Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 352159, 23 pages
doi:10.1155/2012/352159

Research Article

Multiple Bounded Positive Solutions to Integral Type BVPs for Singular Second Order ODEs on the Whole Line

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Received 3 June 2012; Accepted 6 August 2012

Academic Editor: Ziemowit Popowicz

This paper is concerned with the integral type boundary value problems of the second order differential equations with one-dimensional p-Laplacian on the whole line. By constructing a suitable Banach space and a operator equation, sufficient conditions to guarantee the existence of at least three positive solutions of the BVPs are established. An example is presented to illustrate the main results. The emphasis is put on the one-dimensional p-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involved with the function ρ, which makes the solutions un-concave.

1. Introduction

The multipoint boundary-value problems for linear second order ordinary differential equations (ODEs for short) was initiated by Il'in and Moiseev [1]. Since then, more general nonlinear multi-point boundary-value problems (BVPs for short) were studied by several authors, see the text books [2–4] and the references cited therein.

Differential equations governed by nonlinear differential operators have been widely studied. In this setting the most investigated operator is the classical one-dimensional p-Laplacian, that is, $\Phi_p(x) = |x|^{p-2}x$ with $p > 1$. This operator is involved in some models, for example, in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(x')]' = f(t, x, x'), \quad t \in (-\infty, +\infty), \tag{1.1}$$

where $\Phi(x) = |x|^{p-2}x$ with $p > 1$ is a one dimensional p-Laplacian. For a comprehensive bibliography on this subject, see, for example [5–9].

In this paper, we consider the more generalized BVP for second order differential equation on the whole line with p-Laplacian coupled with the integral type BCs, that is the BVP

$$\begin{gathered}
\left[\rho(t)\Phi\left(x'(t)\right)\right]' + f\left(t, x(t), x'(t)\right) = 0, \quad t \in R, \\
\lim_{t\to-\infty} x(t) =: x(-\infty) = \int_{-\infty}^{+\infty} g(s)x(s)ds, \\
\lim_{t\to+\infty} x(t) =: x(+\infty) = \int_{-\infty}^{+\infty} h(s)x(s)ds,
\end{gathered} \tag{1.2}$$

where $f : R^3 \to R$ is a nonnegative Caratheodory function, $g, h : R \to [0, \infty)$ satisfy $g, h \in L^1(R)$, $\rho \in C(R, (0, \infty))$, the integrals in mentioned equations are meant in the sense of Riemann-Stieljes, $\Phi(x) = |x|^{p-2}x$ with $p > 1$ is called a one dimensional p-Laplacian, whose inverse function is denoted by Φ^{-1}.

The purpose is to establish sufficient conditions for the existence of at least three positive solutions of BVP(1.2). The result in this paper generalizes and improves some known ones since the one-dimensional p-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involved with the function ρ, which makes the solutions unconcave and there exists no paper concerned with the existence of at least three positive solutions of this kind of integral BVPs on the whole lines. This paper fills the gap.

The remainder of this paper is organized as follows: the main result (Theorem 2.8) is presented in Section 2, and the example to show the main result is given in Section 3.

2. Main Results

In this section, we first present some background definitions in Banach spaces and state an important three fixed point theorem. Then the main results are given and proved.

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous, and satisfies $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$, or $\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.3. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.4. Let $a, b, c, d, h > 0$ be positive constants, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$\begin{gathered}
P_c = \{x \in P : \|x\| < c\}, \\
P(\gamma, \alpha; a, c) = \{x \in P : \alpha(x) \geq a,\ \gamma(x) \leq c\}, \\
P(\gamma, \theta, \alpha; a, b, c) = \{x \in P : \alpha(x) \geq a,\ \theta(x) \leq b,\ \gamma(x) \leq c\}, \\
Q(\gamma, \beta; , d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\}, \\
Q(\gamma, \beta, \psi; h, d, c) = \{x \in P : \psi(x) \geq h,\ \beta(x) \leq d,\ \gamma(x) \leq c\}.
\end{gathered} \tag{2.1}$$

Lemma 2.5 (see [10]). *Let X be a real Banach space, P be a cone in X, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. Assume that there exists a constant $M > 0$ such that*

$$\alpha(x) \leq \beta(x), \qquad \|x\| \leq M\gamma(x), \quad \forall x \in P. \tag{2.2}$$

Furthermore, suppose that $h, d, a, b, c > 0$ are constants with $d < a$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. If

(C1) $\{x \in P(\gamma, \theta, \alpha; a, b, c) \mid \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Tx) > a$ *for every* $x \in P(\gamma, \theta, \alpha; a, b, c)$;

(C2) $\{x \in Q(\gamma, \theta, \psi; h, d, c) \mid \beta(x) < d\} \neq \emptyset$ *and* $\beta(Tx) < d$ *for every* $x \in Q(\gamma, \theta, \psi; h, d, c)$;

(C3) $\alpha(Tx) > a$ *for* $x \in P(\gamma, \alpha; a, c)$ *with* $\theta(Tx) > b$;

(C4) $\beta(Tx) < d$ *for each* $x \in Q(\gamma, \beta; d, c)$ *with* $\psi(Tx) < h$,

then T has at least three fixed points x_1, x_2 and x_3 such that $\beta(x_1) < d$, $\alpha(x_2) > a$, $\beta(x_3) > d$, $\alpha(x_3) < a$.

Let us list the assumptions

(H1) $g, h : R \to [0, \infty)$ satisfy $\int_{-\infty}^{+\infty} g(s)ds < 1$, $\int_{-\infty}^{+\infty} h(s)ds < 1$ and

$$g(t)\left(1 - \int_{-\infty}^{+\infty} h(s)ds\right) - h(t)\left(1 - \int_{-\infty}^{+\infty} g(s)ds\right) \geq 0, \quad t \in R. \tag{2.3}$$

(H2) $\rho \in C(R), \rho(t) > 0$ for $t \in R$ with $\int_{-\infty}^{0} \Phi^{-1}(1/\rho(t))dt = \int_{0}^{+\infty} \Phi^{-1}(1/\rho(t))dt < +\infty$.

(H3) $f(t, c, 0) \not\equiv 0$ on any finite subinterval of R for each $c \in R$, $f : R^3 \to R$ is a Carathédory function, that is,

(i) $t \to f(t, x, (1/\Phi^{-1}(\rho(t)))y)$ is measurable for any $(x, y) \in R^2$,

(ii) $(x, y) \to f(t, x, (1/\Phi^{-1}(\rho(t)))y)$ is continuous for a.e. $t \in R$,

(iii) for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(R)$ such that $\max\{|u|, |v|\} \leq r$ implies

$$\left| f\left(t, u, \frac{1}{\Phi^{-1}(\rho(t))} v\right) \right| \leq \phi_r(t), \qquad a.e.\ t \in R. \tag{2.4}$$

Choose

$$X = \left\{ x \in C^1(R) : \begin{array}{c} \text{and there exist the limits} \\ \lim_{t \to -\infty} x(t), \\ \lim_{t \to +\infty} x(t), \\ \lim_{t \to -\infty} \Phi^{-1}(\rho(t))x'(t) \\ \text{and } \lim_{t \to +\infty} \Phi^{-1}(\rho(t))x'(t) \end{array} \right\}. \tag{2.5}$$

For $x \in X$, define the norm of x by

$$\|x\| = \max\left\{\sup_{t\in R}|x(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))x'(t)\right\}. \tag{2.6}$$

One can prove that X is a Banach space with the norm $\|x\|$ for $x \in X$.

