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Research Article

Strong Convergence Theorems for a Countable Family of Nonexpansive Mappings in Convex Metric Spaces

Withun Phuengrattana[1, 2] and Suthep Suantai[1, 2]

[1] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[2] *Materials Science Research Center, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Correspondence should be addressed to Suthep Suantai, scmti005@chiangmai.ac.th

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We introduce a new modified Halpern iteration for a countable infinite family of nonexpansive mappings $\{T_n\}$ in convex metric spaces. We prove that the sequence $\{x_n\}$ generated by the proposed iteration is an approximating fixed point sequence of a nonexpansive mapping when $\{T_n\}$ satisfies the AKTT-condition, and strong convergence theorems of the proposed iteration to a common fixed point of a countable infinite family of nonexpansive mappings in CAT(0) spaces are established under AKTT-condition and the SZ-condition. We also generalize the concept of W-mapping for a countable infinite family of nonexpansive mappings from a Banach space setting to a convex metric space and give some properties concerning the common fixed point set of this family in convex metric spaces. Moreover, by using the concept of W-mappings, we give an example of a sequence of nonexpansive mappings defined on a convex metric space which satisfies the AKTT-condition. Our results generalize and refine many known results in the current literature.

1. Introduction

Let C be a nonempty closed convex subset of a metric space (X, d), and let T be a mapping of C into itself. A mapping T is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

In 1967, Halpern [1] introduced the following iterative scheme in Hilbert spaces which was referred to as *Halpern iteration* for approximating a fixed point of T:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where $x_1, u \in C$ are arbitrarily chosen, and $\{\alpha_n\}$ is a sequence in $[0,1]$. Wittmann [2] studied the iterative scheme (1.1) in a Hilbert space and obtained the strong convergence of the iteration. Reich [3] and Shioji and Takahashi [4] extended Wittmann's result to a real Banach space.

The modified version of Halpern iteration was investigated widely by many mathematicians. For instance, Kim and Xu [5] studied the sequence $\{x_n\}$ generated as follows:

$$y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n \quad \forall n \in \mathbb{N}, \tag{1.2}$$

where $x_1, u \in C$ are arbitrarily chosen and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$. They proved the strong convergence of iterative scheme (1.2) in the framework of a uniformly smooth Banach space. In 2007, Aoyama et al. [6] introduced a Halpern iteration for finding a common fixed point of a countable infinite family of nonexpansive mappings in a Banach space as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_n x_n \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $x_1, u \in C$ are arbitrarily chosen, $\{\alpha_n\}$ is a sequence in $[0,1]$, and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a common fixed point of $\{T_n\}$. In 2010, Saejung [7] extended the results of Halpern [1], Wittmann [2], Reich [3], Shioji and Takahashi [4], and Aoyama et al. [6] to the case of a CAT(0) space which is an example of a convex metric space. Recently, Cuntavepanit and Panyanak [8] extended the result of Kim and Xu [5] to a CAT(0) space.

Takahashi [9] introduced the concept of convex metric spaces by using the convex structure as follows. Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0,1]$,

$$d(z, W(x,y,\lambda)) \le \lambda d(z,x) + (1-\lambda)d(z,y), \tag{1.4}$$

for all $z \in X$. A metric space (X,d) together with a convex structure W is called a *convex metric space* which will be denoted by (X,d,W). A nonempty subset C of X is said to be *convex* if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold.

Motivated by the above results, we introduce a new iterative scheme for finding a common fixed point of a countable infinite family of nonexpansive mappings $\{T_n\}$ of C into itself in a convex metric space as follows:

$$y_n = W(u, T_n x_n, \alpha_n), \\ x_{n+1} = W(y_n, T_n y_n, \beta_n) \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where $x_1, u \in C$ are arbitrarily chosen, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$. The main propose of this paper is to prove the convergence theorem of the sequence $\{x_n\}$ generated by (1.5) to a common fixed point of a countable infinite family of nonexpansive mappings in convex metric spaces and CAT(0) spaces under certain suitable conditions.

2. Preliminaries

We recall some definitions and useful lemmas used in the main results.

Lemma 2.1 (see [9, 10]). *Let* (X, d, W) *be a convex metric space. For each* $x, y \in X$ *and* $\lambda, \lambda_1, \lambda_2 \in [0,1]$*, we have the following.*

(i) $W(x, x, \lambda) = x, W(x, y, 0) = y$ *and* $W(x, y, 1) = x$.

(ii) $d(x, W(x, y, \lambda)) = (1-\lambda)d(x, y)$ *and* $d(y, W(x, y, \lambda)) = \lambda d(x, y)$.

(iii) $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$.

(iv) $|\lambda_1 - \lambda_2| d(x, y) \leq d(W(x, y, \lambda_1), W(x, y, \lambda_2))$.

We say that a convex metric space (X, d, W) has the property:

(C) if $W(x, y, \lambda) = W(y, x, 1-\lambda)$ for all $x, y \in X$ and $\lambda \in [0,1]$,

(I) if $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) \leq |\lambda_1 - \lambda_2| d(x, y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in [0,1]$,

(H) if $d(W(x, y, \lambda), W(x, z, \lambda)) \leq (1-\lambda)d(y, z)$ for all $x, y, z \in X$ and $\lambda \in [0,1]$,

(S) if $d(W(x, y, \lambda), W(z, w, \lambda)) \leq \lambda d(x, z) + (1-\lambda)d(y, w)$ for all $x, y, z, w \in X$ and $\lambda \in [0,1]$.

