problem has been studied by many authors; see, for example, [21–24] and references therein. Especially, in [22], Li studied existence of positive solution for fourth-order boundary value problem

$$u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f(t,u),\quad t\in(0,1),$$
$$u(0)=u(1)=0=u''(0)==u''(1)=0,\tag{1.3}$$

by using the fixed point index theorem, where $f\in C([0,\infty),[0,\infty))$.

Motivated by the above-mentioned works [18, 19, 22], in this paper, we study the following fourth-order integral boundary value problem (for short BVP in the sequel) with two parameters:

$$u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f(t,u),\quad t\in(0,1),$$
$$u(0)=u(1)=0,\qquad u''(0)=\int_0^1 u(s)\phi_1(s)ds,\qquad u''(1)=\int_0^1 u(s)\phi_2(s)ds,\tag{1.4}$$

where nonlinear term function $f$ is allowed to change sign. To the best of our knowledge, BVP has not been investigated up to now. In the literature such as above-mentioned paper [18, 19, 22], the nonnegativity on $f$ is a usual assumption. In the present paper, since the function $f$ is not assumed to be nonnegative, the corresponding integral operator doesn't map the cone into cone, and so, there exists difficulty in applying the cone fixed point theorem. On the other hand, owing to the occurrence of parameter $\alpha,\beta$ in this boundary value problem including integral boundary conditions, it is not easy to transform the BVP (1.4) into an integral equation directly. To overcome these difficulties, we first introduce operator spectrum method combined with some analysis technique, next apply the fixed point index theorem, and establish existence of positive solution to BVP (1.4).

Let us begin with listing the following assumption conditions, which will be used in the sequel:

Let $I = [0,1], \mathbf{R} = (-\infty, +\infty), \mathbf{R}_- = (-\infty, 0], \mathbf{R}_+ = [0, +\infty)$.

(H1) $f \in C[I \times \mathbf{R}_+, \mathbf{R}]$ and exists $M \in L^1(0,1) \cap C[(0,1), \mathbf{R}_+]$ such that

$$f(t,u) + M(t) \geq 0, \qquad (t,u) \in (0,1) \times \mathbf{R}_+. \tag{1.5}$$

(H2) $\alpha, \beta \in \mathbf{R}, \beta < 2\pi^2, \alpha \geq -\beta^2/4, \alpha/\pi^4 + \beta/\pi^2 < 1$.

Let $\lambda_1, \lambda_2$ be the roots of the polynomial $p(\lambda) = \lambda^2 + \beta\lambda - \alpha$; namely,

$$\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}. \tag{1.6}$$

By (H2), it is to see that $\lambda_1 \geq \lambda_2 > -\pi^2$.

Let $\Gamma_0 = \pi^4 - \beta\pi^2 - \alpha$. Then (H2) implies $\Gamma_0 > 0$. Let $X = C[0,1]$ be the real Banach space equipped with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Denote by $P$ the set $P = \{u \in X : u(\mathrm{t}) \geq 0, t \in I\}$ in $X$.

## 2. Preliminaries

In this section, we shall give some important preliminary lemmas, which will be used in proving of our main results.

**Lemma 2.1** (see [22, 23]). *Suppose that (H2) holds, then there exist unique $\varphi_i, \psi_i, i = 1,2$ satisfying*

$$\begin{gathered}
-\varphi_i''(t) + \lambda_i \varphi_i(t) = 0, \quad t \in [0,1], \\
\varphi_i(0) = 0, \qquad \varphi_i(1) = 1, \\
-\psi_i''(t) + \lambda_i \psi_i(t) = 0, \quad t \in [0,1], \\
\psi_i(0) = 1, \qquad \psi_i(1) = 0,
\end{gathered} \tag{2.1}$$

*respectively, and $\varphi_i \geq 0, \psi_i \geq 0$ on $[0,1]$, where $\lambda_i$ is as in* (1.6). *Moreover, $\varphi_i, \psi_i$ have the expression*

$$\varphi_i(t) = \begin{cases} \dfrac{\sinh \omega_i t}{\sinh \omega_i}, & \lambda_i > 0, \\ t, & \lambda_i = 0, \\ \dfrac{\sin \omega_i t}{\sin \omega_i}, & -\pi^2 < \lambda_i < 0, \end{cases} \qquad \psi_i(t) = \begin{cases} \dfrac{\sinh \omega_i (1-t)}{\sinh \omega_i}, & \lambda_i > 0, \\ 1-t, & \lambda_i = 0, \\ \dfrac{\sin \omega_i (1-t)}{\sin \omega_i}, & -\pi^2 < \lambda_i < 0, \end{cases} \tag{2.2}$$

*where $\omega_i = \sqrt{|\lambda_i|}, i = 1,2$.*

Let $G_i(t,s)(i=1,2)$ be the Green function of the linear boundary value problem

$$-u''(t)+\lambda_i u(t)=0, \qquad t\in[0,1], \quad u(0)=u(1)=0. \tag{2.3}$$

By [22, 23], $G_i(t,s)$ can be expressed by the formula

$$G_i(t,s)=\frac{1}{\sigma_i}\begin{cases}\varphi_i(t)\psi_i(s), & 0\le t\le s\le 1,\\ \psi_i(t)\varphi_i(s), & 0\le s\le t\le 1,\end{cases} \tag{2.4}$$

where

$$\sigma_i=\begin{cases}\dfrac{\omega_i}{\sinh\omega_i}, & \text{if } \lambda_i>0,\\ 1, & \text{if } \lambda_i=0,\ i=1,2.\\ \dfrac{\omega_i}{\sin\omega_i}, & \text{if} -\pi^2<\lambda_i<0,\end{cases} \tag{2.5}$$

**Lemma 2.2** (see [22, 23]). $G_i=G_i(t,s)(i=1,2)$ *have the following properties:*

(i) $G_i(t,s)>0, \forall t,s\in(0,1)$.

