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Research Article

Existence of Two Positive Periodic Solutions for a Neutral Multi-Delay Logarithmic Population Model with a Periodic Harvesting Rate

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By using an abstract existence result based on a coincidence degree theory for k-set contractive mapping, a new result is obtained for the existence of at least two positive periodic solutions for a neutral multidelay logarithmic population model with a periodic harvesting rate. An example is given to illustrate the effectiveness of the result.

1. Introduction

In recent years, many papers have been published on the existence of positive periodic solutions for neutral delay logarithmic population models by using a topological degree theory for k-set contractive mapping (see, e.g., [1–5]). Recently, Xia [6] obtained some new sufficient conditions for the existence and uniqueness of an almost periodic solution of a multispecies logarithmic population model with feedback controls. However, few papers deal with the existence of multiple positive periodic solutions for neutral multidelay logarithmic population models with harvesting. The main difficulty is hard to obtain a *priori* bounds on solutions for neutral multi-delay models with harvesting.

In this paper, we consider the following neutral multi-delay logarithmic population model of single-species population growth with a periodic harvesting rate

$$\frac{dy(t)}{dt} = y(t)\left[a(t) - \sum_{j=1}^{n} b_j(t)\ln y\left(t-\sigma_j(t)\right) - \sum_{i=1}^{m} c_i(t)\frac{d}{dt}\ln y(t-\tau_i(t))\right] - h(t), \tag{1.1}$$

where $a(t)$, $b_j(t)$, $\sigma_j(t)$ $(j = 1,2,\ldots,n)$, $c_i(t)$, $\tau_i(t)$ $(i = 1,2,\ldots,m)$, $h(t)$ are nonnegative continuous T-periodic functions, and $h(t)$ denotes the harvesting rate. When $h(t) \equiv 0$, (1.1) was considered by [2–5]. When $n = m = 1$, $h(t) \equiv 0$, and $\sigma_1(t)$, $\tau_1(t)$ are constants, (1.1) was considered by [7].

The purpose of this paper is to establish the existence of at least two positive periodic solutions for a neutral multi-delay logarithmic population model (1.1) by using a coincidence degree theory for k-set contractions. Motivated by the work of Chen [8], some novel techniques are employed to find a *priori* bounds on solutions.

2. Preliminaries

We now briefly state the part of the coincidence degree theory for k-set contractive mapping developed by Hetzer [9, 10]. For more details, we refer to [11].

Let Z be a Banach space. For a bounded subset $A \subset Z$, let $\Gamma_Z(A)$ denote the (Kuratowski) measure of noncompactness defined by

$$\Gamma_Z(A) = \inf\left\{\delta > 0 : \exists \text{ a finite number of subsets } A_i \subset A, A = \bigcup_i A_i, \ \operatorname{diam}(A_i) \le \delta\right\}. \tag{2.1}$$

Here, diam (A_i) denotes the maximum distance between the points in the set A_i.

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively and Ω a bounded open subset of X. A continuous and bounded mapping $N : \overline{\Omega} \to Z$ is called k-set contractive if for any bounded $A \subset \overline{\Omega}$, we have

$$\Gamma_Z(N(A)) \le k\Gamma_X(A). \tag{2.2}$$

Also, for a continuous and bounded map $T : X \to Y$, we define

$$l(T) = \sup\{r \ge 0 : \forall \text{ bounded subset } A \subset X, r\Gamma_X(A) \le \Gamma_Y(T(A))\}. \tag{2.3}$$

Let $L : \operatorname{dom} L \subset X \to Z$ be a linear mapping and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim Ker L = codim Im $L < +\infty$ and Im L is closed in Z. If L is a Fredholm mapping of index zero, there then exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that Im P = Ker L, Im L = Ker Q = Im$(I - Q)$. If we define $L_P : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ as the restriction $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$ of L to dom $L \cap$ Ker P, then L_P is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-k-set contractive on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is k-set contractive. Since Im Q is isomorphic to Ker L, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 ([11, Proposition XI.2.]). *Let L be a closed Fredholm mapping of index zero and let $N : \overline{\Omega} \to Z$ be k'-set contractive with*

$$0 \le k' < l(L). \tag{2.4}$$

Then $N : \overline{\Omega} \to Z$ is a L-k-set contraction with constant $k = k'/l(L) < 1$.