From the above properties, it is obvious that the property (C) and (H) imply continuity of a convex structure $W : X \times X \times [0,1] \to X$. Clearly, the property (S) implies the property (H). In [10], Aoyama et al. showed that a convex metric space with the property (C) and (H) has the property (S).

In 1996, Shimizu and Takahashi [11] introduced the concept of uniform convexity in convex metric spaces and studied some properties of these spaces. A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that $d(z, W(x, y, 1/2)) \leq (1-\delta)r$. Obviously, uniformly convex Banach spaces are uniformly convex metric spaces. In fact, the property (I) holds in uniformly convex metric spaces, see [12].

Lemma 2.2. *Property (C) holds in uniformly convex metric spaces.*

Proof. Suppose that (X, d, W) is a uniformly convex metric space. Let $x, y \in X$ and $\lambda \in [0,1]$. It is obvious that the conclusion holds if $\lambda = 0$ or $\lambda = 1$. So, suppose $\lambda \in (0,1)$. By Lemma 2.1(ii), we have

$$\begin{aligned} d(x, W(x, y, \lambda)) &= (1-\lambda)d(x, y), \qquad d(y, W(x, y, \lambda)) = \lambda d(x, y), \\ d(x, W(y, x, 1-\lambda)) &= (1-\lambda)d(x, y), \qquad d(y, W(y, x, 1-\lambda)) = \lambda d(x, y). \end{aligned} \tag{2.1}$$

We will show that $W(x, y, \lambda) = W(y, x, 1-\lambda)$. To show this, suppose not. Put $z_1 = W(x, y, \lambda)$ and $z_2 = W(y, x, 1-\lambda)$. Let $r_1 = (1-\lambda)d(x, y) > 0$, $r_2 = \lambda d(x, y) > 0$,

$\varepsilon_1 = d(z_1, z_2)/r_1$, and $\varepsilon_2 = d(z_1, z_2)/r_2$. It is easy to see that $\varepsilon_1, \varepsilon_2 > 0$. Since (X, d, W) is uniformly convex, we have

$$d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_1(1 - \delta(\varepsilon_1)), \qquad d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_2(1 - \delta(\varepsilon_2)). \tag{2.2}$$

By $\lambda \in (0,1)$, we get $x \neq y$. Since $\delta(\varepsilon_1) > 0$ and $\delta(\varepsilon_2) > 0$, then

$$\begin{aligned} d(x,y) &\leq d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) + d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq r_1(1 - \delta(\varepsilon_1)) + r_2(1 - \delta(\varepsilon_2)) \\ &< r_1 + r_2 \\ &= d(x,y). \end{aligned} \tag{2.3}$$

This is a contradiction. Hence, $W(x, y, \lambda) = W(y, x, 1 - \lambda)$. □

By Lemma 2.2, it is clear that a uniformly convex metric space (X, d, W) with the property (H) has the property (S), and the convex structure W is also continuous.

Next, we recall the special space of convex metric spaces, namely, CAT(0) spaces. Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When unique, this geodesic is denoted $[x, y]$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset Y of X is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$.

If z, x, y are points in a CAT(0) space and if m is the midpoint of the segment $[x, y]$, then the CAT(0) inequality implies

$$d(z, m)^2 \leq \frac{1}{2} d(z, x)^2 + \frac{1}{2} d(z, y)^2 - \frac{1}{4} d(x, y)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [13], which is equivalent to

$$d(z, \lambda x \oplus (1 - \lambda) y)^2 \leq \lambda d(z, x)^2 + (1 - \lambda) d(z, y)^2 - \lambda(1 - \lambda) d(x, y)^2, \tag{CN*}$$

for any $\lambda \in [0,1]$, where $\lambda x \oplus (1-\lambda)y$ denotes the unique point in $[x,y]$. The (CN*) inequality has appeared in [14]. By using the (CN) inequality, it is easy to see that the CAT(0) spaces are uniformly convex. In fact [15], a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. Moreover, if X is CAT(0) space and $x, y \in X$, then for any $\lambda \in [0,1]$, there exists a unique point $\lambda x \oplus (1-\lambda)y \in [x,y]$ such that

$$d(z, \lambda x \oplus (1-\lambda)y) \leq \lambda d(z,x) + (1-\lambda)d(z,y), \tag{2.4}$$

for any $z \in X$. It follows that CAT(0) spaces have convex structure $W(x,y,\lambda) = \lambda x \oplus (1-\lambda)y$. It is clear that the properties (C), (I), and (S) are satisfied for CAT(0) spaces, see [15, 16]. This is also true for Banach spaces.

Let μ be a continuous linear functional on l^∞, the Banach space of bounded real sequences, and let $(a_1, a_2, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_1, a_2, \ldots))$. We call μ a *Banach limit* if μ satisfies $\|\mu\| = \mu(1,1,\ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for each $(a_1, a_2, \ldots) \in l^\infty$. For a Banach limit μ, we know that $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_1, a_2, \ldots) \in l^\infty$. So if $(a_1, a_2, \ldots) \in l^\infty$ with $\lim_{n\to\infty} a_n = c$, then $\mu_n(a_n) = c$, see also [17].