(ii) $G_i(t,s)\le C_iG_i(s,s), \forall t,s\in[0,1], \varphi_i\le C_i, \psi_i\le C_i, t\in[0,1]$.

(iii) $G_i(t,s)\ge \delta_iG_i(t,t)G_i(s,s), \forall t,s\in[0,1], \varphi_i(t)\ge\delta_iG_i(t,t), \psi_i(t)\ge\delta_iG_i(t,t), t\in[0,1]$, *where*

$$C_i=\begin{cases}1, & \textit{if } \lambda_i\ge 0,\\ \dfrac{1}{\sin\omega_i}, & \textit{if } -\pi^2<\lambda_i<0,\end{cases} \qquad \delta_i=\begin{cases}\dfrac{\omega_i}{\sinh\omega_i}, & \textit{if } \lambda_i>0,\\ 1, & \textit{if } \lambda_i=0,\\ \omega_i\sin\omega_i, & \textit{if } -\pi^2<\lambda_i<0.\end{cases} \tag{2.6}$$

Put $D_i=\max_{t\in I}\int_0^1 G_i(t,s)ds, i=1,2$. Set $E_{21}=D_2C_1, E_{12}=D_1C_2$, where $C_i$ is described as before. We need also the following assumptions in the sequel.

(H3) Functions $\phi_i\in C[I,\mathbf{R}_-], i=1,2$, satisfy $D\doteq E_{12}\int_0^1|\phi_1(s)|ds+E_{21}\int_0^1|\phi_2(s)|ds<1$.

Let $h\in C(0,1)\cap L^1(0,1)$, consider the following BVP:

$$\begin{gathered}u^{(4)}(t)+\beta u''(t)-\alpha u(t)=h(t), \quad t\in(0,1),\\ u(0)=u(1)=u''(0)=u''(1)=0.\end{gathered} \tag{2.7}$$

By papers [22, 23], BVP (2.7) has a unique solution $u=Kh$ expressed by

$$\begin{aligned}Kh(t)&=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)h(\tau)\,d\tau\,ds\\ &=\int_0^1\int_0^1 G_2(t,s)G_1(s,\tau)h(\tau)\,d\tau\,ds, \quad t\in[0,1].\end{aligned} \tag{2.8}$$

Let $w = KM$. Since $M \in L^1(0,1) \cap C[(0,1), R_+]$, by Lemma 2.2, it is easy to verify that $w \in P$.

Let

$$g_1(t) = -\int_0^1 G_2(t,s)\varphi_1(s)ds, \qquad t \in [0,1], \tag{2.9}$$

where $\varphi_1$ is as in (2. 1). By Lemmas 2.1 and 2.2, we have $g_1 \in C^2([0,1], R_-)$ and

$$\begin{gathered} -g_1''(t) + \lambda_2 g_1(t) = -\varphi_1(t), \quad t \in [0,1], \\ g_1(0) = g_1(1) = 0. \end{gathered} \tag{2.10}$$

On the other hand, $\varphi_1$ satisfies the following relation:

$$\begin{gathered} -\varphi_1''(t) + \lambda_1 \varphi_1(t) = 0, \quad t \in [0,1], \\ \varphi_1(0) = 0, \qquad \varphi_1(1) = 1. \end{gathered} \tag{2.11}$$

So, from (2.10)–(2.11), it follows that

$$\begin{gathered} g_1''(0) = \lambda_2 g_1(0) + \varphi_1(0) = 0, \\ g_1''(1) = \lambda_2 g_1(1) + \varphi_1(1) = 1. \end{gathered} \tag{2.12}$$

Now, we make the following decomposition:

$$\begin{aligned} g_1^{(4)} + \beta g_1'' - \alpha g_1 &= \left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_2\right) g_1 \\ &= \left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-g_1'' + \lambda_2 g_1\right) \\ &= \frac{d^2\varphi_1}{dt^2} - \lambda_1 \frac{d\varphi_1}{dt} = 0. \end{aligned} \tag{2.13}$$

So by (2.10), (2.12)-(2.13), it follows that

$$\begin{gathered} g_1^{(4)}(t) + \beta g_1''(t) - \alpha g_1(t) = 0, \quad t \in [0,1], \\ g_1(0) = g_1(1) = 0, \qquad g_1''(0) = 0, \qquad g_1''(1) = 1, \\ g_1(t) \le 0, \quad t \in [0,1]. \end{gathered} \tag{2.14}$$

Similarly, by setting

$$g_2(t) = -\int_0^1 G_1(t,s)\varphi_2(s)ds, \qquad t \in [0,1], \tag{2.15}$$

we have

$$\begin{gathered}
g_2^{(4)}(t)+\beta g_2''(t)-\alpha g_2(t)=0, \quad t\in[0,1], \\
g_2(0)=g_2(1)=0, \qquad g_2''(0)=1, \qquad g_2''(1)=0, \\
g_2(t)\le 0, \quad t\in[0,1].
\end{gathered} \tag{2.16}$$

For any $u\in X$, define $u^*$ as

$$u^*(t)=\begin{cases} u(t), & \text{if } u(t)\ge 0, \\ 0, & \text{if } u(t)<0. \end{cases} \tag{2.17}$$

Obviously, $u^*\in P$ for any $u\in X$.