The following lemma [[11, page 213] will play a key role in this paper.

Lemma 2.2. *Let L be a Fredholm mapping of index zero and let $N : \overline{\Omega} \to Z$ be L-k-set contractive on $\overline{\Omega}$, $k < 1$. Suppose*

(i) $Lx \neq \lambda Nx$ *for every* $x \in \operatorname{dom} L \cap \partial\Omega$ *and every* $\lambda \in (0,1)$;

(ii) $QNx \neq 0$ *for every* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(iii) *Brouwer degree* $\deg_B(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

3. Main Result

Let C_T^0 denote the linear space of real valued continuous T-periodic functions on R. The linear space C_T^0 is a Banach space with the usual norm for $x \in C_T^0$ given by $|x|_0 = \max_{t\in R}|x(t)|$. Let C_T^1 denote the linear space of T-periodic functions with the first-order continuous derivative. C_T^1 is a Banach space with norm $|x|_1 = \max\{|x|_0, |x'|_0\}$.

Let $X = C_T^1$ and $Z = C_T^0$ and let $L : X \to Z$ be given by $Lx = dx/dt$. Since $|Lx|_0 = |x'|_0 \leq |x|_1$, we see that L is a bounded (with bound = 1) linear map.

Under the transformation of $y(t) = e^{x(t)}$, (1.1) can be rewritten as

$$x'(t) = a(t) - \sum_{j=1}^{n} b_j(t)x(t-\sigma_j(t)) - \sum_{i=1}^{m} c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t)) - \frac{h(t)}{e^{x(t)}}. \tag{3.1}$$

Next define a nonlinear map $N : X \to Z$ by

$$N(x)(t) = a(t) - \sum_{j=1}^{n} b_j(t)x(t-\sigma_j(t)) - \sum_{i=1}^{m} c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t)) - \frac{h(t)}{e^{x(t)}}. \tag{3.2}$$

Now, if $Lx = Nx$ for some $x \in X$, then the problem (3.1) has a T-periodic solution $x(t)$.

In the following, we denote

$$\overline{g} = \frac{1}{T}\int_0^T g(t)dt, \qquad g^l = \min_{t\in[0,T]} g(t), \qquad g^u = \max_{t\in[0,T]} g(t), \tag{3.3}$$

where $g(t)$ is a continuous nonnegative T-periodic solution.

From now on, we always assume that

(H_1) $a(t), b_j(t) \in C(R,(0,+\infty))$, $\sigma_j(t), c_i(t), \tau_i(t) \in C^1(R,(0,+\infty))$, for all $j \in \{1,2,\ldots,n\}$, for all $i \in \{1,2,\ldots,m\}$.

(H_2) $\sigma_j'(t) < 1, \tau_i'(t) < 1$, for all $t \in R$, and

$$\Gamma(t) = \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1-\sigma_j'(\mu_j(t))} - \sum_{i=1}^{m} \frac{c_i'(\gamma_i(t))}{1-\tau_i'(\gamma_i(t))} > 0, \tag{3.4}$$

where $\mu_j(t)$ is the inverse function of $t-\sigma_j(t)$, $\gamma_i(t)$ is the inverse function of $t-\tau_i(t)$, for all $j \in \{1,2,\ldots,n\}$, for all $i \in \{1,2,\ldots,m\}$.