Lemma 2.3 ([4], Proposition 2). *Let $(a_1, a_2, \ldots) \in l^\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 2.4 ([6], Lemma 2.3). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\delta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \delta_n < \infty$, and let $\{\gamma_n\}$be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \delta_n \quad \forall n \in \mathbb{N}. \tag{2.5}$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([18], Lemma 1). *Let (X,d,W) be a uniformly convex metric space with a continuous convex structure $W : X \times X \times [0,1] \to X$. Then for arbitrary positive number ε and r, there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$d(z, W(x,y,\lambda)) \leq r(1 - 2\min\{\lambda, 1-\lambda\}\eta), \tag{2.6}$$

for all $x, y, z \in X$, $d(z,x) \leq r$, $d(z,y) \leq r$, $d(x,y) \geq r\varepsilon$, and $\lambda \in [0,1]$.

Remark 2.6. The above lemma also holds for a uniformly convex metric space with the property (H).

3. Main Results

The following condition was introduced by Aoyama et al. [6]. Let C be a subset of a complete convex metric space (X,d,W), and let $\{T_n\}$ be a countable infinite family of mappings from

C into itself. We say that $\{T_n\}$ satisfies *AKTT-condition* if

$$\sum_{n=1}^{\infty} \sup\{d(T_{n+1}z, T_n z) : z \in B\} < \infty, \tag{3.1}$$

for each bounded subset B of C. If C is a closed subset and $\{T_n\}$ satisfies AKTT-condition, then we can define a mapping $T : C \to C$ such that $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. In this case, we also say that $(\{T_n\}, T)$ satisfies AKTT-condition. By using the same argument as in [6, Lemma 3.2], we have the following lemma.

Lemma 3.1. *If $(\{T_n\}, T)$ satisfies AKTT-condition, then $\lim_{n\to\infty} \sup\{d(Tz, T_n z) : z \in B\} = 0$ for all bounded subsets B of C.*

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete convex metric space (X, d, W) with the properties (I) and (S). Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence of C generated by (1.5), and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ which satisfy the conditions:*

(C1) $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2) $\beta_n \in (b, 1]$ *for some* $b \in (0,1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition. Then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(y_n, T_n y_n, \beta_n), p) \\ &\leq \beta_n d(y_n, p) + (1-\beta_n) d(T_n y_n, p) \\ &\leq d(y_n, p) \\ &= d(W(u, T_n x_n, \alpha_n), p) \\ &\leq \alpha_n d(u, p) + (1-\alpha_n) d(T_n x_n, p) \\ &\leq \alpha_n d(u, p) + (1-\alpha_n) d(x_n, p) \\ &\leq \max\{d(u, p), d(x_n, p)\}. \end{aligned} \tag{3.2}$$

By induction on n, we obtain that $d(x_n, p) \leq \max\{d(u, p), d(x_1, p)\}$ for all $n \in \mathbb{N}$ and all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{T_n x_n\}$, $\{T_n y_n\}$ are bounded.

It follows by condition (C1) that

$$d(y_n, T_n x_n) = d(W(u, T_n x_n, \alpha_n), T_n x_n) = \alpha_n d(u, T_n x_n) \longrightarrow 0. \tag{3.3}$$

By the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$
\begin{aligned}
d(y_n, y_{n-1}) &= d(W(u, T_n x_n, \alpha_n), W(u, T_{n-1} x_{n-1}, \alpha_{n-1})) \\
&\leq d(W(u, T_n x_n, \alpha_n), W(u, T_n x_{n-1}, \alpha_n)) \\
&\quad + d(W(u, T_n x_{n-1}, \alpha_n), W(u, T_{n-1} x_{n-1}, \alpha_n)) \\
&\quad + d(W(u, T_{n-1} x_{n-1}, \alpha_n), W(u, T_{n-1} x_{n-1}, \alpha_{n-1})) \\
&\leq (1-\alpha_n) d(T_n x_n, T_n x_{n-1}) + (1-\alpha_n) d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + (1-\alpha_n) d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}), \\
d(x_{n+1}, x_n) &= d(W(y_n, T_n y_n, \beta_n), W(y_{n-1}, T_{n-1} y_{n-1}, \beta_{n-1})) \\
&\leq d(W(y_n, T_n y_n, \beta_n), W(y_{n-1}, T_{n-1} y_{n-1}, \beta_n)) \\
&\quad + d(W(y_{n-1}, T_{n-1} y_{n-1}, \beta_n), W(y_{n-1}, T_{n-1} y_{n-1}, \beta_{n-1})) \\
&\leq \beta_n d(y_n, y_{n-1}) + (1-\beta_n) d(T_n y_n, T_{n-1} y_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\leq \beta_n d(y_n, y_{n-1}) + (1-\beta_n)(d(T_n y_n, T_n y_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1})) \\
&\quad + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\leq \beta_n d(y_n, y_{n-1}) + (1-\beta_n)(d(y_n, y_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1})) \\
&\quad + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\leq d(y_n, y_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) M \\
&\quad + d(T_n x_{n-1}, T_{n-1} x_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}),
\end{aligned}
\tag{3.4}
$$

where $M = \max\{\sup_n d(u, T_{n-1} x_{n-1}), \sup_n d(y_{n-1}, T_{n-1} y_{n-1})\}$.

