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Research Article

Solving Nonlinear Partial Differential Equations by the sn-ns Method

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We present the application of the sn-ns method to solve nonlinear partial differential equations. We show that the well-known tanh-coth method is a particular case of the sn-ns method.

1. Introduction

The search of explicit solutions to nonlinear partial differential equations (NLPDEs) by using computational methods is one of the principal objectives in nonlinear science problems. Some powerful methods have been extensively used in the past decade to handle nonlinear PDEs. Some of them are the tanh-method [1], the tanh-coth method [2], the exp-function method [3], the projective Riccati equation method [4], and the Jacobi elliptic functions method. Practically, there is no unified method that could be used to handle all types of nonlinear problems.

The main purpose of this work consists in solving nonlinear polynomial PDE starting from the idea of the projective Riccati equations method. We derive exact solutions to the following equations: Duffing equation, cubic nonlinear Schrodinger equation, Klein-Gordon-Zakharov equations, quadratic Duffing equation, KdV equation, Gardner equation, Boussinesq equation, symmetric regular long wave equation, generalized shallow water wave equation, Klein-Gordon equation with quadratic nonlinearity, Fitzhugh-Nagumo-Huxley equation, and double sine-Gordon equation.

2. The Main Idea

In the search of the traveling wave solutions to nonlinear partial differential equation of the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{2.1}$$

the first step consists in considering the wave transformation

$$u(x,\ t) = v\big(\phi(\xi)\big), \quad \xi = x + \lambda t + \xi_0, \xi_0 = \text{arbitrary constant}, \tag{2.2}$$

for a suitable function $\phi = \phi(\xi)$, where λ is a constant. Usually, $\phi(\xi) = \xi$ (the identity function).

Using (2.2), (2.1) converts to an ordinary differential equation (ODE) with respect to (shortly, w.r.t.) the function $v(\xi)$

$$Q(v,\ v',\ v'', \ldots) = 0, \tag{2.3}$$

with Q being a *polynomial* with respect to variables v, v', v'',....

To find solutions to (2.3), we suppose that $v(\xi)$ can be expressed as

$$v(\xi) = H\big(f(\xi), g(\xi)\big), \tag{2.4}$$

where $H(f, g)$ is a *rational function* in the new variables $f = f(\xi), g = g(\xi)$, which satisfy the system

$$\begin{aligned} f'(\xi) &= r f(\xi) g(\xi), \\ g^2(\xi) &= S\big(f(\xi)\big), \end{aligned} \tag{2.5}$$

with $r \neq 0$ being some constant to be determined and $S(f)$ a rational function in the variable $f = f(\xi)$. We show that the system (2.5) may be solved exactly in certain cases. In fact, taking

$$f(\xi) = \varphi^N(\xi), \tag{2.6}$$

where $\varphi(\xi) \neq 0$ and $N \neq 0$, system (2.5) reduces to

$$\begin{aligned} \varphi'(\xi) &= \frac{r}{N}\varphi(\xi) g(\xi), \\ g^2(\xi) &= S\Big(\varphi^N(\xi)\Big). \end{aligned} \tag{2.7}$$

From (2.7) we get

$$\big(\varphi'(\xi)\big)^2 = \frac{r^2}{N^2}\varphi^2(\xi) S\Big(\varphi^N\Big). \tag{2.8}$$

Equation (2.8) is of elliptic type. Choosing $S(f)$ and N adequately, we may obtain distinct methods to solve nonlinear PDEs. More exactly, suppose we have solved (2.8). Then, in view of (2.5) and (2.6) functions f and g may be computed by formulae

$$f(\xi) = \varphi^N(\xi), \qquad g(\xi) = \frac{f'(\xi)}{rf'(\xi)} = \frac{N}{r}\frac{\varphi'(\xi)}{\varphi(\xi)}. \tag{2.9}$$

To solve (2.3) we try one of the following ansatz:

$$v(\xi) = a_0 + \sum_{j=1}^{n} f^{j-1}\left(a_j f + b_j g\right), \tag{2.10}$$

$$\begin{aligned} v(\xi) &= a_0 + \sum_{j=1}^{n} a_j f^j, \\ v(\xi) &= a_0 + \sum_{j=1}^{n}\left(a_j f^j + b_j f^{-j}\right). \end{aligned} \tag{2.11}$$

We substitute any of these ansatz into (2.3) and we obtain a polynomial equation either in the variables $f = f(\xi)$ and $g = g(\xi)$ or $f = f(\xi)$. We equate the coefficients of $f^i g^j (i, j = 0, 1, 2, 3, \ldots)$ to zero, and we obtain a system of polynomial equations in the variables a_j, b_i, $\lambda, \ldots$. Solving this system with the aid of a symbolic computational package such as Mathematica 8 or Maple 15, we obtain the desired solutions. Sometimes, we replace ξ with $k\xi$ and then the corresponding system is regarded w.r.t. the variables $f = f(k\xi)$ and $g = g(k\xi)$, where $k = \text{const}$.

We also may solve coupled systems of nonlinear equations. Indeed, suppose that we have a coupled system of two equations in the form

$$\begin{aligned} &P(u, \tilde{u}, u_x, \tilde{u}_x, u_t, \tilde{u}_t, u_{xx}, \tilde{u}_{xx}, u_{xt}, \tilde{u}_{xt}, u_{tt}, \tilde{u}_{tt}, \ldots) = 0, \\ &\tilde{P}(u, \tilde{u}, u_x, \tilde{u}_x, u_t, \tilde{u}_t, u_{xx}, \tilde{u}_{xx}, u_{xt}, \tilde{u}_{xt}, u_{tt}, \tilde{u}_{tt}, \ldots) = 0. \end{aligned} \tag{2.12}$$

We first apply the wave transformation

$$\begin{aligned} &u(x,t) = v(\phi(\xi)), \qquad \tilde{u}(x,t) = \tilde{v}\left(\tilde{\phi}(\xi)\right), \\ &\xi = x + \lambda t + \xi_0, \quad \xi_0 = \text{arbitrary constant}, \end{aligned} \tag{2.13}$$

for a suitable pair of functions $\phi = \phi(\xi)$ and $\tilde{\phi} = \tilde{\phi}(\xi)$ in order to obtain a coupled system of two ODEs in the form

$$\begin{aligned} &Q(v, \tilde{v}, v', \tilde{v}', v'', \tilde{v}'', v''', \tilde{v}''', \ldots) = 0, \\ &\tilde{Q}(v, \tilde{v}, v', \tilde{v}', v'', \tilde{v}'', v''', \tilde{v}''', \ldots) = 0, \end{aligned} \tag{2.14}$$

where Q and $\tilde{Q}$ are *polynomials* w.r.t. the variables $v, \tilde{v}, v', \tilde{v}', v'', \tilde{v}'', v''', \tilde{v}''', \ldots$.