(H_3) Let

$$1+\ln\frac{h^l}{\Gamma^u} < \frac{a^u}{\Gamma^l} \le \frac{h^l}{\Gamma^u} < 1. \tag{3.5}$$

(H_4) Let

$$\begin{gathered} T\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)+\sum_{i=1}^{m}c_i^u < 1, \\ \max\left\{1+\ln\frac{h^u}{\sum_{j=1}^{n} b_j^l}, 0\right\} < \frac{a^l-\sum_{i=1}^{m} c_i^u\left(1-\tau_i'\right)^u M_0}{\sum_{j=1}^{n} b_j^u} < R_0, \end{gathered} \tag{3.6}$$

where

$$\begin{gathered} M_0 = \frac{a^u+\sum_{j=1}^{n} b_j^u R_0 + h^u e^{R_0}}{1-\sum_{i=1}^{m} c_i^u\left(1-\tau_i'\right)^u}, \quad R_0 = q+p, \\ q = 2\left|\ln\frac{h^l}{\Gamma^u}\right|, \quad p = \frac{2T\overline{a}+\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)Tq}{1-T\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)-\sum_{i=1}^{m} c_i^u}. \end{gathered} \tag{3.7}$$

We first give some technological lemmas.

Set

$$f(x) = d - x - re^{-x}, \quad x \in (-\infty,+\infty). \tag{3.8}$$

Lemma 3.1. *Assume that d, r are positive constants such that*

$$1+\ln r < d. \tag{3.9}$$

Then there exist x_*^-, x_*^+ such that

$$\begin{gathered} f\left(x_*^-\right) = f\left(x_*^+\right) = 0, \\ x_*^- < \ln r < x_*^+ < d, \\ f(x) > 0 \quad \text{for } x \in \left(x_*^-, x_*^+\right); \qquad f(x) < 0 \quad \text{for } x \in \left(-\infty, x_*^-\right) \cup \left(x_*^+,+\infty\right). \end{gathered} \tag{3.10}$$

If one assumes further that

$$d \le r < 1, \tag{3.11}$$

then the following inequalities also hold:

$$2\ln r < x_*^- < \ln r < x_*^+ \leq 0. \tag{3.12}$$

Proof. Clearly, $f'(x) = r/e^x - 1 = 0$ if and only if $x = \ln r$. Therefore, noticing that $\lim_{x\to\pm\infty} f(x) = -\infty$, we have

$$\sup_{x\in(-\infty,+\infty)} f(x) = d - \ln r - 1 > 0. \tag{3.13}$$

Set

$$g(r) = 2\ln r + \frac{1}{r} - r. \tag{3.14}$$

Since

$$g'(r) = \frac{2}{r} - \frac{1}{r^2} - 1 = -\frac{(1-r)^2}{r^2} < 0 \quad \text{for } 0 < r < 1, \tag{3.15}$$

$g(r)$ is monotonically decreasing on $(0,1]$.

Therefore, we have

$$g(r) > g(1) = 0, \tag{3.16}$$

that is,

$$2\ln r + \frac{1}{r} > r, \tag{3.17}$$

which implies

$$f(2\ln r) = d - 2\ln r - \frac{1}{r} < d - r \leq 0. \tag{3.18}$$

Again, noticing that

$$f(0) = d - r \leq 0, \tag{3.19}$$

by the monotonicity of the function $f(x)$ on the interval $(-\infty, \ln r)$ and $(\ln r, +\infty)$, it is easy to see that the assertion holds. □

Set

$$F(x) = \frac{a^l - \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^u} - x - \frac{h^u}{\sum_{j=1}^n b_j^l} e^{-x},$$
$$G(x) = \frac{\overline{a}}{\sum_{j=1}^n \overline{b}_j} - x - \frac{\overline{h}}{\sum_{j=1}^n \overline{b}_j} e^{-x}, \tag{3.20}$$
$$H(x) = \frac{a^u + \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^l} - x - \frac{h^l}{\sum_{j=1}^n b_j^u} e^{-x}.$$

Lemma 3.2. *Assume that $(H_1), (H_2)$, and (H_4) hold. Then the following assertions hold.*

(1) There exist u^-, u^+ such that

$$F(u^-) = F(u^+) = 0,$$
$$u^- < \ln \frac{h^u}{\sum_{j=1}^n b_j^l} < u^+ < \frac{a^l - \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^u}, \tag{3.21}$$
$$F(x) > 0 \quad \text{for } x \in (u^-, u^+); \qquad F(x) < 0 \quad \text{for } x \in (-\infty, u^-) \cup (u^+, +\infty).$$