Let $x \in X$. Consider the following auxiliary BVP

$$\begin{gathered}
[\rho(t)\Phi(y'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\
y(-\infty) = \int_{-\infty}^{+\infty} g(s)y(s)ds, \\
y(+\infty) = \int_{-\infty}^{+\infty} h(s)y(s)ds.
\end{gathered} \tag{2.7}$$

Lemma 2.6. *Suppose that (H1)–(H3) hold. If $y \in C^1(R)$ such that $[\rho\Phi(y')]' \in L^1(R)$ is a solution of BVP(2.7), then*

(i) *y is bounded and nonnegative on R;*

(ii) *$y(t)$ is concave with respect to $\tau = \int_{-\infty}^{t}\Phi^{-1}(1/\rho(s))ds / \int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds$;*

(iii) *for $k > 0$, it holds that $\min_{t\in[-k,k]} y(t) \geq \mu \sup_{t\in R} y(t)$ with $\mu = \int_{-\infty}^{-k}\Phi^{-1}(1/\rho(s))ds / 2\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds$;*

(iv) *there exists a unique constant $A_x \in [-\int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds, 0]$ such that*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{g(t)\left(1 - \int_{-\infty}^{+\infty} h(s)ds\right) - h(t)\left(1 - \int_{-\infty}^{+\infty} g(s)ds\right)}{1 - \int_{-\infty}^{+\infty} g(s)ds} \\
&\quad \times \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(u, x(u), x'(u))du\right)ds\,dt \\
&\quad + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(u, x(u), x'(u))du\right)ds = 0.
\end{aligned} \tag{2.8}$$

Proof. Since $x \in X$, we get

$$r = \max\left\{\sup_{t\in R}|x(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))x'(t)\right\} < +\infty. \tag{2.9}$$

Then there exists a nonnegative function $\phi_r \in L^1(-\infty, +\infty)$ such that

$$0 \leq f(t, x(t), x'(t)) = f\left(t, x(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x'(t)\right) \leq \phi_r(t), \quad t \in R. \tag{2.10}$$

Then

$$\int_{-\infty}^{+\infty} f(s,x(s),x'(s))ds \text{ is convergent.} \tag{2.11}$$

(i) We know there exist the limits $\lim_{t\to-\infty}\rho(t)\Phi(y'(t))$ and $\lim_{t\to+\infty}\rho(t)\Phi(y'(t))$. We claim that there exists $\tau_0 \in R$ such that $y'(\tau_0) = 0$. In fact, if $y'(t) > 0$ for all $t \in R$, we get $y(-\infty) < y(t) < y(+\infty)$ for all $t \in R$. It follows that

$$y(-\infty) \geq \int_{-\infty}^{+\infty} g(s)dsy(-\infty), \qquad y(+\infty) \leq \int_{-\infty}^{+\infty} h(s)dsy(+\infty). \tag{2.12}$$

Then (H1) implies $y(-\infty) \geq 0 \geq y(+\infty)$, which contradicts to $y'(t) > 0$ for all $t \in R$. Similarly we can prove that $y'(t) < 0$ does not hold. Then there exists $\tau_0 \in R$ such that $y'(\tau_0) = 0$.

Since $[\rho(t)\Phi(y'(t))]' = -f(t,x(t),x'(t)) \leq 0$, we know that $\rho(t)\Phi(y'(t))$ is decreasing on R. Then $\rho(t)\Phi(y'(t)) \geq 0$ for all $t \leq \tau_0$ and $\rho(t)\Phi(y'(t)) \leq 0$ for all $t \geq \tau_0$. Hence

$$y \text{ is increasing for } t \in (-\infty,\tau_0] \text{ and decreasing for } t \in [\tau_0,+\infty). \tag{2.13}$$

One sees that

$$y'(t) = \begin{cases} -\Phi^{-1}\left(\dfrac{1}{\rho(t)}\displaystyle\int_{\tau_0}^{t} f(u,x(u),x'(u))du\right) =: -G(t), & t \geq \tau_0, \\ \Phi^{-1}\left(\dfrac{1}{\rho(t)}\displaystyle\int_{t}^{\tau_0} f(u,x(u),x'(u))du\right) =: H(t), & t \leq \tau_0. \end{cases} \tag{2.14}$$

Since $\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(t))dt < +\infty$ and $\int_{-\infty}^{+\infty} f(t,x(t),x'(t))dt < +\infty$, we see that

$$\text{both } \int_{t}^{+\infty} G(s)ds, \int_{-\infty}^{t} H(s)ds \text{ are convergent.} \tag{2.15}$$

Then we get that

$$y(t) = \begin{cases} y(+\infty) + \displaystyle\int_{t}^{+\infty} G(s)ds, & t \geq \tau_0, \\ y(-\infty) + \displaystyle\int_{-\infty}^{t} H(s)ds, & t \leq \tau_0. \end{cases} \tag{2.16}$$

This tells us that y is bounded on R.

It follows from (2.7) and (2.16) that

$$\begin{aligned} y(-\infty) &= \int_{-\infty}^{\tau_0} g(s)ds\, y(-\infty) + \int_{\tau_0}^{+\infty} g(s)ds\, y(+\infty) + \int_{-\infty}^{\tau_0} g(t)\int_{-\infty}^{t} H(s)ds\,dt \\ &\quad + \int_{\tau_0}^{+\infty} g(t)\int_{t}^{+\infty} G(s)ds\,dt, \\ y(+\infty) &= \int_{-\infty}^{\tau_0} h(s)ds\, y(-\infty) + \int_{\tau_0}^{+\infty} h(s)ds\, y(+\infty) + \int_{-\infty}^{\tau_0} h(t)\int_{-\infty}^{t} H(s)ds\,dt \\ &\quad + \int_{\tau_0}^{+\infty} h(t)\int_{t}^{+\infty} G(s)ds\,dt. \end{aligned} \tag{2.17}$$

Then

$$\begin{aligned} &\left(1 - \int_{-\infty}^{\tau_0} g(s)ds\right) y(-\infty) - \int_{\tau_0}^{+\infty} g(s)ds\, y(+\infty) \\ &\quad = \int_{-\infty}^{\tau_0} g(t)\int_{-\infty}^{t} tH(s)ds\,dt + \int_{\tau_0}^{+\infty} g(t)\int_{t}^{+\infty} G(s)ds\,dt, \\ &\quad - \int_{-\infty}^{\tau_0} h(s)ds\, y(-\infty) + \left(1 - \int_{\tau_0}^{+\infty} h(s)ds\right) y(+\infty) \\ &\quad = \int_{-\infty}^{\tau_0} h(t)\int_{-\infty}^{t} H(s)ds\,dt + \int_{\tau_0}^{+\infty} h(t)\int_{t}^{+\infty} G(s)ds\,dt. \end{aligned} \tag{2.18}$$