We seek solutions to system (2.14) in the forms

$$v(\xi) = a_0 + \sum_{j=1}^{n} f^{j-1}\left(a_j f + b_j g\right),$$
$$\tilde{v}(\xi) = \tilde{a}_0 + \sum_{j=1}^{\tilde{n}} f^{j-1}\left(\tilde{a}_j f + \tilde{b}_j g\right), \tag{2.15}$$

$$v(\xi) = a_0 + \sum_{j=1}^{n} a_j f^j,$$
$$\tilde{v}(\xi) = \tilde{a}_0 + \sum_{j=1}^{\tilde{n}} \tilde{a}_i f^i, \tag{2.16}$$

$$v(\xi) = a_0 + \sum_{j=1}^{n} \left(a_j f^j + b_j f^{-j}\right),$$
$$\tilde{v}(\xi) = \tilde{a}_0 + \sum_{i=1}^{\tilde{n}} \left(\tilde{a}_i f^i + \tilde{b}_i f^{-i}\right). \tag{2.17}$$

The integers n and $\tilde{n}$ are determined by the balancing method.

We substitute any of ansatz (2.15), (2.16), or (2.17) into (2.14), and we obtain a system of two polynomial equations either w.r.t. the variables $f = f(\xi)$ and $g = g(\xi)$ or w.r.t. the variable $f = f(\xi)$. We equate the coefficients of $f^i g^j$ (resp., the coefficients of f^i) $(i, j = 0, 1, 2, 3, \ldots)$ to zero, and we obtain a system of polynomial equations w.r.t the variables a_i, b_j, $\tilde{a}_i$, $\tilde{b}_j$, $\lambda, \ldots$. Solving this system with the aid of a symbolic computational package such as Mathematica or Maple, we obtain the desired solutions. Sometimes, with the aim to add an additional parameter k, we replace ξ with $k\xi$ and then the corresponding system is regarded w.r.t. the variables $f = f(k\xi)$ and $g = g(k\xi)$, where $k = \text{const}$.

The same technique is applied for solving systems of three or more equations.

3. The sn-ns Method and Its Derivation

Let $N = -1$, $r = 1$ and $S(f) = af^{-2} + b + cf^2$, where

$$\Delta = b^2 - 4ac > 0. \tag{3.1}$$

This choice gives us (2.8) in the form

$$\left(\varphi'\right)^2 = a\varphi^4 + b\varphi^2 + c. \tag{3.2}$$

We may express the general solution of this equation in terms of the Jacobi elliptic functions ns or nd as follows:

$$\varphi(\xi) = \pm\sqrt{\frac{-b+\sqrt{\Delta}}{2a}}\,\mathrm{ns}(k(\xi + C) \mid m), \quad C = \text{arbitrary constant}, \tag{3.3}$$

where

$$k = \sqrt{\frac{-b+\sqrt{\Delta}}{2}}, \qquad m = \sqrt{\frac{b^2 - 2ac + b\sqrt{\Delta}}{2ac}}. \tag{3.4}$$

Solution (3.3) is valid for $a > 0$, $b < 0$, and $0 < c \le b^2/4a$.

On the other hand, function $\xi \to \sqrt{-1}\mathrm{ns}\ (\sqrt{-1}k(\xi + C) \mid m)$ is real valued for any real k, m, ξ, and C. We may verify that function

$$\phi(\xi) = \pm\sqrt{-1}\sqrt{\frac{-b+\sqrt{\Delta}}{2a}}\,\mathrm{ns}\left(\sqrt{-1}k(\xi + \xi_0) \mid m\right), \tag{3.5}$$

where k and m are given by (3.4), is a solution to equation

$$(\varphi')^2 = a\varphi^4 - b\varphi^2 + c, \qquad a > 0,\ b < 0,\ 0 < c \le \frac{b^2}{4a}. \tag{3.6}$$

Thus, we always may find a solution to (3.2) when $a > 0$ and $0 < c \le b^2/4a$ for any $b \neq 0$.

Now, let us assume that $a < 0$. It may be verified that a solution to (3.2) is

$$\varphi(\xi) = \pm\sqrt{\frac{-b-\sqrt{\Delta}}{2a}}\,\mathrm{nd}(k(\xi + C) \mid m), \quad C = \text{arbitrary constant}, \tag{3.7}$$

where

$$k = \sqrt{\frac{b+\sqrt{\Delta}}{2}}, \qquad m = \sqrt{-\frac{b^2 - 4ac + b\sqrt{\Delta}}{2ac}}, \quad \text{for } b \neq 0,\ c > 0. \tag{3.8}$$

Observe that for any $A > 0$ function $\xi \to \mathrm{nd}(\sqrt{-A}(\xi + C) | m)$ is real valued for any real values of m, ξ, and C. We conclude that (3.2) has exact solutions for any $a \neq 0$ and $b \neq 0$.

Projective equations are

$$\begin{aligned} f'(\xi) &= f(\xi)g(\xi), \\ g^2(\xi) &= \frac{a}{f^2(\xi)} + b + cf^2(\xi). \end{aligned} \tag{3.9}$$

Taking $C = 0$ in (3.3) we see that solutions to system (3.9) are

$$f(\xi) = \varphi^{-1}(\xi) = \frac{\sqrt{a}}{k}\mathrm{sn}(k\xi \mid m), \qquad g(\xi) = \frac{f'(\xi)}{f(\xi)} = k\mathrm{sn}(k\xi \mid m)\mathrm{cs}(k\xi \mid m)\mathrm{ds}(k\xi \mid m). \tag{3.10}$$

This motivates us to seek solutions to (2.3) in the form

$$v(\xi) = a_0 + \sum_{j=1}^{n}\left[a_j\mathrm{sn}^j(k\xi) + b_j\mathrm{ns}^j(k\xi)\right]. \tag{3.11}$$

Usually, $n = 1, 2$ and then $v(\xi)$ has the forms

$$v(\xi) = a_0 + a_1\mathrm{sn}(k\xi \mid m) + b_1\mathrm{ns}(k\xi \mid m), \tag{3.12}$$

$$v(\xi) = a_0 + a_1\mathrm{sn}(k\xi \mid m) + b_1\mathrm{ns}(k\xi \mid m) + a_2\mathrm{sn}^2(k\xi \mid m) + b_2\mathrm{ns}^2(k\xi \mid m). \tag{3.13}$$

In the case when $m = 1$, this gives the tanh-coth method since $\mathrm{sn}(k\xi \mid 1) = \tanh(k\xi)$ and $\mathrm{ns}(k\xi \mid 1) = \coth(k\xi)$.