(2) There exist x^-, x^+ such that

$$G(x^-) = G(x^+) = 0,$$
$$x^- < \ln \frac{\overline{h}}{\sum_{j=1}^n \overline{b}_j} < x^+ < \frac{\overline{a}}{\sum_{j=1}^n \overline{b}_j}, \tag{3.22}$$
$$G(x) > 0 \quad \text{for } x \in (x^-, x^+); \qquad G(x) < 0 \quad \text{for } x \in (-\infty, x^-) \cup (x^+, +\infty).$$

(3) There exist l^-, l^+ such that

$$H(l^-) = H(l^+) = 0,$$
$$l^- < \ln \frac{h^l}{\sum_{j=1}^n b_j^u} < l^+ < \frac{a^u + \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^l}, \tag{3.23}$$
$$H(x) > 0 \quad \text{for } x \in (l^-, l^+); \qquad H(x) < 0 \quad \text{for } x \in (-\infty, l^-) \cup (l^+, +\infty).$$

(4)

$$l^- < x^- < u^- < u^+ < x^+ < l^+. \tag{3.24}$$

Proof. Noticing that

$$\frac{a^l - \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^u} < \frac{\overline{a}}{\sum_{j=1}^n \overline{b}_j} < \frac{a^u + \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^l},$$
$$\frac{h^l}{\sum_{j=1}^n b_j^u} \le \frac{\overline{h}}{\sum_{j=1}^n \overline{b}_j} \le \frac{h^u}{\sum_{j=1}^n b_j^l}, \tag{3.25}$$

we have

$$F(x) < G(x) < H(x). \tag{3.26}$$

It follows from $(H_1), (H_2),$ and (H_4), (3.25) that

$$1 + \ln \frac{h^u}{\sum_{j=1}^n b_j^l} < \frac{a^l - \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^u},$$
$$1 + \ln \frac{\overline{h}}{\sum_{j=1}^n \overline{b}_j} < \frac{\overline{a}}{\sum_{j=1}^n \overline{b}_j}, \tag{3.27}$$
$$1 + \ln \frac{h^l}{\sum_{j=1}^n b_j^u} < \frac{a^u + \sum_{i=1}^m c_i^u (1-\tau_i')^u M_0}{\sum_{j=1}^n b_j^l}.$$

Therefore, by Lemma 3.1, the assertions (1)–(3) hold. Furthermore, by (3.26) and the assertions (1)–(3), the assertion (4) also holds. □

Lemma 3.3 (see [12]). *L is a Fredholm map of index* 0 *and satisfies*

$$l(L) \ge 1. \tag{3.28}$$

Lemma 3.4 (see [2]). *Suppose* $\sigma \in C_T^1$ *and* $\sigma'(t) < 1$, *for all* $t \in [0,T]$. *Then the function* $t - \sigma(t)$ *has an inverse function* $\mu(t)$ *satisfying* $\mu \in C(R,R)$ *with* $\mu(a+T) = \mu(a) + T$.

Lemma 3.5. *Assume that* (H_1)*–*(H_4) *hold. Let* $k_0 = \sum_{i=1}^m c_i^u (1-\tau_i')^u$, *and*

$$\Omega = \left\{ x \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} x(t) \in (l^- - \delta,\ R_0 + \delta), \\ \min_{t\in[0,T]} x(t) \in (l^- - \delta,\ l^+ + \delta), \\ \max_{t\in[0,T]} |x'(t)| < M_0. \end{array} \right\}, \tag{3.29}$$

where $0 < \delta < l^-$. *Then* $N : \overline{\Omega} \to Z$ *is a* k_0*-set-contractive map.*

Proof. The proof is similar to that of Lemma 3.3 in [12], so we omit it. □

Theorem 3.6. *Assume that* (H_1)*–*(H_4) *hold. Then* (1.1) *has at least two positive* T*-periodic solutions.*