Let $h\in L^1(0,1)\cap C(0,1)$; consider the BVP with integral boundary conditions

$$\begin{gathered}
u^{(4)}(t)+\beta u''(t)-\alpha u(t)=h(t), \qquad t\in(0,1), \\
u(0)=u(1)=0, \\
u''(0)=\int_0^1 [u-w]^*(s)\phi_1(s)ds, \qquad u''(1)=\int_0^1 [u-w]^*(s)\phi_2(s)ds.
\end{gathered} \tag{2.18}$$

Denote operator B on $C[0,1]$ by

$$Bu(t)=g_2(t)\int_0^1 [u-w]^*(s)\phi_1(s)ds+g_1(t)\int_0^1 [u-w]^*(s)\phi_2(s)ds. \tag{2.19}$$

It is easy to see that $B$ maps $C[0,1]$ into $C[0,1]$.

Define operator L: $C^4(0,1)\to C(0,1)$ as follows:

$$Lu=u^{(4)}+\beta u''-\alpha u. \tag{2.20}$$

We need the following Lemma.

**Lemma 2.3.** *Let (H2) holds. Assume that $h\in L^1(0,1)\cap C(0,1)$ and $\phi_i\in C[\mathbf{I},\mathbf{R}_-], i=1,2$. Then $\overline{u}\in C^4(0,1)\cap C^2[0,1]$ is a solution of* (2.18) *if and only if $\overline{u}$ is a solution of operator equation $u=Kh+Bu$ in $C[0,1]$.*

*Proof.* (1) Assume $\overline{u} \in C^4(0,1) \cap C^2[0,1]$ is a solution of (2.18). By (2.14)–(2.20), we have

$$(B\overline{u})(0) = (B\overline{u})(1) = 0, \quad (B\overline{u})''(0) = \int_0^1 [\overline{u} - w]^*(s)\phi_1(s)ds,$$
$$(B\overline{u})''(1) = \int_0^1 [\overline{u} - w]^*(s)\phi_2(s)ds, \tag{2.21}$$
$$L(B\overline{u}) = (Lg_2)\int_0^1 [\overline{u} - w]^*(s)\phi_1(s)ds + (Lg_1)\int_0^1 [\overline{u} - w]^*(s)\phi_2(s)ds = 0.$$

Let $\overline{v} = \overline{u} - B\overline{u}$. Then $L\overline{v}(t) = L\overline{u}(t) - LB\overline{u}(t) = L\overline{u}(t) = h(t), t \in (0,1); \overline{v}(0) = \overline{u}(0) - (B\overline{u})(0) = 0, \overline{v}(1) = \overline{u}(1) - (B\overline{u})(1) = 0$; $\overline{v}''(0) = \overline{u}''(0) - (B\overline{u})''(0) = 0, \overline{v}''(1) = \overline{u}''(1) - (B\overline{u})''(1) = 0$. Thus, by (2.7)-(2.8), we have $\overline{v} = Kh, \overline{v} \in C[0,1]$, and so $\overline{u} = Kh + B\overline{u}, \overline{u} \in C[0,1]$.

(2) Inversely, assume $\overline{u} \in C[0,1]$ satisfies $\overline{u} = Kh + B\overline{u}$. Then $\overline{u} \in C^4(0,1) \cap C^2[0,1]$. By (2.7), (2.8),(2.14)–(2.20), we have

$$LKh = h, \qquad LB\overline{u} = 0, \qquad (Kh)(0) = (Kh)(1) = (Kh)''(0) = (Kh)''(1) = 0,$$
$$(B\overline{u})(0) = (B\overline{u})(1) = 0, \qquad (B\overline{u})''(0) = \int_0^1 [\overline{u} - w]^*(s)\phi_1(s)ds, \tag{2.22}$$
$$(B\overline{u})''(1) = \int_0^1 [\overline{u} - w]^*(s)\phi_2(s)ds.$$

Consequently,

$$L\overline{u} = LKh + LB\overline{u} = h,$$
$$\overline{u}(0) = (Kh)(0) + (B\overline{u})(0) = 0, \qquad \overline{u}(1) = (Kh)(1) + (B\overline{u})(1) = 0,$$
$$\overline{u}''(0) = (Kh)''(0) + (B\overline{u})''(0) = \int_0^1 [\overline{u} - w]^*(s)\phi_1(s)ds, \tag{2.23}$$
$$\overline{u}''(1) = (Kh)''(1) + (B\overline{u})''(1) = \int_0^1 [\overline{u} - w]^*(s)\phi_2(s)ds.$$

Hence, $\overline{u}$ is a solution of (2.18). The proof is complete. □

We have also the following lemma.

**Lemma 2.4.** *Suppose (H3) holds. Then $B : X \to X$ is a bounded operator with $\|B\| \leq D(< 1)$ and $BX \subset P$.*

*Proof.* In view of Lemma 2.2 (ii), by (2.9),(2.15),(2.19) and (H3), noticing that $w \in P$, for any $u \in X$ and $t \in I$, we have

$$\begin{aligned} |(Bu)(t)| &\le |g_2(t)| \left| \int_0^1 [u-w]^* \phi_1(s) \right| ds + |g_1(t)| \left| \int_0^1 [u-w]^* \phi_2(s) \right| ds \\ &\le E_{12} \int_0^1 |u(s)| |\phi_1(s)| ds + E_{21} \int_0^1 |u(s)| |\phi_2(s)| ds \\ &\le D\|u\|. \end{aligned} \tag{2.24}$$

Thus, $||Bu|| \le D||u||$, and so $||B|| \le D(< 1)$.

On the other hand, from $g_i(t) \le 0, \phi_i(t) \le 0, t \in I, i = 1,2$, we have $BX \subset P$. So, Lemma 2.4 is true.