Another possible ansatz suggested from (2.10) is

$$v(\xi) = a_0 + \sum_{j=1}^{n}\mathrm{sn}^{j-1}(k\xi)\left[a_j\mathrm{sn}(k\xi \mid m) + b_j\mathrm{sn}(k\xi \mid m)\mathrm{cs}(k\xi \mid m)\mathrm{ds}(k\xi \mid m)\right]. \tag{3.14}$$

If $n = 1$, 2, this ansatz reads

$$v(\xi) = a_0 + a_1\mathrm{sn}(k\xi \mid m) + b_1\mathrm{sn}(k\xi \mid m) + \mathrm{cs}(k\xi \mid m) + \mathrm{ds}(k\xi \mid m), \tag{3.15}$$

$$\begin{aligned} v(\xi) = a_0 &+ a_1\mathrm{sn}(k\xi \mid m) + b_1\mathrm{sn}(k\xi \mid m)\mathrm{cs}(k\xi \mid m)\mathrm{ds}(k\xi \mid m) \\ &+ a_2\mathrm{sn}^2(k\xi \mid m) + b_2\mathrm{sn}^2(k\xi \mid m)\mathrm{cs}(k\xi \mid m)\mathrm{ds}(k\xi \mid m). \end{aligned} \tag{3.16}$$

We may consider similar ansatz by replacing sn by dn and ns by nd, respectively. We will call this the dn-nd method. Thus, two possible ansatz for this method are

$$\begin{gathered} v(\xi) = a_0 + a_1\mathrm{dn}(k\xi \mid m) + b_1\mathrm{nd}(k\xi \mid m), \\ v(\xi) = a_0 + a_1\mathrm{dn}(k\xi \mid m) + b_1\mathrm{nd}(k\xi \mid m) + a_2\mathrm{dn}^2(k\xi \mid m) + b_2\mathrm{nd}^2(k\xi \mid m). \end{gathered} \tag{3.17}$$

An other useful ansatz to handle other equations of the form (2.3) is

$$v(\xi) = a_0 + a_1\mathrm{cn}(k\xi \mid m) + b_1\mathrm{nc}(k\xi \mid m). \tag{3.18}$$

We will call this the cn-nc method.

We also may try the following ansatz:

$$v(\xi) = \frac{a_0 + a_1 \mathrm{cn}(k\xi \mid m)}{1 + b_1 \mathrm{cn}(k\xi \mid m)}. \tag{3.19}$$

For example, this ansatz may be successfully applied to the cubic-quintic Duffing equation, which is defined by

$$v''(\xi) + pv(\xi) + qv^3(\xi) + rv^5(\xi) = 0. \tag{3.20}$$

4. Examples

In this section we solve various important models related to nonlinear science by the methods described in previous sections.

4.1. Duffing Equation $v''(\xi) + pv(\xi) + qv^3(\xi) = 0$

Let us consider the equation

$$v''(\xi) + pv(\xi) + qv^3(\xi) = 0, \tag{4.1}$$

where p and q are nonzero constants. This equation is very important since some relevant physical models described by a nonlinear PDEs may be studied once this equation is solved. Two of them are related to cubic nonlinear the Schrodinger equation and the Klein-Gordon-Zakharov equations.

To find solutions we multiply (4.1) by $v'(\xi)$ and we integrate it w.r.t. ξ. The resulting equation is

$$\left(\frac{dv}{d\xi}\right)^2 = \frac{q}{2}v^4(\xi) - pv^2(\xi) - 2C, \tag{4.2}$$

where C is the constant of integration. This equation has form (3.2) and we already know that there exists an exact solution to it for any p and q. Instead, we may apply directly the sn-ns method (resp., the dn-nd method) to it. Balancing gives $n = 1$. We seek solutions in the form

$$v(\xi) = a_0 + a_1 \mathrm{sn}(k\xi \mid m) + b_1 \mathrm{ns}(k\xi \mid m). \tag{4.3}$$

Inserting (4.3) into (4.1), we obtain a polynomial equation w.r.t. the variable $\zeta = \mathrm{sn}(k\xi \mid m)$. Equating to zero the coefficients of $\zeta^j\,(j = 0, 1, 2, \ldots)$ yields the following algebraic system:

$$\begin{gathered}
3a_0a_1^2q = 0, \\
a_1\left(a_1^2q + 2k^2m^2\right) = 0, \\
a_1b_1\left(3a_1b_1q + 3a_0^2q - k^2m^2 - k^2 + p\right) = 0, \\
a_0\left(6a_1b_1q + a_0^2q + p\right) = 0, \\
b_1\left(b_1^2q + 2k^2\right) = 0.
\end{gathered} \tag{4.4}$$

Solving system (4.4) gives solutions as follows:

(i) $a_0 = 0$, $a_1 = \sqrt{-2p}m/\sqrt{(m^2+6m+1)q}$, $b_1 = \sqrt{-2p}/\sqrt{(m^2+6m+1)q}$, $k = \sqrt{p}/\sqrt{m^2+6m+1}$,

$$v(\xi) = \pm\frac{\sqrt{-2p}}{\sqrt{(m^2+6m+1)q}}\left(m\,\mathrm{sn}\left(\frac{\sqrt{p}}{\sqrt{m^2+6m+1}}\xi \mid m\right) + \mathrm{ns}\left(\frac{\sqrt{p}}{\sqrt{m^2+6m+1}}\xi \mid m\right)\right), \tag{4.5}$$

(i) $a_0 = 0$, $a_1 = \sqrt{-2p}m/\sqrt{(m^2-6m+1)q}$, $b_1 = -\sqrt{-2p}/\sqrt{(m^2-6m+1)q}$, $k = \sqrt{p}/\sqrt{m^2-6m+1}$,

$$v(\xi) = \pm\frac{\sqrt{-2p}}{\sqrt{(m^2-6m+1)q}}\left(m\,\mathrm{sn}\left(\frac{\sqrt{p}}{m^2-6m+1}\xi \mid m\right) - \mathrm{ns}\left(\frac{\sqrt{p}}{\sqrt{m^2-6m+1}}\xi \mid m\right)\right), \tag{4.6}$$

(ii) $a_0 = 0$, $a_1 = m\sqrt{-2p}/\sqrt{q(m^2+1)}$, $b_1 = 0$, $k = \sqrt{p}/\sqrt{m^2+1}$,

$$v(\xi) = \pm m\sqrt{-\frac{2p}{q(m^2+1)}}\,\mathrm{sn}\left(\sqrt{\frac{p}{m^2+1}}\xi \mid m\right), \tag{4.7}$$

(iii) $a_0 = 0$, $a_1 = 0$, $b_1 = \sqrt{-2p}/\sqrt{q(m^2+1)}$, $k = \sqrt{p}/\sqrt{m^2+1}$,

$$v(\xi) = \pm\frac{\sqrt{-2p}}{\sqrt{q(m^2+1)}}\,\mathrm{ns}\left(\frac{\sqrt{p}}{\sqrt{m^2+1}}\xi \mid m\right). \tag{4.8}$$