Proof. Let $Lx = \lambda Nx$ for $x \in X$, that is,

$$x'(t) = \lambda\left[a(t) - \sum_{j=1}^{n} b_j(t)x\left(t-\sigma_j(t)\right) - \sum_{i=1}^{m} c_i(t)\left(1-\tau_i'(t)\right)x'\left(t-\tau_i(t)\right) - \frac{h(t)}{e^{x(t)}}\right], \quad \lambda \in (0,1). \tag{3.30}$$

Therefore, we have

$$x'(t) = \lambda\left[a(t) - \sum_{j=1}^{n} b_j(t)x\left(t-\sigma_j(t)\right) - \sum_{i=1}^{m} c_i(t)[x(t-\tau_i(t))]' - \frac{h(t)}{e^{x(t)}}\right], \quad \lambda \in (0,1). \tag{3.31}$$

By (3.31), we have

$$\left[x(t) + \lambda\sum_{i=1}^{m} c_i(t)x(t-\tau_i(t))\right]' = \lambda\left[a(t) - \sum_{j=1}^{n} b_j(t)x\left(t-\sigma_j(t)\right) + \sum_{i=1}^{m} c_i'(t)x(t-\tau_i(t)) - \frac{h(t)}{e^{x(t)}}\right]. \tag{3.32}$$

Integrating this identity leads to

$$\int_0^T \left[\sum_{j=1}^{n} b_j(t)x\left(t-\sigma_j(t)\right) - \sum_{i=1}^{m} c_i'(t)x(t-\tau_i(t)) + \frac{h(t)}{e^{x(t)}}\right]dt = \int_0^T a(t)dt. \tag{3.33}$$

By Lemma 3.4, we have

$$\mu_j(T+s) = T + \mu_j(s), \qquad \gamma_i(T+s) = T + \gamma_i(s), \tag{3.34}$$

where $t = \mu_j(s)$ is the inverse function of $s = t - \sigma_j(t)$, and $t = \gamma_i(s)$ is the inverse function of $s = t - \tau_i(t)$, for all $j \in \{1, 2, \ldots, n\}$ and for all $i \in \{1, 2, \ldots, m\}$.

Then

$$\begin{aligned}
\int_0^T \sum_{j=1}^{n} b_j(t)x\left(t-\sigma_j(t)\right)dt &= \sum_{j=1}^{n} \int_{-\sigma_j(0)}^{T-\sigma_j(T)} b_j\left(\mu_j(s)\right)\frac{x(s)}{1-\sigma_j'\left(\mu_j(s)\right)}ds \\
&= \sum_{j=1}^{n} \int_0^T \frac{b_j\left(\mu_j(s)\right)}{1-\sigma_j'\left(\mu_j(s)\right)}x(s)ds, \\
\int_0^T \sum_{i=1}^{m} c_i'(t)x(t-\tau_i(t))dt &= \sum_{i=1}^{m} \int_{-\tau_i(0)}^{T-\tau_i(T)} c_i'\left(\gamma_i(s)\right)\frac{x(s)}{1-\tau_i'\left(\gamma_i(s)\right)}ds \\
&= \sum_{i=1}^{m} \int_0^T \frac{c_i'\left(\gamma_i(s)\right)}{1-\tau_i'\left(\gamma_i(s)\right)}x(s)ds.
\end{aligned} \tag{3.35}$$

From (3.33)–(3.35), we have

$$\int_0^T \left[a(s) - \left(\sum_{j=1}^{n} \frac{b_j(\mu_j(s))}{1-\sigma_j'(\mu_j(s))} - \sum_{i=1}^{m} \frac{c_i'(\gamma_i(s))}{1-\tau_i'(\gamma_i(s))} \right) x(s) - \frac{h(s)}{e^{x(s)}} \right] ds = 0, \tag{3.36}$$

which implies

$$a(\eta) - \Gamma(\eta)x(\eta) - \frac{h(\eta)}{e^{x(\eta)}} = 0, \tag{3.37}$$

for some $\eta \in [0, T]$.