Hence

$$\begin{aligned} y(-\infty) &= \frac{\begin{vmatrix} \int_{-\infty}^{\tau_0} g(t)\int_{-\infty}^{t} H(s)dsdt + \int_{\tau_0}^{+\infty} g(t)\int_{t}^{+\infty} G(s)dsdt & -\int_{\tau_0}^{+\infty} g(s)ds \\ \int_{-\infty}^{\tau_0} h(t)\int_{-\infty}^{t} H(s)dsdt + \int_{\tau_0}^{+\infty} h(t)\int_{t}^{+\infty} G(s)dsdt & 1-\int_{\tau_0}^{+\infty} h(s)ds \end{vmatrix}}{\begin{vmatrix} 1-\int_{-\infty}^{\tau_0} g(s)ds & -\int_{\tau_0}^{+\infty} g(s)ds \\ -\int_{-\infty}^{\tau_0} h(s)ds & 1-\int_{\tau_0}^{+\infty} h(s)ds \end{vmatrix}}, \\ y(+\infty) &= \frac{\begin{vmatrix} 1-\int_{-\infty}^{\tau_0} g(s)ds & \int_{-\infty}^{\tau_0} g(t)\int_{-\infty}^{t} H(s)dsdt + \int_{\tau_0}^{+\infty} g(t)\int_{t}^{+\infty} G(s)dsdt \\ -\int_{-\infty}^{\tau_0} h(s)ds & \int_{-\infty}^{\tau_0} h(t)\int_{-\infty}^{t} H(s)dsdt + \int_{\tau_0}^{+\infty} h(t)\int_{t}^{+\infty} G(s)dsdt \end{vmatrix}}{\begin{vmatrix} 1-\int_{-\infty}^{\tau_0} g(s)ds & -\int_{\tau_0}^{+\infty} g(s)ds \\ -\int_{-\infty}^{\tau_0} h(s)ds & 1-\int_{\tau_0}^{+\infty} h(s)ds \end{vmatrix}}. \end{aligned} \tag{2.19}$$

Since

$$\begin{vmatrix} 1 - \int_{-\infty}^{\tau_0} g(s)ds & -\int_{\tau_0}^{+\infty} g(s)ds \\ -\int_{-\infty}^{\tau_0} h(s)ds & 1 - \int_{\tau_0}^{+\infty} h(s)ds \end{vmatrix} = \begin{vmatrix} 1 - \int_{-\infty}^{+\infty} g(s)ds & -\int_{\tau_0}^{+\infty} g(s)ds \\ 1 - \int_{-\infty}^{+\infty} h(s)ds & 1 - \int_{\tau_0}^{+\infty} h(s)ds \end{vmatrix} > 0, \tag{2.20}$$

we get from (2.19) that $y(-\infty) \geq 0$ and $y(+\infty) \geq 0$. Hence (2.16) implies that

$$y(t) \geq 0, \quad \forall t \in R. \tag{2.21}$$

(ii) We prove that $y(t)$ is concave with respect to τ on R. It is easy to see that $\tau \in C(R, (0,1))$ and

$$\frac{d\tau}{dt} = \Phi^{-1}\left(\frac{1}{\rho(t)}\right)\frac{1}{\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds} > 0. \tag{2.22}$$

Thus

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{dy}{d\tau}\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\frac{1}{\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds}. \tag{2.23}$$

It follows that

$$\rho(t)\Phi\left(\frac{dy}{dt}\right) = \Phi\left(\frac{dy}{d\tau}\right)\Phi\left(\frac{1}{\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds}\right). \tag{2.24}$$

Hence

$$\left[\rho(t)\Phi\left(\frac{dy}{dt}\right)\right]' = \Phi'\left(\frac{dy}{d\tau}\right)\frac{d^2y}{d\tau^2}\frac{d\tau}{dt}\Phi\left(\frac{1}{\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds}\right). \tag{2.25}$$

So

$$\frac{d^2y}{d\tau^2} = \Phi\left(\int_{-\infty}^{+\infty}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\right)\frac{\left[\rho(t)\Phi(dy/dt)\right]'}{\Phi'(dy/d\tau)(d\tau/dt)}. \tag{2.26}$$

Since $[\rho(t)\Phi(y'(t))]' \leq 0$, $\Phi'(y) > 0$ $(y \neq 0)$ and $(d\tau/dt) > 0$, we get that $(d^2y/d\tau^2) \leq 0$. Hence $y(t)$ is concave with respect to τ on R.

(iii) Now, we prove that

$$\min_{t\in[-k,k]} y(t) \geq \mu \sup_{t\in R} y(t). \tag{2.27}$$

Since $d\tau/dt > 0$ for all $t \in R$, there exists the inverse function of $\tau = \tau(t)$. Denote the inverse function of $\tau = \tau(t)$ by $t = t(\tau)$.

It follows from (2.13) that $\sup_{t\in R} y(t) = y(\tau_0)$. One sees

$$\min_{t\in[-k,k]} y(t) = \min\{y(-k), y(k)\}. \tag{2.28}$$

If $\min\{y(-k), y(k)\} = y(k) = y(t(\tau(k)))$, note $\tau(k) < 1$, then for $t \in [-k, k]$, one has

$$y(t) \geq y(t(\tau(k))) = y\left(t\left(\frac{1-\tau(k)+\tau(\tau_0)}{1+\tau(\tau_0)}\frac{\tau(k)}{1-\tau(k)+\tau(\tau_0)} + \frac{\tau(k)}{1+\tau(\tau_0)}\tau(\tau_0)\right)\right). \tag{2.29}$$

Noting that $1 > \tau(k)$ and $y(t)$ is concave with respect to τ, then, for $t \in [-k, k]$,

$$\begin{aligned} y(t) &\geq \frac{1-\tau(k)+\tau(\tau_0)}{1+\tau(\tau_0)} y\left(t\left(\frac{\tau(k)}{1-\tau(k)+\tau(\tau_0)}\right)\right) + \frac{\tau(k)}{1+\tau(\tau_0)} y(t(\tau(\tau_0))) \\ &\geq \int_{-\infty}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds \frac{1}{2\int_{-\infty}^{+\infty} \Phi^{-1}(1/\rho(s))ds} y(\tau_0) \\ &= \mu \sup_{t\in R} y(t). \end{aligned} \tag{2.30}$$

Similarly, if $\min\{y(-k), y(k)\} = y(-k) = (y(t(\tau(-k)))$, note $\tau(-k) < 1$, for $t \in [-k, k]$, one has

$$\begin{aligned} y(t) &\geq y(t(\tau(-k))) \\ &= y\left(t\left(\frac{1+\tau(\tau_0)-\tau(-k)}{1+\tau(\tau_0)}\frac{\tau(-k)}{1+\tau(\tau_0)-\tau(-k)} + \frac{\tau(-k)}{1+\tau(\tau_0)}\tau(\tau_0)\right)\right) \\ &\geq \frac{1+\tau(\tau_0)-\tau(-k)}{1+\tau(\tau_0)} y\left(t\left(\frac{\tau(-k)}{1+\tau(\tau_0)-\tau(-k)}\right)\right) + \frac{\tau(-k)}{1+\tau(\tau_0)} y(t(\tau(\tau_0))) \\ &\geq \int_{-\infty}^{-k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds \frac{1}{2\int_{-\infty}^{+\infty} \Phi^{-1}(1/\rho(s))ds} y(\tau_0) \\ &> \mu \sup_{t\in R} y\,(t). \end{aligned} \tag{2.31}$$

Hence (2.27) holds.