Letting $m \to 1$, we obtain trigonometric and hyperbolic solutions:

$$v(\xi) = \sqrt{-\frac{p}{2q}}\tanh\left(\frac{\sqrt{p}}{2\sqrt{2}}\xi\right) + \sqrt{-\frac{p}{2q}}\coth\left(\frac{\sqrt{p}}{2\sqrt{2}}\xi\right),$$
$$v(\xi) = -\sqrt{-\frac{p}{2q}}\tan\left(\frac{\sqrt{p}}{2}\xi\right) - \sqrt{-\frac{p}{2q}}\cot\left(\frac{\sqrt{p}}{2}\xi\right) = \sqrt{-\frac{2p}{q}}\csc\left(\sqrt{p}\xi\right), \tag{4.9}$$
$$v(\xi) = \pm\sqrt{-\frac{p}{q}}\coth\left(\sqrt{\frac{p}{2}}\xi\right),$$

$$v(\xi) = \pm\sqrt{-\frac{p}{q}}\tanh\left(\sqrt{\frac{p}{2}}\xi\right). \tag{4.10}$$

4.2. Cubic Nonlinear Schrodinger Equation

This equation reads

$$iu_t + u_{xx} + \mu|u|^2 u = 0, \tag{4.11}$$

where $u = u(x,t)$ is a complex-valued function of two real variables x and t and μ is a nonzero real parameter and $i = \sqrt{-1}$. The physical model of the cubic nonlinear Schrodinger equation (shortly, NLS equation) (4.11) and its generalized variants occur in various areas of physics such as nonlinear optics, water waves, plasma physics, quantum mechanics, superconductivity, and the Bose-Einstein condensate theory. It also has applications in optics since it models many nonlinearity effects in a fiber, including but not limited to self-phase modulation, four-wave mixing, second harmonic generation, stimulated Raman scattering, and so forth. For water waves, the NLS equation (4.11) describes the evolution of the envelope of modulated nonlinear wave groups. All these physical phenomena can be better understood with the help of exact solutions for a given μ. When $\mu > 0$, the equation is said to be attractive. If $\mu < 0$ we say that it is repulsive. Recently, Ma and Chen [5] obtained some solutions to (4.11).

We seek solutions to (4.11) in the form

$$u(x,t) = v(\xi)\exp\left(i(\alpha x + \gamma t)\right), \quad \xi = x - 2\alpha t + \xi_0, \tag{4.12}$$

where α and γ are some real constants to be determined. Inserting (4.12) into (4.11) and simplifying, we obtain

$$v''(\xi) - \left(\gamma + \alpha^2\right)v(\xi) + \mu v^3(\xi) = 0. \tag{4.13}$$

This last equation has the form (4.1) with

$$p = -\left(\gamma + \alpha^2\right), \qquad q = \mu. \tag{4.14}$$

Thus, making use of solutions (4.5)–(4.10) for the choices given by (4.14) we obtain exact solutions to the Schrodinger equation (4.11) in the form (4.12).

4.3. *Quadratic Duffing Equation* $v''(\xi) = pv^2(\xi) + qv(\xi) + r$

Let us consider the following second-order and second-degree nonlinear ODE:

$$v''(\xi) = pv^2(\xi) + qv(\xi) + r, \tag{4.15}$$

where p, q, and r are constants and $p \neq 0$. Solutions to this equation may be used to study some important physical models whose associated PDEs may be solved after making the traveling wave transformation (2.2). As we will show in next subsections some examples of nonlinear partial differential equations where this equation arises are the following.

(i) KdV equation: $u_t + 6uu_x + u_{xxx} = 0$.

(ii) Gardner equation (also called combined KdV-mKdV equation): $u_t + \alpha uu_x + \beta u^2 u_x + \gamma u_{xxx} = 0$, $\beta\gamma \neq 0$,

(iii) Boussinesq equation: $u_{tt} + \alpha uu_{xx} + \alpha u_x^2 + \beta u_{xxxx} = 0$,

(iv) symmetric regular long wave equation: $u_{tt} + u_{xx} + uu_{xt} + u_x u_t + u_{xxtt} = 0$,

(v) generalized shallow water wave equation: $u_{xxtt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0$,

(vi) Klein-Gordon equation with quadratic nonlinearity: $u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^2 = 0$.

Balancing gives $n = 2$. We seek solutions to (4.15) in the form (3.13), that is,

$$v(\xi) = a_0 + a_1 \operatorname{sn}(k\xi \mid m) + b_1 \operatorname{ns}(k\xi \mid m) + a_2 \operatorname{sn}^2(k\xi \mid m) + b_2 \operatorname{ns}^2(k\xi \mid m). \tag{4.16}$$

Inserting this ansatz into (4.15) gives the following algebraic system:

$$
\begin{gathered}
a_1\left(k^2m^2 - a_2p\right) = 0,\\
a_2\left(6k^2m^2 - a_2p\right) = 0,\\
4a_2k^2m^2 + 4a_2k^2 + a_1^2p + 2a_0a_2p + a_2q = 0,\\
2a_2b_1p + a_1k^2m^2 + a_1k^2 + 2a_0a_1p + a_1q = 0,\\
b_1\left(k^2 - b_2p\right) = 0,\\
b_2\left(6k^2 - b_2p\right) = 0,\\
2a_0b_2p + 4b_2k^2m^2 + 4b_2k^2 + b_1^2p + b_2q = 0,\\
2a_0b_1p + 2a_1b_2p + b_1k^2m^2 + b_1k^2 + b_1q = 0,\\
2a_1b_1p + 2a_2b_2p - 2a_2k^2 + a_0^2p + a_0q - 2b_2k^2m^2 + r = 0.
\end{gathered} \tag{4.17}
$$

Solving this system, we obtain the following solutions:

(i) $k = (1/2)\sqrt[4]{(q^2 - 4pr)/(m^4 - m^2 + 1)}$, $a_0 = q/2p - (1/2p)(m^2 + 1)\sqrt{(q^2 - 4pr)/(m^4 - m^2 + 1)}$, $a_1 = 0$, $a_2 = (3m^2/2p)\sqrt{(q^2 - 4pr)/(m^4 - m^2 + 1)}$, $b_1 = 0$, $b_2 = 0$,

$$
v(\xi) = a_0 + \frac{3m^2}{2p}\sqrt{\frac{q^2 - 4pr}{m^4 - m^2 + 1}}\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2 - 4pr}{m^4 - m^2 + 1}}\,\xi \mid m\right), \tag{4.18}
$$