Putting $\delta_n = (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M + d(T_n x_{n-1}, T_{n-1}x_{n-1}) + d(T_n y_{n-1}, T_{n-1}y_{n-1})$, we have

$$\begin{aligned}\sum_{n=2}^{\infty}\delta_n &\leq M\sum_{n=2}^{\infty}\left(|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right) + \sum_{n=2}^{\infty}\sup\{d(T_n z, T_{n-1}z) : z \in \{x_k\}\} \\ &\quad + \sum_{n=2}^{\infty}\sup\{d(T_n z, T_{n-1}z) : z \in \{y_k\}\}.\end{aligned} \tag{3.5}$$

Hence, it follows from conditions $(C1)$, $(C2)$, AKTT-condition, and Lemma 2.4 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.6}$$

Now, observe that

$$\begin{aligned}d(x_{n+1}, y_n) &= d(W(y_n, T_n y_n, \beta_n), y_n) \\ &= (1-\beta_n)d(y_n, T_n y_n) \\ &\leq (1-b)\left(d(y_n, T_n x_n) + d(T_n x_n, T_n x_{n+1}) + d(T_n x_{n+1}, T_n y_n)\right) \\ &\leq (1-b)\left(d(y_n, T_n x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_n)\right).\end{aligned} \tag{3.7}$$

We obtain

$$d(x_{n+1}, y_n) \leq \frac{1-b}{b}\left(d(y_n, T_n x_n) + d(x_n, x_{n+1})\right). \tag{3.8}$$

This implies by (3.3) and (3.6) that $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$. Therefore, we have

$$d(x_n, y_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) \longrightarrow 0. \tag{3.9}$$

Since

$$d(T_n x_n, x_n) \leq d(T_n x_n, y_n) + d(y_n, x_n), \tag{3.10}$$

it follows by (3.3) and (3.9) that

$$\lim_{n\to\infty} d(T_n x_n, x_n) = 0. \tag{3.11}$$

By (3.11) and Lemma 3.1, we get

$$\begin{aligned}d(Tx_n, x_n) &\leq d(Tx_n, T_n x_n) + d(T_n x_n, x_n) \\ &\leq \sup\{d(Tz, T_n z) : z \in \{x_k\}\} + d(T_n x_n, x_n) \longrightarrow 0.\end{aligned} \tag{3.12}$$

□

Next, we consider a convergence theorem in CAT(0) spaces. The following two lemmas obtained by Saejung [7] are useful for our main results.

Lemma 3.3. *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0,1)$, the mapping $S_t : C \to C$ defined by $S_t x = tu \oplus (1-t)Tx$ for $x \in C$ has a unique fixed point $x_t \in C$, that is, $x_t = S_t x_t = tu \oplus (1-t)Tx_t$.*

Lemma 3.4. *Let C, T be as the preceding lemma. Then $F(T) \neq \emptyset$ if and only if $\{x_t\}$ remains bounded as $t \to 0$. In this case, the following statements hold:*

(i) *$\{x_t\}$ converges to the unique fixed point z of T which is nearest to u;*

(ii) *$d(u,z)^2 \leq \mu_n d(u,x_n)^2$ for all Banach limit μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Previously, we know that CAT(0) spaces have convex structure $W(x,y,\lambda) = \lambda x \oplus (1-\lambda)y$ and also have the properties (C), (I), and (S). Thus, we have the following result.

Theorem 3.5. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$\begin{aligned} y_n &= \alpha_n u \oplus (1-\alpha_n) T_n x_n, \\ x_{n+1} &= \beta_n y_n \oplus (1-\beta_n) T_n y_n \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.13}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ which satisfy the conditions (C1) and (C2) as in Theorem 3.2. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition. Then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Theorem 3.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence of C generated by (3.13), and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ which satisfy the conditions (C1) and (C2) as in Theorem 3.2. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$ which is nearest to u.*

Proof. By Theorem 3.5, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. For each $t \in (0,1)$, let z_t be a unique point of C such that $z_t = tu \oplus (1-t)Tz_t$. It follows from Lemma 3.4 that $\{z_t\}$ converges to a point $z \in F(T)$ which is nearest to u, and

$$d(u,z)^2 \leq \mu_n d(u,x_n)^2 \quad \text{for all Banach limits } \mu, \tag{3.14}$$

that is, $\mu_n(d(u,z)^2 - d(u,x_n)^2) \leq 0$. Moreover, by Theorem 3.5, we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. It follows that

$$\limsup_{n\to\infty} \left(\left(d(u,z)^2 - d(u,x_{n+1})^2 \right) - \left(d(u,z)^2 - d(u,x_n)^2 \right) \right) = 0. \tag{3.15}$$

By $\lim_{n\to\infty} d(T_n x_n, x_n) = 0$ and Lemma 2.3, we obtain

$$\limsup_{n\to\infty}\Big(d(u,z)^2 - (1-\alpha_n)d(u,T_nx_n)^2\Big) = \limsup_{n\to\infty}\Big(d(u,z)^2 - d(u,x_n)^2\Big) \le 0. \tag{3.16}$$