By (2.7)-(2.8), it follows from $w = KM$ that

$$\begin{gathered} w^{(4)}(t) + \beta w''(t) - \alpha w(t) = M(t), \quad t \in (0,1), \\ w(0) = w(1) = w''(0) = w''(1) = 0. \end{gathered} \tag{2.25}$$

For any $u \in X$, let $\bar{\mathrm{f}}u(t) = f(t,[u-w]^*(t)), t \in [0,1]$ and $Gu(t) = \bar{\mathrm{f}}u(t) + M(t), t \in (0,1)$. Under conditions (H1)–(H3), consider the following auxiliary BVP:

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = Gu(t), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = \int_0^1 [u-w]^*(s)\phi_1(s)ds, \qquad u''(1) = \int_0^1 [u-w]^*(s)\phi_2(s)ds. \end{gathered} \tag{2.26}$$

Notice that $w(t)$ satisfies (2.25), it is easy to see that $\overline{u} \in C^4(0,1) \cap C^2[0,1]$ is a solution of (2.26) if and only if $\overline{u} - w \in C^4(0,1) \cap C^2[0,1]$ is a solution of the following BVP:

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t,u^*(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \qquad u''(0) = \int_0^1 u^*(s)\phi_1(s)ds, \qquad u''(1) = \int_0^1 u^*(s)\phi_2(s)ds. \end{gathered} \tag{2.27}$$

Thus, if and only if $\overline{u}(t) \ge w(t), t \in [0,1]$, then $\overline{u} - w$ is a solution of BVP (1.4).

Now, by Lemma 2.3, $\overline{u} \in C^4(0,1) \cap C^2[0,1]$ is a solution of (2.26) if $\overline{u} \in X$ is a fixed point of the operator $KG + B$. So, we only need focusing our attention on the existence of the fixed point of $KG + B$.

For the remainder of this section, we give the definition of positive solution.

By a positive solution of BVP (1.4), we mean a function $u \in C^4(0,1) \cap C^2[0,1]$ such that $u(t) \ge 0, t \in [0,1]$, $u(t) > 0, t \in (0,1)$, and $u$ satisfies (1.4). □

## 3. Main Results

We introduce now some notations, which will be used in the sequel.

Let $C_1$, $\delta_1$, and $D$ be as described in Lemma 2.2 and (H3), respectively. We also set

$$d_1 = \int_0^1 \int_0^1 G_2(s,\tau)M(\tau)d\tau\, ds, \qquad b_0 = \frac{C_1^2 d_1}{\delta_1(1-D)}, \qquad f_0 = \liminf_{u\to+0} \inf_{t\in I} \frac{f(t,u)}{u},$$
$$f^\infty = \overline{\lim_{u\to+\infty}} \max_{t\in I} \frac{f(t,u)}{u}. \tag{3.1}$$

We also need the following assumption.

(H4) There exists a number $r_0 \in (b_0, +\infty)$, and $\Gamma_1 \geq r_0/\Gamma_0(r_0 - b_0)$ such that

$$f(t,u) + M(t) \geq \Gamma_1 u, \qquad \forall (t,u) \in (0,1) \times [0, r_0]. \tag{3.2}$$

We are now in a position to state and prove our main results on the existence.

**Theorem 3.1.** *Suppose that (H1)–(H4) hold. If $f^\infty = 0$, then BVP* (1.4) *has a positive solution.*

*Proof.* By Lemma 2.4 together with (H3), we have $\|B\| \leq D(<1)$. By operator spectrum theorem, we know that $(I-B)^{-1}$ exists and is bounded. Furthermore, by Neumann expression, $(I-B)^{-1}$can be expressed by

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n. \tag{3.3}$$

Noticing that $BP \subset P$ and from (3.3), we have

$$(I - B)^{-1} u = \sum_{n=0}^{\infty} B^n u \geq u, \qquad \forall u \in P, \tag{3.4}$$

$$\left\| (I - B)^{-1} \right\| \leq \sum_{n=0}^{\infty} \|B\|^n = \frac{1}{1 - \|B\|} \leq \frac{1}{1 - D}. \tag{3.5}$$

Thus, from the reversibility of $I - B$, we have

$$u = KGu + Bu, \quad u \in X \Longleftrightarrow u = (I - B)^{-1} KGu, \qquad u \in X. \tag{3.6}$$

The following proof will be divided into five steps.

*Step 1.* We will show that $(I - B)^{-1} KG : P \to P$ is completely continuous.

(1) $KG$ maps $P$ into $P$.

For any $u \in P$, it follows from (H1) that $f(t,[u-w]^*(t)) \in X$, and so $(Gu)(t) \geq 0, t \in (0,1), Gu \in C(0,1) \cap L^1(0,1)$. By (H1)-(H2) together with Lemma 2.2, for any $t \in [0,1]$, we have

$$\begin{aligned} 0 \leq (KGu)(t) &= \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau)\left[f\left(\tau,[u-w]^*(\tau)\right) + M(\tau)\right) d\tau\, ds \\ &\leq \rho\eta + \rho \int_0^1 M(\tau) d\tau < +\infty, \end{aligned} \tag{3.7}$$

where $\eta = \max_{t\in[0,1]} |f(t,[u-w]^*(t))| < \infty, \rho = \max_{t,s,\tau\in[0,1]} G_1(t,s)G_2(s,\tau) < \infty$.

From the continuity of $G_1(t,s)$, it is easy to see that $KGu \in X$, and so $KGu \in P$.

(2) $KG$ is a compact operator on $P$.

Assume that $U$ is a arbitrary bounded set in $P$. Then there exists a $L_0 > 0$ such that $\|u\| \leq L_0$ for all $u \in U$. Also, we have $\|[u-w]^*\| \leq L_0$ for all $u \in U$ since $w \in P$. Consequently,

$$0 \leq (KGu)(t) \leq \rho b + \rho \int_0^1 M(\tau) d\tau < +\infty, \qquad \forall w \in P, \tag{3.8}$$

where $b = \max_{(t,u)\in I\times[0,L_0]} |f(t,u)|,\ \rho = \max_{t,s,\tau\in[0,1]} G_1(t,s)G_2(s,\tau)$. That means $\{KGu \mid u \in U\}$ is a uniformly bounded set in $P$.