(ii) $k = (1/2)\sqrt[4]{(q^2 - 4pr)/(m^4 - m^2 + 1)}$, $a_0 = -q/2p - (1/2p)(m^2 + 1)\sqrt{(q^2 - 4pr)/(m^4 - m^2 + 1)}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = (3/2p)\sqrt{(q^2 - 4pr)/(m^4 - m^2 + 1)}$,

$$
v(\xi) = a_0 + \frac{3}{2p}\sqrt{\frac{q^2 - 4pr}{m^4 - m^2 + 1}}\,\mathrm{ns}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2 - 4pr}{m^4 - m^2 + 1}}\,\xi \mid m\right), \tag{4.19}
$$

(iii) $k = (1/2)\sqrt[4]{(q^2-4pr)/(m^4+14m^2+1)}$, $a_0 = -q/2p - (1/2p)(m^2+1)\sqrt{(q^2-4pr)/(m^4-m^2+1)}$, $a_1 = 0$, $a_2 = (3m^2/2p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, b_1=0,$b_2 = (3/2p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$,

$$v(\xi) = a_0 + \frac{3}{2p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\left(3m^2\mathrm{sn}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right) + \mathrm{ns}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right)\right), \tag{4.20}$$

(iv) $k = (\sqrt{-1}/2)\sqrt[4]{(q^2-4pr)/(m^4-m^2+1)}$, $a_0 = (1/2p)[(m^2+1)\sqrt{(q^2-4pr)/(m^4-m^2+1)} - q]$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = (-3/2p)\sqrt{(q^2-4pr)/(m^4-m^2+1)}$,

$$v(\xi) = a_0 - \frac{3}{2p}\sqrt{\frac{q^2-4pr}{m^4-m^2+1}}\mathrm{ns}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2-4pr}{m^4-m^2+1}}\xi \mid m\right), \tag{4.21}$$

(v) $k = (\sqrt{-1}/2)\sqrt[4]{(q^2-4pr)/(m^4-m^2+1)}$, $a_0 = q/2p - (1/2p)(m^2+1)\sqrt{(q^2-4pr)/(m^4-m^2+1)}$, $a_1 = 0$, a_2=$-(3m^2/2p)\sqrt{(q^2-4pr)/(m^4-m^2+1)}$, $b_1 = 0$, $b_2 = 0$,

$$v(\xi) = a_0 - \frac{3m^2}{2p}\sqrt{\frac{q^2-4pr}{m^4-m^2+1}}\mathrm{sn}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2-4pr}{m^4-m^2+1}}\xi \mid m\right), \tag{4.22}$$

(vi) $k = (\sqrt{-1}/2)\sqrt[4]{(q^2-4pr)/(m^4+14m^2+1)}$, $a_0 = q/2p - (1/2p)(m^2+1)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $a_1 = 0$, $a_2 = -(3m^2/2p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $b_1 = 0$, $b_2 = -(3/2p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$,

$$v(\xi) = a_0 - \frac{3}{2p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\left(3m^2\mathrm{sn}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right) + \mathrm{ns}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right)\right), \tag{4.23}$$

Letting $m \to 1$, we obtain trigonometric and hyperbolic solutions:

$$v(\xi) = a_0 + \frac{3}{2p}\sqrt{q^2-4pr}\tanh^2\left(\frac{1}{2}\sqrt[4]{q^2-4pr}\xi\right).$$

$$v(\xi) = a_0 + \frac{3}{2p}\sqrt{q^2-4pr}\coth^2\left(\frac{1}{2}\sqrt[4]{q^2-4pr}\xi\right). \tag{4.24}$$

$$v(\xi) = a_0 + \frac{3}{4p}\sqrt{q^2-4pr}\left(3\tanh^2\left(\frac{1}{4}\sqrt[4]{q^2-4pr}\xi\right) + \coth^2\left(\frac{1}{4}\sqrt[4]{q^2-4pr}\xi\right)\right).$$

Now, let us seek solutions in the ansatz form (3.16), that is,

$$\begin{aligned} v(\xi) = {} & a_0 + a_1\operatorname{sn}(k\xi \mid m) + b_1\operatorname{ns}(k\xi \mid m)\operatorname{cn}(k\xi \mid m)\operatorname{dn}(k\xi \mid m) \\ & + a_2\operatorname{sn}^2(k\xi \mid m) + b_2\operatorname{cn}(k\xi \mid m)\operatorname{dn}(k\xi \mid m). \end{aligned} \tag{4.25}$$

Inserting this ansatz into (4.15) and solving the resulting algebraic system yields the following solutions to (4.15):

(i) $k = \sqrt{-1}\sqrt[4]{(q^2-4pr)/(m^4+14m^2+1)}$, $a_0 = -q/2p + (1/2p)(m^2+1)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $a_1 = 0$, $a_2 = -(3m^2/p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $b_1 = 0$, $b_2 = \pm(3m/p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$,

$$\begin{aligned} v(\xi) = {} & -\frac{q}{2p} + \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}} \\ & - \frac{3m^2}{p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\operatorname{sn}^2\left(\sqrt{-1}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right) \\ & \pm \frac{3m}{p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\operatorname{cn}\left(\sqrt{-1}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right) \\ & \times \operatorname{dn}\left(\sqrt{-1}\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right), \end{aligned} \tag{4.26}$$

(ii) $k = \sqrt[4]{(q^2-4pr)/(m^4+14m^2+1)}$, $a_0 = -q/2p - (1/2p)(m^2+1)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $a_1 = 0$, $a_2 = (3m^2/p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$, $b_1 = 0$, $b_2 = \pm(3m^2/p)\sqrt{(q^2-4pr)/(m^4+14m^2+1)}$,

$$\begin{aligned}
v(\xi) = &-\frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}} \\
&- \frac{3m^2}{p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\operatorname{sn}^2\left(\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right) \\
&\pm \frac{3m}{p}\sqrt{\frac{q^2-4pr}{m^4+14m^2+1}}\operatorname{cn}\left(\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right)\operatorname{dn}\left(\sqrt[4]{\frac{q^2-4pr}{m^4+14m^2+1}}\xi \mid m\right).
\end{aligned} \tag{4.27}$$

We may obtain other solutions by making use of the dn-nd method.

4.4. KdV Equation

This is the equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4.28}$$

If $u = v(\xi)$, $\xi = x + \lambda t + \xi_0$, this equation takes the form

$$\lambda v'(\xi) + 6v(\xi)v'(\xi) + v'''(\xi) = 0. \tag{4.29}$$

Integrating this equation w.r.t. ξ yields

$$v''(\xi) = -3v^2(\xi) + \lambda v(\xi) + C, \tag{4.30}$$

where C is the constant of integration. Equation (4.30) has the form (4.15) with

$$p = -3, \qquad q = \lambda, \qquad r = C. \tag{4.31}$$

Exact solutions to KdV equation may be derived from (4.18)–(4.25) and (4.31).

The KdV equation may also be solved by the Weierstrass elliptic functions method.