Finally, we show that $\lim_{n\to\infty} d(x_n, z) = 0$. By the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned}
d(x_{n+1},z)^2 &= d(\beta_n y_n \oplus (1-\beta_n)T_ny_n, z)^2 \\
&\le (\beta_n d(y_n,z) + (1-\beta_n)d(T_ny_n,z))^2 \\
&\le d(y_n,z)^2 = d(\alpha_n u \oplus (1-\alpha_n)T_nx_n, z)^2 \\
&\le \alpha_n d(u,z)^2 + (1-\alpha_n)d(T_nx_n,z)^2 - \alpha_n(1-\alpha_n)d(u,T_nx_n)^2 \\
&\le \alpha_n d(u,z)^2 + (1-\alpha_n)d(x_n,z)^2 - \alpha_n(1-\alpha_n)d(u,T_nx_n)^2 \\
&= (1-\alpha_n)d(x_n,z)^2 + \alpha_n\Big(d(u,z)^2 - (1-\alpha_n)d(u,T_nx_n)^2\Big).
\end{aligned} \tag{3.17}$$

This implies by $\sum_{n=1}^{\infty}\alpha_n = \infty$, inequality (3.16), and Lemma 2.4 that $\lim_{n\to\infty} d(x_n,z)^2 = 0$. Hence, $\{x_n\}$ converges to $z \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ which is nearest to u. □

Corollary 3.7 (see [7], Theorem 8). *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T_nx_n \quad \forall n \in \mathbb{N}, \tag{3.18}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ which satisfies the condition (C1) as in Theorem 3.2. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$ which is nearest to u.

Proof. By putting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.6, we obtain the desired result. □

In 2009, Song and Zheng [19] introduced a condition in Banach spaces for a countable infinite family of nonexpansive mappings which is different from AKTT-condition and also give some examples of a family of mappings that satisfies this condition. Now, we state this condition in CAT(0) spaces, and it is referred as SZ-condition as follows. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Suppose that $\{T_n\}$ is a family of nonexpansive mappings from C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. We say that $\{T_n\}$ satisfies *SZ-condition* if, for any bounded subset K of C, there exists a nonexpansive mapping T of C into itself such that

$$\limsup_{n\to\infty}\{d(T(T_nx), T_nx) : x \in K\} = 0, \qquad F(T) = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.19}$$

Theorem 3.8. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and satisfies SZ-condition.*

Suppose that $\{x_n\}$ *is a sequence of* C *defined by* (3.13) *with* $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$ *which satisfy the following conditions:*

(C3) $0 < \alpha_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C4) $\lim_{n\to\infty}\beta_n = 1$.

Then $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_n\}$ *which is nearest to* u.

Proof. As in the proof of Theorem 3.2, we have that $\{x_n\}$ and $\{T_n x_n\}$ are bounded. Since $\{T_n\}$ satisfies SZ-condition, there exists a nonexpansive mapping T of C into itself such that $\lim_{n\to\infty}\sup\{d(T(T_n x), T_n x) : x \in \{x_k\}\} = 0$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned} d(x_{n+1}, T_n x_n) &= d(\beta_n y_n \oplus (1-\beta_n) T_n y_n, T_n x_n) \\ &\le \beta_n d(y_n, T_n x_n) + (1-\beta_n) d(T_n y_n, T_n x_n) \\ &\le \beta_n d(y_n, T_n x_n) + (1-\beta_n) d(y_n, x_n) \\ &= \beta_n d(\alpha_n u \oplus (1-\alpha_n) T_n x_n, T_n x_n) + (1-\beta_n) d(\alpha_n u \oplus (1-\alpha_n) T_n x_n, x_n) \\ &\le \beta_n \alpha_n d(u, T_n x_n) + (1-\beta_n)(\alpha_n d(u, x_n) + (1-\alpha_n) d(T_n x_n, x_n)). \end{aligned} \tag{3.20}$$

It follows from condition $(C3)$ and $(C4)$ that

$$\lim_{n\to\infty} d(x_{n+1}, T_n x_n) = 0. \tag{3.21}$$

Since

$$\begin{aligned} d(x_{n+1}, T x_{n+1}) &\le d(x_{n+1}, T_n x_n) + d(T_n x_n, T(T_n x_n)) + d(T(T_n x_n), T x_{n+1}) \\ &\le 2d(x_{n+1}, T_n x_n) + \sup\{d(T(T_n x), T_n x) : x \in \{x_k\}\}, \end{aligned} \tag{3.22}$$

this implies by (3.21) and SZ-condition, we have

$$\lim_{n\to\infty} d(x_n, T x_n) = 0. \tag{3.23}$$

From $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and

$$d(x_n, T_n x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n), \tag{3.24}$$

it follows that

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0. \tag{3.25}$$

By using the same arguments and techniques as those of Theorem 3.6, we can show that $\{x_n\}$ converges to a common fixed point of $\{T_n\}$ which is nearest to u. □

Corollary 3.9. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfies SZ-condition. Suppose that $\{x_n\}$ is a sequence of C defined by* (3.18) *with $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies the condition* $(C3)$ *as in Theorem 3.8. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$ which is nearest to u.*

Proof. By putting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.8, we obtain the desired result. □

4. W-Mapping in Convex Metric Spaces

In Theorems 3.2, 3.5, and 3.6 and Corollary 3.7, to obtain a convergence result, we have to assume that $(\{T_n\}, T)$ satisfies AKTT-condition. In general, one cannot apply these results for a sequence of nonexpansive mappings. However, we give an example of a sequence $\{T_n\}$ of nonexpansive mappings satisfying AKTT-condition.

Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself, where C is a convex subset of a convex metric space (X, d, W). We now define mappings $U_{n;1}, U_{n;2}, \ldots, U_{n;n}$ and S_n as follows. For $\{\lambda_n\}$ a sequence in $[0,1]$ and $x \in X$,

$$\begin{aligned} U_{n;n}x &= W(T_n x, x, \lambda_n), \\ U_{n;n-1}x &= W(T_{n-1}U_{n;n}x, x, \lambda_{n-1}), \\ U_{n;n-2}x &= W(T_{n-2}U_{n;n-1}x, x, \lambda_{n-2}), \\ &\vdots \\ U_{n;k}x &= W(T_k U_{n;k+1}x, x, \lambda_k), \\ U_{n;k-1}x &= W(T_{k-1}U_{n;k}x, x, \lambda_{k-1}), \\ &\vdots \\ U_{n;2}x &= W(T_2 U_{n;3}x, x, \lambda_2), \\ S_n x = U_{n;1}x &= W(T_1 U_{n;2}x, x, \lambda_1). \end{aligned} \tag{4.1}$$

Such a mapping S_n is called the *W-mapping* generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$.

In 2007, Shimizu [18] generalized W-mapping which was introduced by Takahashi [20] from Banach spaces to convex metric spaces. Then, the following result is obtained by using the same proof as in of [18, Lemma 2].

Lemma 4.1. *Let C be a nonempty closed convex subset of a uniformly convex metric space (X, d, W) with a continuous convex structure $W : X \times X \times [0,1] \to X$. Let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{N} F(T_n) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_n < 1$ for every $n = 1, 2, \ldots, N$. Let S_N be the W-mapping of C into itself generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $F(S_N) = \bigcap_{n=1}^{N} F(T_n)$.*

Next, we consider the W-mapping given by a countable infinite family of nonexpansive mappings in a uniformly convex metric space.

Lemma 4.2. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X,d,W) with the property (H). Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for every $n \in \mathbb{N}$. Then for every $x \in C$, and $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n;k}x$ exists.*

Proof. Let $x \in C$ and $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Fix $k \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ with $n > k$, we have

$$\begin{aligned}
d(U_{n+1;k}x, U_{n;k}x) &= d(W(T_kU_{n+1;k+1}x, x, \lambda_k), W(T_kU_{n;k+1}x, x, \lambda_k)) \\
&\leq \lambda_k d(T_kU_{n+1;k+1}x, T_kU_{n;k+1}x) \\
&\leq \lambda_k d(U_{n+1;k+1}x, U_{n;k+1}x) \\
&= \lambda_k d(W(T_{k+1}U_{n+1;k+2}x, x, \lambda_{k+1}), W(T_{k+1}U_{n;k+2}x, x, \lambda_{k+1})) \\
&\leq \lambda_k\lambda_{k+1} d(U_{n+1;k+2}x, U_{n;k+2}x) \\
&\;\;\vdots \\
&\leq \lambda_k\lambda_{k+1}\cdots\lambda_{n-1} d(U_{n+1;n}x, U_{n;n}x) \\
&= \lambda_k\lambda_{k+1}\cdots\lambda_{n-1} d(W(T_nU_{n+1;n+1}x, x, \lambda_n), W(T_nx, x, \lambda_n)) \\
&\leq \lambda_k\lambda_{k+1}\cdots\lambda_n d(T_nU_{n+1;n+1}x, T_nx) \\
&\leq \lambda_k\lambda_{k+1}\cdots\lambda_n d(U_{n+1;n+1}x, x) \\
&= \lambda_k\lambda_{k+1}\cdots\lambda_n d(W(T_{n+1}x, x, \lambda_{n+1}), x) \\
&= \lambda_k\lambda_{k+1}\cdots\lambda_{n+1} d(T_{n+1}x, x) \\
&\leq \lambda_k\lambda_{k+1}\cdots\lambda_{n+1}\left(d(T_{n+1}x, p) + d(p, x)\right) \\
&\leq 2d(p,x)b^{n-k+2}.
\end{aligned} \tag{4.2}$$

Thus for $m > n$,

$$\begin{aligned}
d(U_{m;k}x, U_{n;k}x) &\leq d(U_{m;k}x, U_{m-1;k}x) + d(U_{m-1;k}x, U_{m-2;k}x) + \cdots + d(U_{n+1;k}x, U_{n;k}x) \\
&\leq 2d(p,x)b^{(m-1)-k+2} + 2d(p,x)b^{(m-2)-k+2} + \cdots + 2d(p,x)b^{n-k+2} \\
&= 2d(p,x)\sum_{j=n}^{m-1} b^{j-k+2}.
\end{aligned} \tag{4.3}$$

It follows that $\{U_{n;k}x\}$ is a Cauchy sequence. Hence, $\lim_{n\to\infty} U_{n;k}x$ exists. □