On the other hand, the continuity of $G_1$ on $I \times I$ yields that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta$, the following inequality

$$|G_1(t_2,s) - G_1(t_1,s)| < \varepsilon \tag{3.9}$$

holds for all $s \in I$, and so,

$$\begin{aligned} |(KGu)(t_2) - (KGu)(t_1)| &\leq \varepsilon \int_0^1 \int_0^1 G_2(s,\tau)(Gu)(\tau) d\tau ds \\ &\leq \varepsilon \left[ b e_2 + e_2 \int_0^1 M(\tau) d\tau \right], \end{aligned} \tag{3.10}$$

for any $u \in U$, where $e_2 = \max_{s,\tau\in[0,1]} G_2(s,\tau) < +\infty, b = \max_{(t,u)\in I\times[0,L_0]} |f(t,u)| < +\infty$. That is, $\{KGu \mid u \in U\}$is equicontinuous.

Hence, in view of Arzela-Ascoli theorem, we know that the operator $KG$ is compact on $P$.

(3) Now, we show that the operator $KG$ is continuous.

Indeed, for any sequence $\{u_n\}$ in $P$ with $u_n \to u$ and any $t \in I$, we have

$$\begin{aligned}
\left|[u_n - w]^*(t) - [u - w]^*(t)\right| &= \frac{1}{2}|[|u_n(t) - w(t)| + (u_n(t) - w(t))] - [|u(t) - w(t)| + u(t) - w(t)]| \\
&= \frac{1}{2}|[|u_n(t) - w(t)| - |u(t) - w(t)| + u_n(t) - u(t)]| \\
&\le \frac{1}{2}\{|[|u_n(t) - w(t)| - |u(t) - w(t)|]| + |u_n(t) - u(t)|\} \\
&\le |u_n(t) - u(t)|.
\end{aligned} \tag{3.11}$$

Thus, $\|[u_n - w]^* - [u - w]^*\| \to 0$, and, by Lemma 2.2, it follows from the continuity of $f$ that

$$\begin{aligned}
&\|KGu_n - KGu\| \\
&\quad \le C_1 \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) \left| f(\tau, [u_n - w]^*(\tau)) - f(\tau, [u - w]^*(\tau)) \right| d\tau\, ds \longrightarrow 0.
\end{aligned} \tag{3.12}$$

By (1)–(3) we obtain that $KG : P \to P$ is completely continuous.

Now, from (3.4), we have $(I - B)^{-1} : P \to P$ is continuous, and so, $(I - B)^{-1}KG : P \to P$ is completely continuous.

Now we set

$$Q = (I - B)^{-1}KG, \qquad q_1(t) = \frac{\delta_1}{C_1} G_1(t,t), \quad t \in I, \tag{3.13}$$

where $\delta_1, C_1$ are described in Lemma 2.2. Set

$$P_0 = \left\{u \in P : u(t) \ge (1 - \|B\|) q_1(t) \|u\|, t \in I\right\}. \tag{3.14}$$

Obviously, $P_0$ is a cone in $X$.

*Step 2.* $Q : P \to P_0$.

In fact, for any $u \in P$ and every $t, \sigma$ in $I$, by Lemma 2.2, we have

$$\begin{aligned}
(KGu)(t) &= \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau)(Gu)(\tau) d\tau\, ds \\
&\ge \delta_1 G_1(t,t) \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau)(Gu)(\tau) d\tau\, ds \\
&\ge \frac{\delta_1}{C_1} G_1(t,t) \int_0^1 \int_0^1 G_1(\sigma,s) G_2(s,\tau)(Gu)(\tau) d\tau\, ds \\
&= q_1(t)(KGu)(\sigma).
\end{aligned} \tag{3.15}$$

Thus, we have

$$(KGu)(t) \geq q_1(t)\|KGu\|, \quad t \in I. \tag{3.16}$$

Since $KGP \subset P$, by (3.4) together with (3.16) for every $t \in I$, we have

$$\begin{aligned}\left((I-B)^{-1}KGu\right)(t) &\geq (KGu)(t)\\ &\geq q_1(t)\|KGu\|.\end{aligned} \tag{3.17}$$

On the other hand, since $\|(I-B)^{-1}(KGu)\| \leq \|(I-B)^{-1}\| \cdot \|KGu\|$, by (3.5), we have

$$\begin{aligned}\|KGu\| &\geq \frac{1}{\left\|(I-B)^{-1}\right\|}\left\|(I-B)^{-1}KGu\right\|\\ &\geq (1-\|B\|)\left\|(I-B)^{-1}KGu\right\|.\end{aligned} \tag{3.18}$$

Inequality (3.17) together with (3.18) implies for every $t \in I$

$$\left((I-B)^{-1}KGu\right)(t) \geq (1-\|B\|)q_1(t)\left\|(I-B)^{-1}KGu\right\|, \tag{3.19}$$

namely, $(Qu)(t) \geq (1-\|B\|)q_1(t)\|Qu\|, t \in I$. Thus, we obtain that $Q$ maps $P$ into $P_0$.

*Step 3.* We shall deduce that for any $u \in P_0$ and $t \in I$, the following inequality holds:

$$u(t) - w(t) \geq \left(1 - \frac{b_0}{\|u\|}\right)u(t), \tag{3.20}$$

where $b_0 = C_1^2 d_1/\delta_1(1-D)$.