4.5. Gardner Equation

This equation reads

$$u_t + \alpha uu_x + \beta u^2u_x + \gamma u_{xxx} = 0, \quad \beta\gamma \neq 0. \tag{4.32}$$

This equation reduces to mKdV equation if we apply the similarity transformations

$$\tilde{t} = \gamma t, \qquad \tilde{x} = x + \frac{\alpha^2}{4\beta}t, \qquad \tilde{u} = \sqrt{\frac{|\beta|}{|\gamma|}}\left(u + \frac{\alpha}{2\beta}\right) \tag{4.33}$$

to the Gardner equation (4.32). We obtain the mKdV equation

$$\tilde{u}_{\tilde{t}} + \varepsilon \tilde{u}_{\tilde{x}}^2 \tilde{u} + \gamma \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0, \tag{4.34}$$

where $\varepsilon = \operatorname{sign}(\beta\gamma) = \pm 1$.

This means that if $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x})$ is a solution to the mKdV equation, then the function defined by

$$u = u(t,x) = \sqrt{\frac{|\gamma|}{|\beta|}}\tilde{u}\left(\gamma t, x + \frac{\alpha^2}{4\beta}t\right) - \frac{\alpha}{2\beta} \tag{4.35}$$

is a solution to Gardner equation (4.32). For solutions to MkdV equation, see [6].

4.6. Boussinesq Equation

This equation reads

$$u_{tt} + \alpha u u_{xx} + \alpha u_x^2 + \beta u_{xxxx} = 0, \qquad \alpha\beta \neq 0. \tag{4.36}$$

After the traveling wave transformation $u = v(\xi)$, $\xi = x + \lambda t + \xi_0$ and integrating twice w.r.t ξ, we obtain the following ODE:

$$v''(\xi) = -\frac{\alpha}{2\beta}v^2(\xi) - \frac{\lambda^2}{\beta}v(\xi) - \frac{D\xi + C}{\beta}, \tag{4.37}$$

where C and D are the constants of integration. Setting $D = 0$, we obtain an equation of the form (4.15) with

$$p = -\frac{\alpha}{2\beta}, \qquad q = -\frac{\lambda^2}{\beta}, \qquad r = -\frac{C}{\beta}. \tag{4.38}$$

As we can see, exact solutions to the Boussinesq equation are calculated from (4.18)–(4.25) and (4.38).

4.7. Symmetric Regular Long Wave Equation

This equation is defined by

$$u_{tt} + u_{xx} + uu_{xt} + u_x u_t + u_{xxtt} = 0. \tag{4.39}$$

Let $u = v(\xi)$, $\xi = x + \lambda t + \xi_0$. Applying this transformation and integrating twice w.r.t. ξ, we obtain the ODE

$$v''(\xi) = -\frac{1}{2\lambda} v^2(\xi) - \frac{\lambda^2 + 1}{\lambda^2} v(\xi) - \frac{D\xi + C}{\lambda^2}, \tag{4.40}$$

where C and D are the constants of integration. Setting $D = 0$, we obtain an equation of the form (4.15) with

$$p = -\frac{1}{2\lambda}, \qquad q = -\frac{\lambda^2 + 1}{\lambda^2}, \qquad r = -\frac{C}{\lambda^2}. \tag{4.41}$$

Thus, exact solutions to symmetric regular long wave equation are easily found from (4.18)–(4.25) taking into account (4.41).

4.8. Generalized Shallow Water Wave Equation

This equation is given by

$$u_{xxtt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0. \tag{4.42}$$

Let

$$u = u(x, t) = V(\xi), \quad V(\xi) = \int v(\xi) d\zeta, \ \xi = x + \lambda t. \tag{4.43}$$

Substituting (4.43) into (4.42) and integrating once w.r.t. ξ, we obtain

$$v''(\xi) = -\frac{\alpha + \beta}{2\lambda} v^2(\xi) + \frac{\lambda + 1}{\lambda^2} v(\xi) + \frac{C}{\lambda^2}, \tag{4.44}$$

where C is the constant of integration. Equation (4.15) has the form (4.15) with

$$p = -\frac{\alpha + \beta}{2\lambda}, \qquad q = \frac{\lambda + 1}{\lambda^2}, \qquad r = \frac{C}{\lambda^2}. \tag{4.45}$$

It is evident that we may find exact solutions to generalized shallow water wave equation (4.42) from (4.18)–(4.25) for the choices given by (4.45).

4.9. Klein-Gordon Equation with Quadratic Nonlinearity

The general Klein-Gordon equation has the form

$$u_{tt} - \alpha^2 u_{xx} + \beta u + f(u) = 0. \tag{4.46}$$

In the case when $f(u) = -\gamma u^2$ we obtain the so-called Klein-Gordon equation with quadratic nonlinearity:

$$u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^2 = 0. \tag{4.47}$$

Let $u = v(\xi)$, $\xi = x + \lambda t + \xi_0$. After this traveling wave transformation, (4.47) reduces to

$$v''(\xi) = -\frac{\gamma}{\alpha^2 - \lambda^2} v^2(\xi) + \frac{\beta}{\alpha^2 - \lambda^2} v(\xi), \quad \alpha^2 \neq \lambda^2, \tag{4.48}$$

which is an equation of the form (4.15) with

$$p = -\frac{\gamma}{\alpha^2 - \lambda^2}, \qquad q = \frac{\beta}{\alpha^2 - \lambda^2}, \qquad r = 0. \tag{4.49}$$

Again, exact solutions to (4.47) are obtained from (4.18)–(4.25) taking into account (4.49).

4.10. Fitzhugh-Nagumo-Huxley Equation

This equation reads

$$u_t - u_{xx} + u(1-u)(\alpha - u) = 0, \quad \alpha = \text{const}. \tag{4.50}$$

This equation is an important model in the study of neuron axon [7]. Let

$$u = u(x,t) = v(\xi), \qquad \xi = x + \lambda t + \xi_0. \tag{4.51}$$

The corresponding ODE is

$$v''(\xi) - \lambda v'(\xi) - v(\xi)(v(\xi) - 1)(v(\xi) - \alpha) = 0. \tag{4.52}$$

Application of the sn-ns method gives only trivial solutions since we get $\lambda = 0$. Instead, we may use other methods. If we apply the tanh-coth method or the exp method, we obtain nontrivial solutions. Indeed, balancing gives $n = 1$. Following the tanh-coth method, we try the ansatz

$$v(\xi) = a_0 + a_1 \tanh(k\xi) + b_1 \coth(k\xi). \tag{4.53}$$

Inserting (4.53) into (4.50) and solving the corresponding algebraic system gives the following solutions to (4.50):