Using the above lemma, one can define mappings $U_{\infty;k}$ and S of C into itself as

$$U_{\infty;k}x = \lim_{n\to\infty} U_{n;k}x, \qquad Sx = \lim_{n\to\infty} S_nx = \lim_{n\to\infty} U_{n;1}x, \tag{4.4}$$

for every $x \in C$. Such a mapping S is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 4.3. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for every $n \in \mathbb{N}$. Let S be the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Then, S is a nonexpansive mapping and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Proof. First, we show that S is a nonexpansive mapping. For $x, y \in C$, we have

$$\begin{aligned}
d(S_n x, S_n y) &= d(W(T_1 U_{n;2} x, x, \lambda_1), W(T_1 U_{n;2} y, y, \lambda_1)) \\
&\leq \lambda_1 d(T_1 U_{n;2} x, T_1 U_{n;2} y) + (1 - \lambda_1) d(x, y) \\
&\leq \lambda_1 d(U_{n;2} x, U_{n;2} y) + (1 - \lambda_1) d(x, y) \\
&\ \ \vdots \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_{n-1} d(U_{n;n} x, U_{n;n} y) + (1 - \lambda_1 \lambda_2 \cdots \lambda_{n-1}) d(x, y) \\
&= \lambda_1 \lambda_2 \cdots \lambda_{n-1} d(W(T_n x, x, \lambda_n), W(T_n y, y, \lambda_n)) + (1 - \lambda_1 \lambda_2 \cdots \lambda_{n-1}) d(x, y) \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_{n-1} \lambda_n d(T_n x, T_n y) + \lambda_1 \lambda_2 \cdots \lambda_{n-1} (1 - \lambda_n) d(x, y) \\
&\quad + (1 - \lambda_1 \lambda_2 \cdots \lambda_{n-1}) d(x, y) \\
&\leq d(x, y).
\end{aligned} \tag{4.5}$$

This implies that S_n is a nonexpansive mapping, and we have $d(Sx, Sy) = \lim_{n \to \infty} d(S_n x, S_n y) \leq d(x, y)$. Thus, S is also a nonexpansive mapping.

Finally, we show that $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then, it is obvious that $U_{n;k} p = p$ for all $n, k \in \mathbb{N}$ with $n > k$. So we have $U_{\infty;k} p = p$ for all $k \in \mathbb{N}$. Therefore, we have $Sp = U_{\infty;1} p = p$, and hence, $\bigcap_{n=1}^{\infty} F(T_n) \subseteq F(S)$. We now show that $F(S) \subseteq \bigcap_{n=1}^{\infty} F(T_n)$. Let $x \in F(S)$ and let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then we have

$$\begin{aligned}
d(S_n p, S_n x) &= d(U_{n;1} p, U_{n;1} x) \\
&= d(p, W(T_1 U_{n;2} x, x, \lambda_1)) \\
&\leq \lambda_1 d(p, T_1 U_{n;2} x) + (1 - \lambda_1) d(p, x) \\
&\leq \lambda_1 d(p, U_{n;2} x) + (1 - \lambda_1) d(p, x) \\
&\ \ \vdots \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_{k-1} d(p, U_{n;k} x) + (1 - \lambda_1 \lambda_2 \cdots \lambda_{k-1}) d(p, x) \\
&= \lambda_1 \lambda_2 \cdots \lambda_{k-1} d(p, W(T_k U_{n;k+1} x, x, \lambda_k)) + (1 - \lambda_1 \lambda_2 \cdots \lambda_{k-1}) d(p, x) \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_{k-1} \lambda_k d(p, T_k U_{n;k+1} x) + \lambda_1 \lambda_2 \cdots \lambda_{k-1} (1 - \lambda_k) d(p, x)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\lambda_1\lambda_2\cdots\lambda_{k-1})d(p,x)\\
&= \lambda_1\lambda_2\cdots\lambda_k d(p,T_kU_{n;k+1}x) + (1-\lambda_1\lambda_2\cdots\lambda_k)d(p,x)\\
&\le \lambda_1\lambda_2\cdots\lambda_k d(p,U_{n;k+1}x) + (1-\lambda_1\lambda_2\cdots\lambda_k)d(p,x)\\
&\ \vdots\\
&\le \lambda_1\lambda_2\cdots\lambda_{n-1} d(p,U_{n;n}x) + (1-\lambda_1\lambda_2\cdots\lambda_{n-1})d(p,x)\\
&= \lambda_1\lambda_2\cdots\lambda_{n-1} d(p,W(T_nx,x,\lambda_n)) + (1-\lambda_1\lambda_2\cdots\lambda_{n-1})d(p,x)\\
&\le \lambda_1\lambda_2\cdots\lambda_{n-1}\lambda_n d(p,T_nx) + \lambda_1\lambda_2\cdots\lambda_{n-1}(1-\lambda_n)d(p,x)\\
&\quad + (1-\lambda_1\lambda_2\cdots\lambda_{n-1})d(p,x)\\
&= \lambda_1\lambda_2\cdots\lambda_n d(p,T_nx) + (1-\lambda_1\lambda_2\cdots\lambda_n)d(p,x)\\
&\le d(p,x).
\end{aligned}
\tag{4.6}
$$