In fact, in view of Lemma 2.2 and the symmetry of $G_1(t,s)$, we have

$$G_1(t,s) = G_1(s,t) \leq C_1 G_1(t,t), \quad \forall t, s \in I. \tag{3.21}$$

Thus, keeping in mind that $d_1 = \int_0^1\int_0^1 G_2(s,\tau)M(\tau)d\tau\, ds$, it follows from $w = KM$ that

$$\begin{aligned}w(t) &= \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)M(\tau)d\tau\, ds\\ &\leq C_1 G_1(t,t)\int_0^1\int_0^1 G_2(s,\tau)M(\tau)d\tau\, ds\\ &= \frac{C_1^2}{\delta_1}d_1 q_1(t), \quad t \in I.\end{aligned} \tag{3.22}$$

On the other hand, from $u \in P_0$, it follows that

$$u(t) \geq q_1(t)(1 - \|B\|)\|u\| \geq q_1(t)(1 - D)\|u\|, \quad t \in I. \tag{3.23}$$

Thus, by (3.22)-(3.23), we have

$$w(t) \leq \frac{C_1^2 d_1}{\delta_1(1 - D)\|u\|}, \tag{3.24}$$

and so,

$$u(t) - w(t) \geq \left(1 - \frac{C_1^2 d_1}{\delta_1(1 - D)\|u\|}\right)u(t) = \left(1 - \frac{b_0}{\|u\|}\right)u(t), \quad t \in I, \tag{3.25}$$

where $b_0 = C_1^2 d_1/(\delta_1(1 - D))$.

*Step 4.* By (H4), we have

$$f(t, u) + M(t) \geq \Gamma_1 u, \quad (t, u) \in (0, 1) \times [0, r_0]. \tag{3.26}$$

Let $\phi_0 = \sin \pi t$. By (2.7)-(2.8), we easily know that $\sin \pi t$ is a positive eigenfunction of operator $K$ with respect to positive eigenvalue $\Gamma_0$, that is, $K\phi_0 = \Gamma_0 \phi_0$.

Now, we show that $\phi_0 \in P_0$, that is, $\phi_0(t) \geq (1 - \|B\|)q_1(t)\|\phi_0\|, t \in I$. We discuss it in three different cases.

(1) $\lambda_1 = 0$. In this case, $G_1(t, t) = t(1 - t), t \in I$, and $C_1 = \delta_1 = 1$.

(i) If $t \in [0, 1/2]$, then $\pi t \in [0, \pi/2]$. By Jordan's inequality, we have

$$\sin \pi t \geq \frac{2}{\pi} \cdot \pi t = 2t, \quad t \in \left[0, \frac{1}{2}\right]. \tag{3.27}$$

(ii) If $t \in [1/2, 1]$, by setting $x = 1 - t$, we have $x \in [0, 1/2]$. Then from (3. 12), it follows that

$$\sin \pi t = \sin \pi(1 - x) = \sin \pi x \geq 2x = 2(1 - t), \quad t \in \left[\frac{1}{2}, 1\right]. \tag{3.28}$$

Thus, by (i)-(ii) above, we have

$$\sin \pi t \geq 2t(1 - t) = 2G_1(t, t) = 2q_1(t), \quad t \in I. \tag{3.29}$$

(2) $\lambda_1 > 0$. In this case, $G_1(t, t) = (\sinh \omega_1 t \cdot \sinh \omega_1(1 - t))/\omega_1 \sinh \omega_1, t \in I$, and $C_1 = 1, \delta_1 = \omega_1/\sinh \omega_1$.

(i) If $t \in [0,1/2]$, by setting $\varphi(t) = (\cosh(\omega_1/2))t - (\sinh \omega_1 t/\omega_1), t \in [0,1/2]$, we have

$$\varphi'(t) = \cosh\frac{\omega_1}{2} - \cosh\omega_1 t \geq \cosh\frac{\omega_1}{2} - \cosh\frac{\omega_1}{2} = 0, \quad t \in \left[0, \frac{1}{2}\right]. \tag{3.30}$$

From $\phi(0) = 0$, it follows that $0 \leq \sinh\omega_1 t/\omega_1 \leq (\cosh(\omega_1/2))t, t \in [0,1/2]$. Keeping in mind that $0 < \sinh\omega_1(1-t)/\sinh\omega_1 \leq 1$ for all $t \in [0,1/2]$, it follows immediately that

$$G_1(t,t) \leq \left(\cosh\frac{\omega_1}{2}\right)t, \quad t \in \left[0, \frac{1}{2}\right]. \tag{3.31}$$

(ii) If $t \in [1/2,1]$, by setting $x = 1-t$, we have $x \in [0,1/2]$. From (2)(i) above, it follows that

$$G_1(t,t) = G_1(1-x,1-x) = G_1(x,x) \leq (\cosh(\omega_1/2))x = (\cosh(\omega_1/2))(1-t), \quad t \in \left[\frac{1}{2}, 1\right]. \tag{3.32}$$

Hence, by (2)(i)-(ii) above, we have

$$G(t,t) \leq \cosh\frac{1}{\omega_1} \cdot \begin{cases} t, & t \in \left[0, \dfrac{1}{2}\right], \\ 1-t, & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{3.33}$$

On the other hand, by (3.27)-(3.28), we have

$$\sin\pi t \geq 2 \cdot \begin{cases} t, & t \in \left[0, \dfrac{1}{2}\right], \\ 1-t, & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{3.34}$$

Thus, we have immediately

$$\sin\pi t \geq \frac{2}{\cosh(\omega_1/2)} G_1(t,t) = \frac{2C_1}{\delta_1\cosh(\omega_1/2)} q_1(t) = \frac{2\sinh\omega_1}{\omega_1\cosh(\omega_1/2)} q_1(t) = \frac{2\sinh(\omega_1/2)}{\omega_1/2} q_1(t), \quad t \in I. \tag{3.35}$$

It is easy to verity that $\sinh\ (\omega_1 2)/\omega_1/2 \geq 1$. Hence, $\sin\pi t \geq 2q_1(t), t \in I$.