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left(1+\tanh\left[\frac{1}{2\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]\right),\\
u(x,t) &= \frac{1}{2}\left(1+\coth\left[\frac{1}{2\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]\right),\\
u(x,t) &= \frac{1}{4}\left(2+\tanh\left[\frac{1}{4\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]+\coth\left[\frac{1}{4\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]\right),\\
u(x,t) &= \frac{\alpha}{2}\left(1+\tanh\left[\frac{\alpha}{2\sqrt{2}}\left(x-\frac{2-\alpha}{\sqrt{2}}t+\xi_0\right)\right]\right),\\
u(x,t) &= \frac{\alpha}{2}\left(1+\coth\left[\frac{\alpha}{2\sqrt{2}}\left(x-\frac{2-\alpha}{\sqrt{2}}t+\xi_0\right)\right]\right),\\
u(x,t) &= \frac{\alpha}{4}\left(2+\tanh\left[\frac{\alpha}{4\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]+\coth\left[\frac{\alpha}{4\sqrt{2}}\left(x-\frac{2\alpha-1}{\sqrt{2}}t+\xi_0\right)\right]\right).
\end{aligned} \tag{4.54}$$

4.11. Double Sine-Gordon Equation

Our last example deals with the double sine-Gordon equation. This equation in a normalized form reads

$$u_{tt}-u_{xx}+\sin(u)-\frac{1}{2}\sin(2u)=0. \tag{4.55}$$

This equation is an important model in the study of the DNA molecule [8].

The application of the tanh-coth method gives only the trivial solution $u = 0$. If we apply the sn-ns method, we obtain nontrivial solutions. Indeed, let

$$u = 2\arctan(v), \qquad v = v(\xi), \qquad \xi = x+\lambda t+\xi_0. \tag{4.56}$$

Inserting ansatz (4.56) into (4.55) gives the ODE

$$2v^3(\xi)-2\left(\lambda^2-1\right)v(\xi)\left(v'(\xi)\right)^2+\left(\lambda^2-1\right)v''(\xi)+\left(\lambda^2-1\right)v^2(\xi)v''(\xi)=0. \tag{4.57}$$

Let

$$v(\xi) = a_0+a_1\operatorname{sn}(k\xi \mid m)+b_1\operatorname{ns}(k\xi \mid m). \tag{4.58}$$

Substituting (4.58) into (4.57) and solving the corresponding algebraic system gives the following solutions to (4.55):

(i) $a_0 = 0$, $a_1 = (\sqrt{2}/2)\sqrt{m^2+1}\sqrt{-1}$, $b_1 = 0$, $\lambda = \sqrt{k^2(m^2-1)^2 - 2(m^2+1)}/k(m^2-1)$,

$$v(\xi) = \frac{\sqrt{2}}{2}\sqrt{m^2+1}\sqrt{-1}\,\mathrm{sn}\left(k\left(x + \frac{\sqrt{k^2(m^2-1)^2 - 2(m^2+1)}}{k(m^2-1)}t + \xi_0\right) \mid m\right), \tag{4.59}$$

(ii) $a_0 = 0$, $a_1 = 0$, $b_1 = (\sqrt{2}/2m)\sqrt{m^2+1}\sqrt{-1}$, $\lambda = \sqrt{k^2(m^2-1)^2 - 2(m^2+1)}/k(m^2-1)$,

$$v(\xi) = \frac{\sqrt{2}}{2m}\sqrt{m^2+1}\sqrt{-1}\,\mathrm{ns}\left(k\left(x + \frac{\sqrt{k^2(m^2-1)^2 - 2(m^2+1)}}{k(m^2-1)}t + \xi_0\right) \mid m\right), \tag{4.60}$$

(iii) $a_0 = 0$, $a_1 = \sqrt{m(m^2-6m+1)}/2\sqrt{2}(m-1)$, $b_1 = -\sqrt{m(m^2-6m+1)}/2\sqrt{2}m(m-1)$, $\lambda = -\sqrt{k^2(m+1)^4 - 2(m^2-6m+1)}/k(m+1)^2$,

$$\begin{gathered} v(\xi) = \frac{\sqrt{m(m^2-6m+1)}}{2\sqrt{2}(m-1)}\mathrm{sn}(k\xi \mid m) - \frac{\sqrt{m(m^2-6m+1)}}{2\sqrt{2}m(m-1)}\mathrm{ns}(k\xi \mid m), \\ \xi = x - \frac{\sqrt{k^2(m+1)^4 - 2(m^2-6m+1)}}{k(m+1)^2}t + \xi_0. \end{gathered} \tag{4.61}$$

5. Comparison with Other Methods to Solve Nonlinear PDEs

There are some other powerful and systematical approaches for solving nonlinear partial differential equations, such as the expansion along the integrable ODE [9, 10], the transformed rational function method [11], and the multiple expfunction method [12]. Even about linear DEs, there is some recent study on solution representations [13] and the linear superposition principle has been applied to bilinear equations [14].

5.1. Multiple Exp Function Method

Let us consider the following equation:

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \tag{5.1}$$

Equation (5.1) is also called the Sawada-Kotera equation [15]. In a recent work [16], the authors obtained multisoliton solutions to (5.1) by using Hirota's bilinear approach.

Introducing the potential w, defined by

$$u = w_x, \tag{5.2}$$

(5.1) may be written in the form

$$w_{tx} + w_{xxxxxx} + 30w_x w_{xxxx} + 30w_{xx}w_{xxx} + 180w_x^2 w_{xx} = 0. \tag{5.3}$$

Integrating (5.3) once w.r.t x and taking the constant of integration equal to zero, the following partial differential equation is obtained:

$$w_t + 60w_x^3 + 30w_x w_{xxx} + w_{xxxxx} = 0. \tag{5.4}$$

We will call (5.4) the potential Caudrey-Dodd-Gibbon equation associated with (5.1).

In view of the multiple exp method one-soliton solutions to (5.4) have the form

$$w(x,t) = \frac{a_0 + a_1 \exp(\eta)}{1 + b_1 \exp(\eta)}, \quad \eta = \eta(x,t) = kx - \omega t, \tag{5.5}$$

where a_0, a_1, and b_1 are some constants to be determined. Inserting ansatz (5.5) into (5.4) and simplifying, we obtain the following polynomial equation in the variable $\zeta = \exp(\eta)$:

$$\begin{aligned} &k^5 - \omega + 2\left(-15a_0b_1k^4 + 15a_1k^4 - 13b_1k^5 - 2b_1\omega\right)\zeta \\ &\quad + 6\left(20a_0b_1^2k^4 - 20a_1b_1k^4 + 10a_0^2b_1^2k^3 - 20a_0a_1b_1k^3 + 10a_1^2k^3 + 11b_1^2k^5 - b_1^2\omega\right)\zeta^2 \\ &\quad + 2b_1^2\left(-15a_0b_1k^4 + 15a_1k^4 - 13b_1k^5 - 2b_1\omega\right)\zeta^3 + b_1^4\left(k^5 - \omega\right)\zeta^4 = 0. \end{aligned} \tag{5.6}$$