(iv) Finally, we prove the uniqueness of A_x. Define

$$\begin{aligned} H_x(c) = {} & \int_{-\infty}^{+\infty} \frac{g(t)\left(1-\int_{-\infty}^{+\infty} h(s)ds\right) - h(t)\left(1-\int_{-\infty}^{+\infty} g(s)ds\right)}{1-\int_{-\infty}^{+\infty} g(s)ds} \\ & \times \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(c + \int_{s}^{+\infty} f(u, x(u), x'(u))du\right) ds\, dt \\ & + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(c + \int_{s}^{+\infty} f(u, x(u), x'(u))du\right) ds. \end{aligned} \tag{2.32}$$

Then (H1) implies that $H_x \in C(R, R)$ is increasing on R and $H_x(0) \geq 0$. Let

$$\overline{c} = -\int_{-\infty}^{+\infty} f(u, x(u), x'(u))du, \tag{2.33}$$

then $H_x(\overline{c}) \leq 0$. By mean value theorem, there exists an unique $A_x \in [\overline{c}, 0]$ satisfying $H_x(A_x) = 0$. Then (iv) holds. This completes the proof of the lemma. □

Choose $1 \geq k > 0$ and

$$\mu = \frac{\int_{-\infty}^{-k} \Phi^{-1}(1/\rho(s))ds}{2\int_{-\infty}^{\infty} \Phi^{-1}(1/\rho(s))ds}. \tag{2.34}$$

Define the cone $P \subseteq X$ by

$$P = \left\{ x \in X : \begin{array}{c} x(t) \geq 0,\ t \in R, \\ \min\limits_{t\in[-k,k]} x(t) \geq \mu \max\limits_{t\in R} x(t) \end{array} \right\}. \tag{2.35}$$

It is easy to see that P is a cone in X.

Define the operator $T : P \to X$ by

$$\begin{aligned}(Tx)(t) = {}& \frac{\int_{-\infty}^{+\infty} g(u) \int_{-\infty}^{u} \Phi^{-1}(1/\rho(s))\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right)ds\,du}{1 - \int_{-\infty}^{+\infty} g(s)ds} \\ &+ \int_{-\infty}^{t} \Phi^{-1}\frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right)ds,\end{aligned} \tag{2.36}$$

where A_x satisfies

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{g(u)\left(1 - \int_{-\infty}^{+\infty} h(s)ds\right) - h(u)\left(1 - \int_{-\infty}^{+\infty} g(s)ds\right)}{1 - \int_{-\infty}^{+\infty} g(s)ds} \\ &\quad\times \int_{-\infty}^{u} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right)ds\,du \\ &\quad+ \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right)ds = 0.\end{aligned} \tag{2.37}$$

It follows from Lemma 2.6(iv) that $T : P \to X$ is well defined and $A_x \in [-\int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds, 0]$. It is easy to show that

$$\begin{aligned}&\left[\rho(t)\Phi\left((Tx)'(t)\right)\right]' + f\left(t, x(t)x'(t)\right) = 0, \quad t \in R, \\ &(Tx)(-\infty) = \int_{-\infty}^{+\infty} g(s)(Tx)(s)ds, \\ &(Tx)(+\infty) = \int_{-\infty}^{+\infty} h(s)(Tx)(s)ds.\end{aligned} \tag{2.38}$$

It follows from Lemma 2.6(i) and (iii) that $Tx \in P$ for all $x \in P$. Then $T : P \to P$ is well defined.

Lemma 2.7. *Suppose (H1)–(H3) hold. Then T is completely continuous.*

Proof. First, we prove that T is continuous.

We claim that the function $A_x : P \to [0,+\infty)$ is continuous in x.

Let $\{x_n\} \in P$ with $x_n \to x_0 \in P$ as $n \to \infty$ in P. Let $\{A_{x_n}\}$ $(n = 1,2,\ldots)$ be the constants determined by

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{g(t)\left(1-\int_{-\infty}^{+\infty} h(s)ds\right)-h(t)\left(1-\int_{-\infty}^{+\infty} g(s)ds\right)}{1-\int_{-\infty}^{+\infty} g(s)ds}\\ &\quad\times\int_{-\infty}^{t}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_{x_n}+\int_{s}^{+\infty} f\left(r,x_n(r),x_n'(r)\right)dr\right)ds\,dt\\ &\quad+\int_{-\infty}^{+\infty}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_{x_n}+\int_{s}^{+\infty} f\left(r,x_n(r),x_n'(r)\right)dr\right)ds=0\end{aligned} \tag{2.39}$$

Since $x_n \to x_0$ in P as $n \to \infty$, there exists an $r > 0$ such that $\|x_n\| \le r$. The fact f is a Carathédory function means that there exists a nonnegative function $\phi_r \in L^1(-\infty,+\infty)$ such that

$$0 \le f\left(t,x_n(t),x_n'(t)\right) = f\left(t,x_n(t),\frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x_n'(t)\right) \le \phi_r(t), \quad t\in R. \tag{2.40}$$

Then

$$\int_{-\infty}^{+\infty} f\left(s,x_n(s),x_n'(s)\right)ds \text{ is convergent.} \tag{2.41}$$

So

$$A_{x_n} \in \left[-\int_{-\infty}^{+\infty} f\left(s,x_n(s),x_n'(s)\right)ds,0\right] \subseteq \left[-\int_{-\infty}^{+\infty}\phi_r(s)ds,0\right], \tag{2.42}$$

which means that $\{A_{x_n}\}$ is uniformly bounded.

Suppose that $\{A_{x_n}\}$ does not converge to A_{x_0}. By the bounded property, we know that there exist two subsequences $\{A^{(1)}_{x_{n_k}}\}$ and $\{A^{(2)}_{x_{n_k}}\}$ of $\{A_{x_{n_k}}\}$ with $A^{(1)}_{x_{n_k}} \to c_1$ and $A^{(2)}_{x_{n_k}} \to c_2$ and $c_1 \ne c_2$.

By the construction of A_{x_n}, $(n = 1, 2, \ldots)$, we have

$$\int_{-\infty}^{+\infty} \frac{g(t)\left(1-\int_{-\infty}^{+\infty} h(s)ds\right) - h(t)\left(1-\int_{-\infty}^{+\infty} g(s)ds\right)}{1-\int_{-\infty}^{+\infty} g(s)ds} \times \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_{x_{n_k}}^{(1)} + \int_{s}^{+\infty} f\left(u, x_{n_k}(u), x'_{n_k}(u)\right)du\right)ds\,dt + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_{x_{n_k}}^{(1)} + \int_{s}^{+\infty} f\left(u, x_{n_k}(u), x'_{n_k}(u)\right)du\right)ds = 0. \tag{2.43}$$

Let $k \to \infty$, using Lebesgue's dominated convergence theorem, the above equality implies

$$\int_{-\infty}^{+\infty} \frac{g(t)\left(1-\int_{-\infty}^{+\infty} h(s)ds\right) - h(t)\left(1-\int_{-\infty}^{+\infty} g(s)ds\right)}{1-\int_{-\infty}^{+\infty} g(s)ds} \times \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(c_1 + \int_{s}^{+\infty} f\left(u, x_0(u), x'_o(u)\right)du\right)ds\,dt + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(c_1 + \int_{s}^{+\infty} f\left(u, x_0(u), x'_o(u)\right)du\right)ds = 0. \tag{2.44}$$

By Lemma 2.6 (iv), we get $c_1 = A_{x_0}$. Similarly, $c_2 = A_{x_0}$. Thus $c_1 = c_2$, a contradiction. So, for any $x_n \to x_0$, one has $A_{x_n} \to A_{x_0}$, which means $A_x : P \to R$ is continuous.

Since A_x is continuous, together with the continuity of $(x, y) \to f(t, x, (1/\Phi^{-1}(\rho(t)))y)$, we get that T is continuous.

Second, we show that T is maps bounded subsets into bounded sets.