Taking $n \to \infty$, we obtain

$$
\begin{aligned}
d(Sp,Sx) &\le \lambda_1\lambda_2\cdots\lambda_{k-1} d(p,W(T_kU_{\infty;k+1}x,x,\lambda_k)) + (1-\lambda_1\lambda_2\cdots\lambda_{k-1})d(p,x)\\
&\le \lambda_1\lambda_2\cdots\lambda_{k-1}\lambda_k d(p,T_kU_{\infty;k+1}x) + \lambda_1\lambda_2\cdots\lambda_{k-1}(1-\lambda_k)d(p,x)\\
&\quad + (1-\lambda_1\lambda_2\cdots\lambda_{k-1})d(p,x)\\
&= \lambda_1\lambda_2\cdots\lambda_k d(p,T_kU_{\infty;k+1}x) + (1-\lambda_1\lambda_2\cdots\lambda_k)d(p,x)\\
&\le d(p,x).
\end{aligned}
\tag{4.7}
$$

Since $p \in \bigcap_{n=1}^{\infty} F(T_n) \subseteq F(S)$, we have $d(Sp,Sx) = d(p,x)$. Then, for $\lambda_n \in (0,1)$, $n \in \mathbb{N}$, we have

$$
d(p,T_kU_{\infty;k+1}x) = d(p,x), \qquad d(p,W(T_kU_{\infty;k+1}x,x,\lambda_k)) = d(p,x), \tag{4.8}
$$

for every $k \in \mathbb{N}$. Suppose that $T_kU_{\infty;k+1}x \neq x$. Then $d(T_kU_{\infty;k+1}x,x) > 0$. It follows by Lemma 2.5, we have

$$
d(p,W(T_kU_{\infty;k+1}x,x,\lambda_k)) < d(p,x). \tag{4.9}
$$

This is a contradiction. Hence, $T_kU_{\infty;k+1}x = x$. Since $U_{n;k+1}x = W(T_{k+1}U_{n;k+2}x,x,\lambda_{k+1})$, we have

$$
U_{\infty;k+1}x = \lim_{n\to\infty} U_{n;k+1}x = W(T_{k+1}U_{\infty;k+2}x,x,\lambda_{k+1}) = x. \tag{4.10}
$$

So, we have $x = T_kU_{\infty;k+1}x = T_kx$ for every $k \in \mathbb{N}$. This implies that $x \in \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, we have $F(S) \subseteq \bigcap_{n=1}^{\infty} F(T_n)$. □

Lemma 4.4. *Suppose that* $X, C, \{T_n\}, \{\lambda_n\}$ *are as in Lemma 4.3. Let* S_n *and* S *be the* W*-mappings generated by* $T_1, T_2, \ldots, T_n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$*, and* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$*, respectively. Then* $(\{S_n\}, S)$ *satisfies AKTT-condition, and* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.

Proof. Let B be a bounded subset of C and $x \in B$. For $p \in \bigcap_{n=1}^{\infty} F(T_n)$, we have

$$\begin{aligned} d(S_{n+1}x, S_n x) &= d(U_{n+1;1}x, U_{n;1}x) \\ &= d(W(T_1 U_{n+1;2}x, x, \lambda_1), W(T_1 U_{n;2}x, x, \lambda_1)) \\ &\le \lambda_1 d(T_1 U_{n+1;2}x, T_1 U_{n;2}x) \\ &\le \lambda_1 d(U_{n+1;2}x, U_{n;2}x) \\ &\vdots \\ &\le \lambda_1 \lambda_2 \cdots \lambda_{n-1} d(U_{n+1;n}x, U_{n;n}x) \\ &= \lambda_1 \lambda_2 \cdots \lambda_{n-1} d(W(T_n U_{n+1;n+1}x, x, \lambda_n), W(T_n x, x, \lambda_n)) \\ &\le \lambda_1 \lambda_2 \cdots \lambda_n d(U_{n+1;n+1}x, x) \\ &= \lambda_1 \lambda_2 \cdots \lambda_n d(W(T_{n+1}x, x, \lambda_{n+1}), x) \\ &\le \lambda_1 \lambda_2 \cdots \lambda_{n+1} d(T_{n+1}x, x) \\ &\le \lambda_1 \lambda_2 \cdots \lambda_{n+1} \left(d(T_{n+1}x, p) + d(p, x)\right) \\ &\le 2\lambda_1 \lambda_2 \cdots \lambda_{n+1} d(p, x) \\ &\le 2b^{n+1} d(p, x). \end{aligned} \tag{4.11}$$

This implies

$$\sum_{n=1}^{\infty} \sup\{d(S_{n+1}x, S_n x) : x \in B\} < \infty. \tag{4.12}$$

Thus, $(\{S_n\}, S)$ satisfies AKTT-condition. Moreover, from Lemmas 4.1–4.3, we obtain that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. □

Remark 4.5. Lemmas 4.2 and 4.3 were proved in Banach spaces by Shimoji and Takahashi [21], and Lemma 4.4 was proved in Banach spaces by Peng and Yao [22].

Remark 4.6. Suppose that $X, C, \{T_n\}, \{\lambda_n\}$ are as in Lemma 4.3. Let S_n and S be the W-mappings generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, respectively. By Lemma 4.4, we know that $(\{S_n\}, S)$ satisfies the AKTT-condition and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Therefore, in Theorems 3.2, 3.5, and 3.6 and Corollary 3.7, the mapping T_n can be also replaced by S_n without assuming the AKTT-condition and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.

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