Therefore, by (H_2), we have

$$\frac{a^u}{\Gamma^l} - x(\eta) - \frac{h^l/\Gamma^u}{e^{x(\eta)}} \geq 0. \tag{3.38}$$

By (H_3) and Lemma 3.1, we have

$$2\ln\frac{h^l}{\Gamma^u} < x(\eta) \leq 0. \tag{3.39}$$

Set

$$q := 2\left|\ln\frac{h^l}{\Gamma^u}\right|, \tag{3.40}$$

then we have

$$|x(t)| \leq |x(\eta)| + \int_0^T |x'(t)|dt \leq q + \int_0^T |x'(t)|dt, \tag{3.41}$$

which implies

$$|x|_0 \leq q + \int_0^T |x'(t)|dt. \tag{3.42}$$

It follows from (3.30) that

$$\begin{aligned}\int_0^T |x'(t)|dt &= \lambda\int_0^T\left|a(t)-\sum_{j=1}^n b_j(t)x(t-\sigma_j(t))-\sum_{i=1}^m c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t))-\frac{h(t)}{e^{x(t)}}\right|dt \\ &\le \lambda\int_0^T\left|a(t)-\sum_{j=1}^n b_j(t)x(t-\sigma_j(t))-\sum_{i=1}^m c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t))\right|dt+\int_0^T\frac{h(t)}{e^{x(t)}}dt.\end{aligned} \tag{3.43}$$

By (3.36) and (H_2), we have

$$\int_0^T\frac{h(t)}{e^{x(t)}}dt \le \int_0^T a(t)dt+\int_0^T\Gamma(t)dt|x|_0 = T\overline{a}+T\overline{\Gamma}|x|_0. \tag{3.44}$$

By this and (3.43), we obtain

$$\int_0^T|x'(t)|dt \le T\left[2\overline{a}+\overline{\Gamma}|x|_0+\sum_{j=1}^n\overline{b}_j|x|_0\right]+\sum_{i=1}^m\int_0^T|c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t))|dt. \tag{3.45}$$

Meanwhile,

$$\sum_{i=1}^m\int_0^T|c_i(t)(1-\tau_i'(t))x'(t-\tau_i(t))|dt=\sum_{i=1}^m\int_0^T c_i(\gamma_i(s))|x'(s)|ds \le \sum_{i=1}^m c_i^u\int_0^T|x'(s)|ds. \tag{3.46}$$

Substituting (3.42) and (3.46) into (3.45) gives

$$\begin{aligned}\int_0^T|x'(t)|dt &\le T\left[2\overline{a}+\overline{\Gamma}|x|_0+\sum_{j=1}^n\overline{b}_j|x|_0\right]+\sum_{i=1}^m c_i^u\int_0^T|x'(s)|ds \\ &\le T\left[2\overline{a}+\left(\overline{\Gamma}+\sum_{j=1}^n\overline{b}_j\right)\left(q+\int_0^T|x'(t)|dt\right)\right]+\sum_{i=1}^m c_i^u\int_0^T|x'(s)|ds \\ &\le T\left[2\overline{a}+\left(\overline{\Gamma}+\sum_{j=1}^n\overline{b}_j\right)q\right]+\left[T\left(\overline{\Gamma}+\sum_{j=1}^n\overline{b}_j\right)+\sum_{i=1}^m c_i^u\right]\int_0^T|x'(t)|dt.\end{aligned} \tag{3.47}$$

Since

$$T\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)+\sum_{i=1}^{m}c_i^u<1, \tag{3.48}$$

we have

$$\int_0^T\left|x'(t)\right|dt<\frac{2T\overline{a}+\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)Tq}{1-T\left(\overline{\Gamma}+\sum_{j=1}^{n}\overline{b}_j\right)-\sum_{i=1}^{m}c_i^u}:=p. \tag{3.49}$$