Let $D \subseteq P$ be bounded. Then, there exists $r > 0$ such that $D \subseteq \{x \in P : \|x\| \le r\}$. Hence

$$r = \max\left\{\sup_{t\in R}|x(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))x'(t)\right\} < +\infty. \tag{2.45}$$

Then there exists a nonnegative function $\phi_r \in L^1(-\infty, +\infty)$ such that

$$0 \le f\left(t, x(t), x'(t)\right) = f\left(t, x(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x'(t)\right) \le \phi_r(t), \quad t \in R. \tag{2.46}$$

Then

$$\int_{-\infty}^{+\infty} f\left(t, x(t), x'(t)\right)dt \le \int_{-\infty}^{+\infty} \phi_r(t)dt := L. \tag{2.47}$$

Thus $|A_x| \leq L$ for all $x \in D$. Therefore,

$$\begin{aligned}(Tx)(t) &= \frac{\int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{t} \Phi^{-1}(1/\rho(s))\Phi^{-1}\left(A_x + \int_s^{+\infty} f(u, x(u), x'(u))du\right)ds\,dt}{1 - \int_{-\infty}^{+\infty} g(s)ds} \\ &\quad + \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{+\infty} f(u, x(u), x'(u))du\right)ds \\ &\leq \left(\frac{\int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{t} \Phi^{-1}(1/\rho(s))ds\,dt}{1 - \int_{-\infty}^{+\infty} g(s)ds} + \int_{-\infty}^{t} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\right)\Phi^{-1}(2L) \\ &\leq \left(\frac{\int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{t} \Phi^{-1}(1/\rho(s))ds\,dt}{1 - \int_{-\infty}^{+\infty} g(s)ds} + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\right)\Phi^{-1}(2L) =: M_1.\end{aligned} \tag{2.48}$$

On the other hand, we have

$$\Phi^{-1}(\rho(t))\left|(Tx)'(t)\right| = \left|\Phi^{-1}\left(A_x + \int_t^{\infty} f(u, x(u), x'(u))du\right)\right| \leq \Phi^{-1}(2L) =: M_2. \tag{2.49}$$

Then

$$\|(Tx)\| = \max\left\{\sup_{t\in R}(Tx)(t), \sup_{t\in R}\Phi^{-1}(\rho(t))(Tx)'(t)\right\} < \infty. \tag{2.50}$$

So, $\{Tx : x \in D\}$ is bounded.

Third, given a bounded set $D \subseteq P$, we prove that both $\{Tx : x \in D\}$ and $\{\Phi^{-1}(\rho(t))(Tx)' : x \in D\}$ are equicontinuous on each finite subinterval on R.

Then, there exists $r > 0$ such that $D \subseteq \{x \in P : \|x\| \leq r\}$. Hence

$$r = \max\left\{\sup_{t\in R}|x(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))x'(t)\right\} < +\infty. \tag{2.51}$$

Then there exists a nonnegative function $\phi_r \in L^1(-\infty, +\infty)$ such that

$$0 \leq f(t, x(t), x'(t)) = f\left(t, x(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x'(t)\right) \leq \phi_r(t), \quad t \in R. \tag{2.52}$$

Then

$$\left|A_x + \int_t^{+\infty} f(t, x(t), x'(t))dt\right| \leq \int_{-\infty}^{+\infty} \phi_r(t)dt := L. \tag{2.53}$$

For any $\epsilon > 0$, since Φ^{-1} is uniformly continuous on $[-L, L]$, there exists $\delta_1 > 0$ such that

$$\left|\Phi^{-1}(u) - \Phi^{-1}(v)\right| < \epsilon, \quad u, v \in [-L, L],\ |u - v| < \delta_1. \tag{2.54}$$

For any $a, b \in R$, $t_1, t_2 \in [a, b]$ and $x \in D$ with $t_1 \le t_2$, we have

$$\left|A_x + \int_{t_1}^{+\infty} f(t, x(t), x'(t))dt - A_x - \int_{t_2}^{+\infty} f(t, x(t), x'(t))dt\right| \le \int_{t_1}^{t_2} \phi_r(s)ds. \tag{2.55}$$

Then there exists $\delta > 0$ such that

$$\left|A_x + \int_{t_1}^{+\infty} f(t, x(t), x'(t))dt - A_x - \int_{t_2}^{+\infty} f(t, x(t), x'(t))dt\right| < \delta_1, \quad t_1, t_2 \in [a, b],\ |t_1 - t_2| < \delta. \tag{2.56}$$

Hence $t_1, t_2 \in [a, b]$, $|t_1 - t_2| < \delta$ imply that

$$\begin{aligned} &\left|\Phi^{-1}(\rho(t_1))(Tx)(t_1) - \Phi^{-1}(\rho(t_2))(Tx)(t_2)\right| \\ &\quad = \left|\Phi^{-1}\left(A_x + \int_{t_1}^{+\infty} f(t, x(t), x'(t))dt\right) - \Phi^{-1}\left(A_x + \int_{t_2}^{+\infty} f(t, x(t), x'(t))dt\right)\right| < \epsilon. \end{aligned} \tag{2.57}$$

It follows that $\{Tx : x \in D\}$ is equicontinuous on each finite subinterval on R.

On the other hand, we have

$$\begin{aligned} |(Tx)(t_1) - (Tx)(t_2)| &= \int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{+\infty} f(u, x(u), x'(u))du\right)ds \\ &\le \int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\Phi^{-1}(2L) \\ &\longrightarrow 0 \quad \text{uniformly as } t_1 \longrightarrow t_2. \end{aligned} \tag{2.58}$$

Then $\{\Phi^{-1}(\rho(t))(Tx)' : x \in D\}$ is equicontinuous on each finite subinterval on R.

At last given a bounded set $D \subseteq P$, we show that both $\{Tx : x \in D\}$ and $\{\Phi^{-1}(\rho(t))(Tx)' : x \in D\}$ are equiconvergent at $\pm\infty$, respectively.

Then, there exists $r > 0$ such that $D \subseteq \{x \in P : \|x\| \le r\}$. Hence

$$r = \max\left\{\sup_{t\in R}|x(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))x'(t)\right\} < +\infty. \tag{2.59}$$

Then there exists a nonnegative function $\phi_r \in L^1(-\infty, +\infty)$ such that

$$0 \le f(t, x(t), x'(t)) = f\left(t, x(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x'(t)\right) \le \phi_r(t), \quad t \in R. \tag{2.60}$$

Then

$$\left| A_x + \int_t^{+\infty} f(t, x(t), x'(t))dt \right| \le \int_{-\infty}^{+\infty} \phi_r(t)dt := L. \tag{2.61}$$

For any $\epsilon > 0$, since Φ^{-1} is uniformly continuous on $[-L, L]$, there exists $\delta_1 > 0$ such that

$$\left|\Phi^{-1}(u) - \Phi^{-1}(v)\right| < \epsilon, \quad u, v \in [-L, L],\ |u - v| < \delta_1. \tag{2.62}$$

Since

$$|\rho(t)\Phi((Tx)'(t)) - A_x| = \int_t^{\infty} f(u, x(u), x'(u))du \le \int_t^{\infty} \phi_r(u)du \longrightarrow 0 \tag{2.63}$$

uniformly as $t \to \infty$, we get that there exists $T > 0$ such that

$$|\rho(t)\Phi((Tx)'(t)) - A_x| < \delta_1, \quad t \ge T. \tag{2.64}$$

Hence $t > T$ implies that

$$\left|\Phi^{-1}(\rho(t))(Tx)'(t) - \Phi^{-1}(A_x)\right| = \left|\Phi^{-1}(\rho(t)\Phi((Tx)'(t))) - \Phi^{-1}(A_x)\right| < \epsilon. \tag{2.65}$$

Furthermore, we get

$$\begin{aligned}
&\left|(Tx)(t) - \frac{\int_{-\infty}^{+\infty} g(t)\int_{-\infty}^{t}\Phi^{-1}(1/\rho(s))\Phi^{-1}\left(A_x + \int_s^{+\infty} f(u, x(u), x'(u))du\right)ds\,dt}{1 - \int_{-\infty}^{+\infty} g(s)ds}\right. \\
&\qquad \left. + \int_{-\infty}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{+\infty} f(u, x(u), x'(u))du\right)ds\right| \\
&\quad = \int_t^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x + \int_s^{\infty} f(u, x(u), x'(u))du\right)ds \\
&\quad \le \int_t^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\Phi^{-1}(2L) \\
&\quad \longrightarrow 0 \text{ uniformly as } t \longrightarrow +\infty.
\end{aligned} \tag{2.66}$$

Hence $\{Tx : x \in D\}$ and $\{\Phi^{-1}(\rho(t))(Tx)' : x \in D\}$ are equiconvergent at $+\infty$.