(3) $-\pi^2 < \lambda_1 < 0$. In this case, $G_1(t,t) = \sin\omega_1 t \cdot \sin\omega_1(1-t)/\omega_1\sin\omega_1, t \in I$, and $C_1 = 1/\sin\omega_1$, $\delta_1 = \omega_1\sin\omega_1$.

(i) If $t \in [0, 1/2]$, then $0 \le \sin\omega_1 t \le \sin\pi t, 0 < \sin\omega_1(1-t) \le 1$. Thus,

$$G_1(t,t) \le \frac{\sin\pi t}{\omega_1 \sin\omega_1}, \qquad t \in \left[0, \frac{1}{2}\right]. \tag{3.36}$$

(ii) If $t \in [1/2, 1]$, from (i), by letting $x = 1-t$, then we have $x \in [0, 1/2]$, and

$$\begin{aligned} G_1(t,t) &= G_1(1-x, 1-x) = G_1(x,x) \\ &\le \frac{\sin\pi x}{\omega_1 \sin\omega_1} = \frac{\sin\pi(1-t)}{\omega_1 \sin\omega_1} = \frac{\sin\pi t}{\omega_1 \sin\omega_1}, \quad t \in \left[\frac{1}{2}, 1\right]. \end{aligned} \tag{3.37}$$

Thus, (3)(i)-(ii) above implies that

$$\sin\pi t \ge \omega_1 \sin\omega_1 G_1(t,t) = \omega_1 \sin\omega_1 \frac{C_1}{\delta_1} q_1(t) = \frac{1}{\sin\omega_1} q_1(t) \ge q_1(t), \quad t \in I. \tag{3.38}$$

Summing up (1)–(3) keeping in mind that $\|\phi_0\| = 1$, we have

$$\phi_0(\mathrm{t}) = \sin\pi t \ge q_1(t) = q_1(t)\|\phi_0\| \ge (1 - \|B\|) q_1(t) \|\phi_0\|, \quad t \in I, \tag{3.39}$$

that is, $\phi \in P_0$.

Now, set $\Omega_{r_0} = \{u \in P_0 : \|u_0\| < r_0\}$. We claim that

$$u \ne Qu + \lambda\phi_0, \quad \forall\lambda \ge 0,\ u \in \partial\Omega_{r_0}. \tag{3.40}$$

Indeed, if not, then exists a $u_0 \in \partial\Omega_{r_0}$ and $\lambda_0 \ge 0$ with $u_0 = Qu_0 + \lambda_0\phi_0$. Without loss of generality, assume that $\lambda_0 > 0$ (otherwise, by proving later on, we will know that the theorem is true). By $u_0 \in \partial\Omega_{r_0}$, we have $\|u_0\| = r_0$, and so, it follows from (3.25) that

$$r_0 \ge u_0(t) \ge u_0(t) - w(t) \ge \left(1 - \frac{b_0}{r_0}\right) u_0(t) \ge 0, \quad t \in I, \tag{3.41}$$

since $r_0 > b_0$.

Thus, by (3.26) and (3.41), we have

$$\begin{aligned} f\left(t, [u_0 - w]^*(t)\right) + M(t) &= f(t, (u_0 - w)(t)) + M(t) \\ &\ge \Gamma_1 \left(1 - \frac{b_0}{r_0}\right) u_0(t), \qquad t \in (0,1). \end{aligned} \tag{3.42}$$

Therefore, by (3.4), (3.16), we have

$$\begin{aligned} u_0 &= Qu_0 + \lambda_0\phi_0 = (I-B)^{-1}KGu_0 + \lambda_0\phi_0 \\ &\geq KGu_0 + \lambda_0\phi_0 \\ &= \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\left[f\left(\tau,[u-w]^*(\tau)\right)+M(\tau)\right]d\tau\, ds + \lambda_0\phi_0 \\ &\geq \Gamma_1\left(1-\frac{b_0}{r_0}\right)\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)u_0(\tau)d\tau\, ds + \lambda_0\phi_0 \\ &= \Gamma_1\left(1-\frac{b_0}{r_0}\right)Ku_0 + \lambda_0\phi_0. \end{aligned} \tag{3.43}$$

Thus, $u_0 \geq \lambda_0\phi_0$. Let $\lambda^* = \sup\{\lambda \mid u_0 \geq \lambda\phi_0\}$. Then $\lambda^* \geq \lambda_0$, and $u_0 \geq \lambda^*\phi_0$. By $KP \subset P$ and $K\phi_0 = \Gamma_0\phi_0$, it follows that

$$Ku_0 \geq \lambda^* K\phi_0 = \lambda^*\Gamma_0\phi_0. \tag{3.44}$$

Thus, by (3.43), we have

$$u_0 \geq \Gamma_1\left(1-\frac{b_0}{r_0}\right)\Gamma_0\lambda^*\phi_0 + \lambda_0\phi. \tag{3.45}$$

The hypothesis in (H4) yields $\Gamma_1(1-b_0/r_0)\Gamma_0 \geq 1$, and so $u_0 \geq (\lambda^* + \lambda_0)\phi_0$, which contradicts to the definition of $\lambda^*$( noticing that $\lambda_0 > 0$). This shows that (3.40) fulfils. Therefore, in terms of the fixed point index theorem on cone ([25]), we have

$$i(Q,\Omega_{r_0},P_0) = 0. \tag{3.46}$$

*Step 5.* Let $d_2 = \max_{t\in[0,1]}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)d\tau\, ds, \Gamma_2 = (1-D)/d_2$. By hypothesis $f^\infty = 0$, we have $f^\infty < (\Gamma_2 - \varepsilon_0)$ for a fixed $\varepsilon_0 \in (0,\Gamma_2)$, and so, there exists $R_1 > 0$ such that

$$f(t,u) < (\Gamma_2 - \varepsilon_0)u, \qquad t \in I \tag{3.47}$$

holds when $u \geq R_1$.