Then,

$$|x|_0\le q+\int_0^T\left|x'(t)\right|dt\le q+p:=R_0. \tag{3.50}$$

Again from (3.30), we get

$$\left|x'\right|_0<a^u+\sum_{j=1}^{n}b_j^u|x|_0+\sum_{i=1}^{m}c_i^u\left(1-\tau_i'\right)^u\left|x'\right|_0+h^ue^{R_0}. \tag{3.51}$$

Since $\sum_{i=1}^{m}c_i^u(1-\tau_i')^u<1$, we have

$$\left|x'\right|_0<\frac{a^u+\sum_{j=1}^{n}b_j^uR_0+h^ue^{R_0}}{1-\sum_{i=1}^{m}c_i^u\left(1-\tau_i'\right)^u}:=M_0. \tag{3.52}$$

Choose t_M, $t_m\in[0,T]$, such that

$$x(t_M)=\max_{t\in[0,T]}x(t),\qquad x(t_m)=\min_{t\in[0,T]}x(t). \tag{3.53}$$

Then, it is clear that

$$x'(t_M)=0,\qquad x'(t_m)=0. \tag{3.54}$$

From this and (3.30), we obtain that

$$a(t_M)=\sum_{j=1}^{n}b_j(t_M)x\left(t_M-\sigma_j(t_M)\right)+\sum_{i=1}^{m}c_i(t_M)\left(1-\tau_i'(t_M)\right)x'\left(t_M-\tau_i(t_M)\right)+\frac{h(t_M)}{e^{x(t_M)}}, \tag{3.55}$$

$$a(t_m)=\sum_{j=1}^{n}b_j(t_m)x\left(t_m-\sigma_j(t_m)\right)+\sum_{i=1}^{m}c_i(t_m)\left(1-\tau_i'(t_m)\right)x'\left(t_m-\tau_i(t_m)\right)+\frac{h(t_m)}{e^{x(t_m)}}. \tag{3.56}$$

It follows from (3.55) that

$$a(t_M) - \sum_{i=1}^{m} c_i(t_M)\left(1 - \tau_i'(t_M)\right) M_0 - \sum_{j=1}^{n} b_j(t_M) x(t_M) - \frac{h(t_M)}{e^{x(t_M)}} \leq 0, \tag{3.57}$$

which implies

$$\frac{a^l - \sum_{i=1}^{m} c_i^u \left(1 - \tau_i'\right)^u M_0}{\sum_{j=1}^{n} b_j^u} - x(t_M) - \frac{h^u}{\sum_{j=1}^{n} b_j^l} e^{-x(t_M)} \leq 0. \tag{3.58}$$

By the assertion (1) of Lemma 3.2, we have

$$x(t_M) \leq u^- \quad \text{or} \quad x(t_M) \geq u^+. \tag{3.59}$$

It follows from (3.56) that

$$a(t_m) + \sum_{i=1}^{m} c_i(t_m)\left(1 - \tau_i'(t_m)\right) M_0 - \sum_{j=1}^{n} b_j(t_m) x(t_m) - \frac{h(t_m)}{e^{x(t_m)}} \geq 0, \tag{3.60}$$

which implies

$$\frac{a^u + \sum_{i=1}^{m} c_i^u \left(1 - \tau_i'\right)^u M_0}{\sum_{j=1}^{n} b_j^l} - x(t_m) - \frac{h^l}{\sum_{j=1}^{n} b_j^u} e^{-x(t_m)} \geq 0. \tag{3.61}$$

By the assertion (3) of Lemma 3.2, we have

$$l^- \leq x(t_m) \leq l^+. \tag{3.62}$$

Hence, it follows from (3.50), (3.59), and (3.62) that

$$\begin{gathered} x(t_M) \in \left[l^-, u^-\right] \cup \left[u^+, R_0\right], \\ x(t_m) \in \left[l^-, l^+\right]. \end{gathered} \tag{3.63}$$

Clearly, $l^\pm, u^\pm$ are independent of λ. Now, let us consider $QN(x)$ with $x \in R$. Note that

$$QN(x) = \overline{a} - \sum_{j=1}^{n} \overline{b}_j x - \frac{\overline{h}}{e^x}. \tag{3.64}$$