Similarly we can how that $\{Tx : x \in D\}$ and $\{\Phi^{-1}(\rho(t))(Tx)' : x \in D\}$ are equiconvergent at $-\infty$. We omit the details.

Therefore, $T : P \to P$ is equiconvergent at $\pm\infty$. So the operator $T : P \to P$ is completely continuous.

Define the functionals on P by

$$\begin{aligned}
\gamma(y) &= \sup_{t\in R} \Phi^{-1}(\rho(t))\left|y'(t)\right|, \quad y \in P,\\
\beta(y) &= \sup_{t\in R}\left|y(t)\right|, \quad y \in P,\\
\theta(y) &= \sup_{t\in R}\left|y(t)\right|, \quad y \in P,\\
\alpha(y) &= \min_{t\in[-k,k]}\left|y(t)\right|, \quad y \in P,\\
\psi(y) &= \min_{t\in[-k,k]}\left|y(t)\right|, \quad y \in P.
\end{aligned} \tag{2.67}$$

It is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P and $\alpha(y) \le \beta(y)$ for all $y \in P$.

For $e_1, e_2, c > 0$ and $1 \ge k > 0$, define

$$\begin{aligned}
M &= \max\left\{ \frac{\int_{-\infty}^{+\infty}(1/\Phi^{-1}(\rho(s)))ds}{1-\int_{-\infty}^{+\infty} g(s)ds}, 1 \right\},\\
L_1 &= \int_{-k}^{0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(2\sqrt{|s|}\right)ds,\\
L_2 &= \int_{0}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}(2\sqrt{s})ds,\\
Q &= \min\left\{ \frac{\Phi(c)}{4+\pi}, \frac{\Phi(c)}{4+\pi}\Phi\left(\frac{1-\int_{-\infty}^{+\infty} g(s)ds}{\int_{-\infty}^{+\infty}(1/\Phi^{-1}(\rho(s)))ds}\right)\right\},\\
W &= \Phi\left(\frac{e_2}{\mu}\right)\max\left\{\Phi\left(\frac{1}{L_1}\right), \Phi\left(\frac{1}{L_2}\right)\right\},\\
E &= \frac{\Phi(e_1)}{4+\pi},\\
\delta(t) &= \begin{cases} \dfrac{1}{t^2}, & |t| \ge 1,\\ \dfrac{1}{\sqrt{|t|}}, & |t| \le 1, \end{cases}\\
\mu &= \frac{\int_{-\infty}^{-k}\Phi^{-1}(1/\rho(s))ds}{2\int_{-\infty}^{+\infty}\Phi^{-1}(1/\rho(s))ds}.
\end{aligned} \tag{2.68}$$

□

Theorem 2.8. *Suppose that (H1)–(H3) hold. Given positive constants e_1, e_2, c and $k \in (0,1)$, let Q, W, and E be as above. If*

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0, \qquad Q \geq W \tag{2.69}$$

and

(A1) $f(t, u, (1/\Phi^{-1}(\rho(t)))v) \leq \delta(t)Q$ for all $t \in R$, $u \in [0, c]$, $v \in [-c, c]$;

(A2) $f(t, u, (1/\Phi^{-1}(\rho(t)))v) \geq \delta(t)W$ for all $t \in [-k, k]$, $u \in [e_2, e_2/\mu]$, $v \in [-c, c]$;

(A3) $f(t, u, (1/\Phi^{-1}(\rho(t)))v) \leq \delta(t)E$ for all $t \in R$, $u \in [0, e_1]$, $v \in [-c, c]$;

then BVP(1.2) *has at least three positive solutions x_1, x_2, x_3 such that*

$$\sup_{t\in R} x_1(t) < e_1, \qquad \min_{t\in[-k,k]} x_2(t) > e_2, \qquad \sup_{t\in R} x_3(t) > e_1, \qquad \min_{t\in[-k,k]} x_3(t) < e_2. \tag{2.70}$$

Proof. We prove that all conditions in Lemma 2.5 are satisfied.

(i) By the definitions, it is easy to show that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P and $\alpha(y) \leq \beta(y)$ for all $y \in P$. One sees $x \in P$ is a positive solution of BVP (1.2) if and only if x is a solution of the operator equation $x = Tx$.

(ii) For $y \in P$, we have

$$\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{t} y'(s)ds + y(-\infty) \right| \\
&\leq \int_{-\infty}^{t} |y'(s)|ds + |y(-\infty)| \\
&\leq \int_{-\infty}^{t} \frac{1}{\Phi^{-1}(\rho(s))} \left|\Phi^{-1}(\rho(s))y'(s)\right| ds + \int_{-\infty}^{+\infty} g(s)|y(s)|ds \\
&\leq \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds \sup_{t\in R} \Phi^{-1}(\rho(t))|y'(t)| \int_{-\infty}^{+\infty} g(s)ds \sup_{t\in R} |y(t)|.
\end{aligned} \tag{2.71}$$

It follows that

$$\sup_{t\in R} |y(t)| \leq \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds \sup_{t\in R} \Phi^{-1}(\rho(t))|y'(t)| \int_{-\infty}^{+\infty} g(s)ds \sup_{t\in R} |y(t)|. \tag{2.72}$$

Then

$$\sup_{t\in R} |y(t)| \leq \frac{\int_{-\infty}^{+\infty} (1/\Phi^{-1}(\rho(s)))ds}{1 - \int_{-\infty}^{+\infty} g(s)ds} \sup_{t\in R} \Phi^{-1}(\rho(t))|y'(t)|. \tag{2.73}$$

Hence

$$\|y\| = \max\left\{\sup_{t\in R}|y(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))|y'(t)|\right\}$$
$$\leq \max\left\{\frac{\int_{-\infty}^{+\infty}(1/\Phi^{-1}(\rho(s)))ds}{1-\int_{-\infty}^{+\infty}g(s)ds}, 1\right\}\sup_{t\in R}\Phi^{-1}(\rho(t))|y'(t)|. \tag{2.74}$$

It follows that $\|y\| \leq M\gamma(y)$ for all $y \in P$.

(iii) Corresponding to Lemma 2.5,

$$c = c, \qquad h = \mu e_1, \qquad d = e_1, \qquad a = e_2, \qquad b = \frac{e_2}{\mu}. \tag{2.75}$$

Now, we prove that all other conditions of Lemma 2.5 hold. One sees that $0 < d < a$. The remainder is divided into five steps.

Step 1. Prove that $T: \overline{P_c} \to \overline{P_c}0$.