Equating the coefficients of different powers of ζ to zero gives an algebraic system. Solving it with either Mathematica 8 or Maple 15, we obtain

$$w = k^5, \qquad a_1 = b_1(a_0 + k). \tag{5.7}$$

Thus, the following is a one-soliton solution (or one wave solution in Ma's terminology) to (5.4):

$$w(x,t) = \frac{a_0 + b_1(a_0 + k)\exp(\eta)}{1 + b_1 \exp(\eta)}, \qquad \eta = kx - k^5 t. \tag{5.8}$$

In view of (5.2) one-soliton solution to Caudrey-Dodd-Gibbon equation (5.1) is

$$u(x,t) = \frac{b_1 k^2 \exp(kx - k^5 t)}{(1 + b_1 \exp(kx - k^5 t))^2}. \tag{5.9}$$

Observe that solution (5.9) is the same solution obtained in [16] by using Hirota's bilinear method. We conclude that Hirota's method and multiple exp method give the same result for one-soliton solutions. Let us examine two-soliton solutions. In view of Ma's method the two-soliton solutions to (5.4) are of the form

$$w(x,t) = \frac{k_1\eta_1 + k_2\eta_2 + R(k_1+k_2)\eta_1\eta_2}{\Delta + \eta_1 + \eta_2 + R\eta_1\eta_2}, \quad \Delta \in \{0,1\},$$
$$\eta_1 = \exp(k_1x - \omega_1 t), \qquad \eta_2 = \exp(k_2x - \omega_2 t). \tag{5.10}$$

Let $\Delta = 1$. Inserting ansatz (5.10) into (5.4) and simplifying, we obtain the following polynomial equation w.r.t the variables $\zeta_1 = \exp(\eta_1)$ and $\zeta_2 = \exp(\eta_2)$:

$$\begin{aligned}
&c_1^2c_2k_2R\left(k_2^5-\omega_2\right)\zeta_1^2\zeta_2 + c_1c_2^2k_1R\left(k_1^5-\omega_1\right)\zeta_1\zeta_2^2\\
&\quad + \Big(c_1c_2(k_1+k_2)\left(k_1^5+5k_2k_1^4+10k_2^2k_1^3+10k_2^3k_1^2+5k_2^4k_1+k_2^5-\omega_1-\omega_2\right)R\\
&\quad + c_1c_2(k_1-k_2)\left(k_1^5-5k_2k_1^4+10k_2^2k_1^3-10k_2^3k_1^2+5k_2^4k_1-k_2^5-\omega_1+\omega_2\right)\Big)\zeta_1\zeta_2\\
&\quad + c_1k_1\left(k_1^5-\omega_1\right)\zeta_1 + c_2k_2\left(k_2^5-\omega_2\right)\zeta_2 = 0.
\end{aligned} \tag{5.11}$$

Equating the coefficients of $\zeta_1^2\zeta_2$, $\zeta_1\zeta_2^2$, $\zeta_1\zeta_2$, ζ_1, and ζ_2 to zero gives a system of algebraic equations. Solving it, we obtain the following nontrivial solution:

$$w_1 = k_1^5, \qquad w_2 = k_2^5, \qquad R = \frac{(k_1-k_2)^2\left(k_1^2-k_1k_2+k_2^2\right)}{(k_1+k_2)^2\left(k_1^2+k_1k_2+k_2^2\right)}. \tag{5.12}$$

We see that solution (5.12) coincides with solutions obtained in [16]. The same is valid if we set $\Delta = 0$. We conclude that Ma's method does not give any new solutions compared with Hirota's method.

5.2. The Transformed Rational Function Method

Given a nonlinear ODE in the unknown $u = u(x,t)$, $u = u(x,y,t)$ or $u = u(x,y,z,t)$. we search for traveling wave solutions determined by

$$u^{(r)}(\xi) = v(\eta),$$
$$v(\eta) = \frac{p(\eta)}{q(\eta)} = \frac{p_m\eta^m + p_{m-1}\eta^{m-1} + \cdots + p_0}{q_n\eta^n + q_{n-1}\eta^{n-1} + \cdots + q_0}, \tag{5.13}$$

where m and n are two natural numbers and p_i $(0 \le i \le m), q_j$ $(0 \le j \le n)$ are normally constants but could be functions of the independent variables and η is a solution to equation

$$A\eta'' + B\eta' + C\eta^2 + D\eta + E = 0, \tag{5.14}$$

with A, B, C, D, and E being some constants. Observe that equations

$$\begin{gathered}\eta' = \eta^2, \\ \eta' = \alpha + \eta^2 \quad (\text{Riccati equation}),\end{gathered} \tag{5.15}$$

$$v''(\xi) = pv^2(\xi) + qv(\xi) + r \quad (\text{quadratic Duffing equation (4.15)}) \tag{5.16}$$

are particular cases of (5.14). This method was applied in [11] to solve the (3+1)- dimensional Jimbo-Miwa equation

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \tag{5.17}$$

which converts into the nonlinear ODE

$$a^3 b u^{(4)} + 6a^2 b u' u'' - (2b\omega + 3ac)u'' = 0 \tag{5.18}$$

after the traveling wave transformation

$$u(x, y,\ z,\ t) = u(\xi), \quad \xi = ax + by + cz - \omega t. \tag{5.19}$$

If we integrate (5.14) once w.r.t x, then the following equation follows:

$$v''(\xi) = -\frac{3}{a}v^2(\xi) + \frac{2b\omega + 3ac}{a^3 b}v(\xi), \quad \text{where } u(\xi) = \int v(\xi)d\xi. \tag{5.20}$$

Observe that (5.20) is a quadratic Duffing equation (5.16) with $p = -3/a$, $q = (2b\omega+3ac)/a^3 b$, and $r = 0$.

Solutions to this equation are given by (4.18)–(4.27). These solutions were not reported in [11]. On the other hand, it is clear from (4.24) that the sn-ns method covers the solutions obtained in [11].

6. Conclusions

We successfully obtained exact solutions for some important physical models by techniques based on projective equations. Mainly, we have used the sn-ns method. In our opinion, this is the most appropriate of all methods we have studied since it provides elliptic function solutions as well as trigonometric and hyperbolic solutions. However, in the cases when the sn-ns method does not work (this occurs for the Fitzhugh-Nagumo-Huxley equation), we may try other methods, such as the tanh-coth method. On the other hand, there are some equations for which the tanh-coth method gives only trivial solutions (this is the case of the double sine-Gordon equation).

We think that some of the results we obtained are new in the open literature. Other results concerning exact solutions of nonlinear PDEs may be found in [6, 15, 17–52].

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