Let $C = \max\{|f(t,u)| : (t,u) \in I\times[0,R_1]\}$. Then

$$f(t,u) \leq (\Gamma_2-\varepsilon_0)u + C, (t,u) \in I\times\mathbf{R}_+. \tag{3.48}$$

Let $d_3 = \max_{t\in[0,1]}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)M(\tau)d\tau\, ds$, let $E_0 = (Cd_2 + d_3)/\varepsilon_0 d_2$. Take $R_0 > max\{r_0, E_0\}$. Set $\Omega_{R_0} = \{u \in P_0 : \|u\| < R_0\}$. We shall show that

$$\lambda u \neq Qu, \qquad u \in \partial\Omega_{R_0}, \qquad \lambda \geq 1. \tag{3.49}$$

Suppose on the contradiction that there exist $u_0 \in \partial\Omega_{R_0}$ and $\lambda_0 \geq 1$ with $\lambda_0 u_0 = Qu_0$. Then $\|u_0\| = R_0 > r_0 > b_0$. By (3.48), we have

$$\begin{aligned} f(t, [u_0 - w]^*(t)) &\leq (\Gamma_2 - \varepsilon_0)[u_0 - w]^*(t) + C \\ &\leq (\Gamma_2 - \varepsilon_0)u_0(t) + C, \qquad t \in I. \end{aligned} \tag{3.50}$$

Hence,

$$\begin{aligned} (KGu_0)(t) &= \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)\big(f(\tau, [u_0 - w]^*(\tau)) + M(\tau)\big)d\tau\, ds \\ &\leq \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)((\Gamma_2 - \varepsilon_0)u_0(\tau) + C + M(\tau))d\tau\, ds \\ &\leq (\Gamma_2 - \varepsilon_0)d_2\|u_0\| + Cd_2 + d_3, \qquad t \in I. \end{aligned} \tag{3.51}$$

So, $\|KGu_0\| \leq (\Gamma_2 - \varepsilon_0)d_2\|u_0\| + Cd_2 + d_3$. Thus, from (3.5) and $\Gamma_2 = (1 - D)/d_2$, it follows that

$$\|u_0\| \leq \|\lambda_0 u_0\| = \left\|(I - B)^{-1}KGu_0\right\| \leq \frac{1}{1 - D}\|KGu_0\| \leq \frac{d_2(\Gamma_2 - \varepsilon_0)}{1 - D}\|u_0\| + \frac{Cd_2 + d_3}{1 - D}. \tag{3.52}$$

Then, $R = \|u_0\| \leq E_0$, which contradicts to the choice of $R_0$. Hence, (3.49) holds. Therefore, the fixed point index theorem ([25]) implies

$$i(Q, \Omega_{R_0}, P_0) = 1. \tag{3.53}$$

By (3.46)–(3.53), applying additivity of fixed point index [25], we have

$$i\left(Q, \Omega_{R_0} \setminus \overline{\Omega}_{r_0}, P_0\right) = i(Q, \Omega_{R_0}, P_0) - i(Q, \Omega_{r_0}, P_0) = 1. \tag{3.54}$$

Therefore, $Q$ has a fixed point $\overline{u} \in \Omega_{R_0} \setminus \overline{\Omega}_{r_0}$. Hence, $\overline{v} = \overline{u} - w$ is a solution of BVP (1.4).

Now, from $\overline{u} \notin \overline{\Omega}_{r_0}$, we have $\|\overline{u}\| > r_0 (> b_0)$, and so, (3.20) together with the fact that $\overline{u} \in P_0$ gives

$$\overline{v}(t) = \overline{u}(t) - w(t) \geq \left(1 - \frac{b_0}{\|\overline{u}\|}\right)\overline{u}(t) \geq \left(1 - \frac{b_0}{r_0}\right)\overline{u}(t) \geq \left(1 - \frac{b_0}{r_0}\right)(1 - \|B\|)q_1(t)r_0, \quad t \in I. \tag{3.55}$$

Thus, $\overline{v}(t) \geq 0, t \in [0,1]$. Moreover, $\overline{v}(t) > 0, t \in (0,1)$ from $q_1(t) > 0, t \in (0,1)$. That means that $\overline{v}$ is a positive solution of BVP (1.4). The proof is completed. □

**Corollary 3.2.** *Let* $(H2), (H3)$ *hold. Assume that* $f \in C[I \times \mathbf{R}_+, \mathbf{R}_+]$. *If* $f_0 > 1/\Gamma_0$ , $f^\infty = 0$, *then BVP* (1.4) *has a positive solution.*

*Proof.* Let us take $M(t) = 0$ in Theorem 3.1. Then $d_1 = 0$, and so $b_0 = 0$. By $f_0 > 1/\Gamma_0$, we can take a $\Gamma_1 \in (1/\Gamma_0, \infty)$ such that $f_0 > \Gamma_1$. Then there exists a $r_0 \in (0, +\infty)$ such that

$$f(t, u) \geq \Gamma_1 u, (t, u) \in I \times [0, r_0]. \tag{3.56}$$

Hence, all hypotheses in Theorem 3.1 are satisfied, and the conclusion of Corollary 3.2 follows. This completes the proof. □

*Remark 3.3.* Even in the case that $M(t) = 0$, the conclusion of Corollary 3.2 is still new.

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