For $y \in \overline{P_c}$, we have $\|y\| \leq c$. Then $0 \leq y(t) \leq c$ and $-c \leq \Phi^{-1}(\rho(t))y'(t) \leq c$ for all $t \in R$. So (A1) implies that

$$f(t, y(t), y'(t)) = f\left(t, y(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))y'(t)\right) \leq \delta(t)Q, \quad t \in R. \tag{2.76}$$

We have

$$\begin{aligned}\Phi^{-1}(\rho(t))\left|(Ty)'(t)\right| &= \left|\Phi^{-1}\left(A_y + \int_t^{+\infty} f(u, y(u), y'(u))du\right)\right| \\ &\leq \Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(u, y(u), y'(u))|du\right) \\ &\leq \Phi^{-1}\left(\int_{-\infty}^{+\infty}\delta(s)Qdr\right) \\ &= \Phi^{-1}(Q)\Phi^{-1}(4+\pi) \\ &\leq c.\end{aligned} \tag{2.77}$$

Similarly to (ii), we can show that

$$\begin{aligned}0 \leq (Ty)(t) &\leq M\sup_{t\in R}\Phi^{-1}(\rho(t))\left|(Ty)'(t)\right| \\ &\leq M\Phi^{-1}\left(\int_{-\infty}^{+\infty}\delta(s)Qdr\right) \\ &= M\Phi^{-1}(Q)\Phi^{-1}(4+\pi) \\ &\leq c.\end{aligned} \tag{2.78}$$

It follows that

$$\|Ty\| = \max\left\{\sup_{t\in R}\left|(Ty)(t)\right|, \ \sup_{t\in R}\Phi^{-1}(\rho(t))\left|(Ty)'(t)\right|\right\} \le c. \tag{2.79}$$

Then $T: \ \overline{P_c} \to \overline{P_c}$.

Step 2. Prove that

$$\{y \in P(\gamma,\theta,\alpha;a,b,c) \mid \alpha(y) > a\} = \left\{y \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right) \mid \alpha(y) > e_2\right\} \ne \emptyset \tag{2.80}$$

and $\alpha(Ty) > e_2$ for every $y \in P(\gamma,\theta,\alpha;e_2,e_2/\mu,c)$.

Choose $y(t) = e_2/2\mu$ for all $t \in R$. Then $y \in P$ and

$$\alpha(y) = \frac{e_2}{2\mu} > e_2, \qquad \theta(y) = \frac{e_2}{2\mu} \le \frac{e_2}{\mu}, \qquad \gamma(y) = 0 < c. \tag{2.81}$$

It follows that $\{y \in P(\gamma,\theta,\alpha;a,b,c) \mid \alpha(y) > a\} \ne \emptyset$.

For $y \in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$\alpha(y) = \min_{t\in[-k,k]} y(t) \ge e_2, \qquad \theta(y) = \sup_{t\in R} y(t) \le \frac{e_2}{\mu}, \qquad \gamma(y) = \sup_{t\in R}\left|y'(t)\right| \le c. \tag{2.82}$$

Then

$$e_2 \le y(t) \le \frac{e_2}{\mu}, \quad t \in [-k,k], \quad \Phi^{-1}(\rho(t))\left|y'(t)\right| \le c. \tag{2.83}$$

Thus (A2) implies that

$$f(t,y(t),y'(t)) \ge \delta(t)W, \quad t \in [-k,k]. \tag{2.84}$$

Similarly to Lemma 2.6(i), we know that there exists $\tau_0 \in (0,1)$ such that $(Ty)'(\tau_0) = 0$. Then

$$(Ty)(t) = \begin{cases} (Ty)(+\infty) + \displaystyle\int_t^{+\infty}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_{\tau_0}^{s} f(u,y(u),y'(u))\,du\right)ds, & t \ge \tau_0, \\ (Ty)(-\infty) + \displaystyle\int_{-\infty}^{t}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_{s}^{\tau_0} f(u,y(u),y'(u))\,du\right)ds, & t \le \tau_0. \end{cases} \tag{2.85}$$

Since

$$\alpha(Ty) = \min_{t\in[-k,k]}(Ty)(t) \ge \mu\sup_{t\in R}(Ty)(t) = \mu(Ty)(\tau_0), \tag{2.86}$$

if $\tau_0 \geq 0$, we get

$$\begin{aligned}
\alpha(Ty) &\geq \mu\left[(Ty)(-\infty) + \int_{-\infty}^{\tau_0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_s^{\tau_0} f(u, y(u), y'(u))\,du\right)ds\right] \\
&> \mu\left[\int_{-k}^{0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_s^{0} f(u, y(u), y'(u))\,du\right)ds\right] \\
&\geq \mu\left[\int_{-k}^{0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_s^{0} \delta(u)W\,du\right)ds\right] \\
&\geq \mu\left[\int_{-k}^{0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(2\sqrt{|s|}\right)ds\right]\Phi^{-1}(W) \\
&\geq e_2.
\end{aligned} \tag{2.87}$$

If $\tau_0 < 0$, we get

$$\begin{aligned}
\alpha(Ty) &\geq \mu\left[(Ty)(+\infty) + \int_{\tau_0}^{+\infty} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_{\tau_0}^{s} f(u, y(u), y'(u))\,du\right)ds\right] \\
&> \mu\left[\int_{\tau_0}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_{\tau_0}^{s} f(u, y(u), y'(u))\,du\right)ds\right] \\
&\geq \mu\left[\int_{0}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_0^{s} \delta(u)W\,du\right)ds\right] \\
&\geq \mu\left[\int_{0}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(2\sqrt{t}\right)ds\right]\Phi^{-1}(W) \\
&\geq e_2.
\end{aligned} \tag{2.88}$$

This completes Step 2.

Step 3. Prove that $\{y \in Q(\gamma, \theta, \psi; h, d, c) \mid \beta(y) < d\} = \{y \in Q(\gamma, \theta, \psi; \mu e_1, e_1, c) | \beta(y) < e_1\} \neq \emptyset$ and

$$\beta(Ty) < e_1 \quad \text{for every } y \in Q(\gamma, \theta, \psi; h, d, c) = Q(\gamma, \theta, \psi; \mu e_1, e_1, c). \tag{2.89}$$

Choose $y(t) = \mu e_1$. Then $y \in P$, and

$$\psi(y) = \mu e_1 \geq h, \qquad \beta(y) = \theta(y) = \mu e_1 < e_1 = d, \qquad \gamma(y) = 0 \leq c. \tag{2.90}$$

It follows that $\{y \in Q(\gamma, \theta, \psi; h, d, c) \mid \beta(y) < d\} \neq \emptyset$.

For $y \in Q(\gamma, \theta, \psi; h, d, c)$, one has that

$$\psi(y) = \min_{t\in[-k,k]} y(t) \geq h = \mu e_1, \qquad \theta(y) = \sup_{t\in R} y(t) \leq d = e_1, \qquad \gamma(y) = \sup_{t\in R}\left|y'(t)\right| \leq c. \tag{2.91}$$

Hence we get that

$$0 \leq y(t) \leq e_1, \quad t \in R; \quad -c \leq \Phi^{-1}(\rho(t))y'(t) \leq c, \quad t \in R. \tag{2.92}$$

Then (A3) implies that

$$f(t, y(t), y'(t)) \leq \delta(t)E, \quad t \in R. \tag{2.93}$$

So

$$\begin{aligned}
\beta(Ty) &= M \sup_{t\in R} \Phi^{-1}(\rho(t))\left|(Ty)'(t)\right| \\
&\leq M\left|\Phi^{-1}\left(A_y + \int_t^{+\infty} f(u, y(u), y'(u))du\right)\right| \\
&\leq M\Phi^{-1}\left(\int_{-\infty}^{+\infty} \left|f(u, y(u), y'(u))\right|du\right) \\
&\leq M\Phi^{-1}\left(\int_{-\infty}^{+\infty} \delta(u)Edu\right) \\
&\leq M\Phi^{-1}(4+\pi)\Phi^{-1}(E) \\
&= e_1 = d.
\end{aligned} \tag{2.94}$$

This completes Step 3.

Step 4. Prove that $\alpha(Ty) > a$ for $y \in P(\gamma, \alpha; a, c)$ with $\theta(Ty) > b$;

For $y \in P(\gamma, \alpha; a, c) = P(\gamma, \alpha; e_2, c)$ with $\theta(Ty) > b = e_2/\mu$, we have that $\alpha(y) = \min_{t\in[-k,k]} y(t) \geq e_2$ and $\gamma(y) = \sup_{t\in R}\Phi^{-1}(\rho(t))|y'(t)| \leq c$ and $\sup_{t\in R}(Ty)(t) > e_2/\mu$. Then

$$\alpha(Ty) = \min_{t\in[-k,k]} (Ty)(t) \geq \mu\beta(Ty) > \mu\frac{e_2}{\mu} = e_2 = a. \tag{2.95}$$

This completes Step 4.

Step 5. Prove that $\beta(Ty) < d$ for each $y \in Q(\gamma, \beta; d, c)$ with $\psi(Ty) < h$.

For $y \in Q(\gamma, \beta; d, c)$ with $\psi(Ty) < d$, we have $\gamma(y) = \sup_{t\in R}\Phi^{-1}(\rho(t))|y'(t)| \leq c$ and $\beta(y) = \sup_{t\in R} y(t) \leq d = e_1$ and $\psi(Ty) = \min_{t\in[-k,k]}(Ty)(t) < h = e_1\mu$. Then

$$\beta(Ty) = \sup_{t\in R}(Ty)(t) \leq \frac{1}{\mu}\min_{t\in[-k,k]}(Ty)(t) < \frac{1}{\mu}e_1\mu = e_1 = d. \tag{2.96}$$

This completes the Step 5.

Then Lemma 2.5 implies that T has at least three fixed points x_1, x_2, and x_3 such that

$$\beta(x_1) < e_1, \qquad \alpha(x_2) > e_2, \qquad \beta(x_3) > e_1, \qquad \alpha(x_3) < e_2. \tag{2.97}$$

Hence BVP(1.2) has three decreasing positive solutions x_1, x_2 and x_3 such that (2.70) holds. The proof is complete. □

3. Examples

Now, we present an example, whose three positive solutions cannot be obtained by theorems in known papers, to illustrate the main results.

Example 3.1. Consider the following BVP

$$\begin{gathered} \left[e^{3t^2}\left(x'(t)\right)^3\right]' + f\left(t, x(t), x'(t)\right) = 0, \quad t \in R, \\ x(-\infty) = 0, \\ x(+\infty) = 0. \end{gathered} \tag{3.1}$$

Corresponding to BVP(1.2), one sees that $\phi(x) = x^3$, $\phi^{-1}(x) = x^{1/3}$, $g(t) = h(t) \equiv 0$, $\rho(t) = e^{3t^2}$, $f : R \times R \times R \to [0, \infty)$ is nonnegative and continuous and is defined by

$$\begin{gathered} f(t, x, y) = \delta(t)\left(f_0(x) + g_0\left(e^{t^2}|y|\right)\right), \\ \delta(t) = \begin{cases} \dfrac{1}{t^2}, & |t| \geq 1, \\ \dfrac{1}{\sqrt{|t|}}, & |t| \leq 1. \end{cases} \end{gathered} \tag{3.2}$$

Choose $k = 1$, $e_1 = 50$, $e_2 = 250$, $c = 40000$. By direct computation, we see that Q, W, and E are given by

$$\begin{aligned} M &= \max\left\{\frac{\int_{-\infty}^{+\infty}\left(1/\Phi^{-1}(\rho(s))\right)ds}{1 - \int_{-\infty}^{+\infty} g(s)ds}, 1\right\} = \frac{\sqrt{\pi}}{2}, \\ L_1 &= \int_{-k}^{0} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(2\sqrt{|s|}\right)ds = 2^{1/3}\int_{-1}^{0} e^{-s^2}|s|^{1/6}ds > 0.06, \\ L_2 &= \int_{0}^{k} \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}(2\sqrt{s})ds = 2^{1/3}\int_{0}^{1} e^{-s^2}s^{1/6}ds > 0.06, \end{aligned}$$

$$\mu = \frac{\int_{-\infty}^{-k} \Phi^{-1}(1/\rho(s))ds}{2\int_{-\infty}^{+\infty} \Phi^{-1}(1/\rho(s))ds} = \frac{\int_{-\infty}^{-1} e^{-s^2}ds}{\sqrt{\pi}} > 0.37,$$

$$Q = \min\left\{\frac{\Phi(c)}{4+\pi}, \frac{\Phi(c)}{4+\pi}\Phi\left(\frac{1-\int_{-\infty}^{+\infty} g(s)ds}{\int_{-\infty}^{+\infty}(1/\Phi^{-1}(\rho(s)))ds}\right)\right\} = \frac{64\times 10^{12}}{4+\pi} > 8.53\times 10^{12},$$

$$W = \Phi\left(\frac{e_2}{\mu}\right)\max\left\{\Phi\left(\frac{1}{L_1}\right), \Phi\left(\frac{1}{L_2}\right)\right\} < 16.67^3 \times 675.7^3 < 3.43\times 10^{11},$$

$$E = \frac{\Phi(e_1)}{4+\pi} = \frac{2500}{4+\pi} > 333.33. \tag{3.3}$$

One can show that

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0, \qquad Q \geq W. \tag{3.4}$$

Suppose that

$$f_0(x) = \begin{cases} 166.67, & x \in [0,50], \\ 166.67 + \dfrac{44.36\times 10^{11} - 166.67}{250-50}(x-50), & x \in [50,250], \\ 44.36\times 10^{11}, & x \in [250,40000], \\ 44.36\times 10^{11} e^{x-40000}, & x \geq 40000, \end{cases} \tag{3.5}$$

$$|g_0(y)| \leq 10, \quad \forall y \in R.$$

From

$$f\left(t, x, e^{-t^2}y\right) = \delta(t)\left[f_0(x) + g_0(y)\right], \tag{3.6}$$

it is easy to show that

(A_1) $f(t,u,e^{-t^2}v) \leq 8.53\times 10^{12}\delta(t)$ for all $t \in R, u \in [0,400000], v \in [-40000,40000]$;

(A_2) $f(t,u,e^{-t^2}v) \geq 3.43\times 10^{11}\delta(t)$ for all $t \in [-1,1], u \in [250,1000], v \in [-40000,40000]$;

(A_3) $f(t,u,e^{-t^2}v) \leq 333.33\delta(t)$ for all $t \in R, u \in [0,50], v \in [-40000,40000]$;

then Theorem 2.8 implies that BVP(3.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\begin{aligned} &\sup_{t\in R} x_1(t) < 50, \qquad \min_{t\in[-1,1]} x_2(t) > 250, \\ &\sup_{t\in R} x_3(t) > 50, \qquad \min_{t\in[-1,1]} x_3(t) < 250. \end{aligned} \tag{3.7}$$

Remark 3.2. Example 3.1 implies that there is a large number of functions that satisfy the conditions of Theorem 2.8. In addition, the conditions of Theorem 2.8 are also easy to check.

Acknowledgments

This work was supported by Natural Science Foundation of Guangdong provinces Grant (no S2011010001900) and the Foundation for High-level talents in Guangdong Higher Education Project.

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