It follows from the assertion (2) of Lemma 3.2 that $QN(x) = 0$ has two distinct solutions:

$$\widetilde{u}_1 = x^-, \qquad \widetilde{u}_2 = x^+. \tag{3.65}$$

By the assertion (4) of Lemma 3.2, one can take $v^-, v^+ > 0$ such that

$$u^- < v^- < v^+ < u^+. \tag{3.66}$$

Let

$$\Omega_1 = \left\{ x \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} x(t) \in (l^- - \delta, v^-), \\ \min_{t\in[0,T]} x(t) \in (l^- - \delta, l^+ + \delta), \\ \max_{t\in[0,T]} |x'(t)| < M_0. \end{array} \right\},$$
$$\Omega_2 = \left\{ x \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} x(t) \in (v^+, R_0 + \delta), \\ \min_{t\in[0,T]} x(t) \in (l^- - \delta, l^+ + \delta), \\ \max_{t\in[0,T]} |x'(t)| < M_0. \end{array} \right\}. \tag{3.67}$$

Then Ω_1, Ω_2 are bounded open subsets of X. Clearly, $\Omega_i \subset \Omega$ $(i = 1,2)$, where Ω as defined in Lemma 3.5. It follows from Lemma 3.5 that $N : \overline{\Omega}_i \to Z$ is a k_0-set-contractive map $(i = 1,2)$. Therefore, it follows from Lemmas 2.1 and 3.3 that $N : \overline{\Omega}_i \to Z$ is L-k-set contractive on $\overline{\Omega}_i$ $(i = 1,2)$ with $k = k_0/l(L) \leq k_0 < 1$.

By the assertion (4) of Lemma 3.2, (3.52), (3.63), (3.65), and (3.66), it is easy to verify that Ω_i satisfies the assumptions (i) and (ii) in Lemma 2.2 $(i = 1,2)$. By the assertion (2) of Lemma 3.2, a direct computation gives

$$\deg_B\{JQN, \Omega_1 \cap \operatorname{Ker} L, 0\} = \operatorname{sgn}\left(-\sum_{j=1}^{n} \overline{b}_j + \frac{\overline{h}}{e^{x^-}} \right) = 1,$$
$$\deg_B\{JQN, \Omega_2 \cap \operatorname{Ker} L, 0\} = \operatorname{sgn}\left(-\sum_{j=1}^{n} \overline{b}_j + \frac{\overline{h}}{e^{x^+}} \right) = -1. \tag{3.68}$$

Here, J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 2.2 $(i = 1,2)$. Hence, (3.1) has at least two T-periodic solutions $x_i^*(t)$ and $x_i^* \in \operatorname{dom} L \cap \overline{\Omega}_i$ $(i = 1,2)$. Since $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$, x_i^* $(i = 1,2)$ are different. Let $y_i^*(t) = e^{x_i^*(t)}$ $(i = 1,2)$. Then $y_i^*(t)$ $(i = 1,2)$ are two different positive T-periodic solutions of (1.1). The proof is complete. □

Example 3.7. Consider the following equation:

$$\frac{dy(t)}{dt} = y(t)\left[a(t) - b(t)\ln y(t - \sigma(t)) - c(t)\frac{d}{dt}\ln y(t - \tau(t))\right] - h(t), \tag{3.69}$$

where

$$a(t) = \left(\frac{1}{4.2e} + \frac{1}{4.2e}\sin^2 t\right)\epsilon, \qquad b(t) = (2.35 + 0.15\sin t)\epsilon,$$
$$c(t) = (0.9 + 0.1\cos t)\epsilon, \qquad \sigma(t) \equiv 2\pi, \qquad \tau(t) \equiv 4\pi, \qquad h = \frac{3\epsilon}{2e} + \frac{\epsilon}{2e}\sin t, \tag{3.70}$$

and the constant $\epsilon > 0$. Clearly, (H_1) is satisfied.

Let $\mu(t)$ be the inverse function of $t - \sigma(t)$, and $\gamma(t)$ be the inverse function of $t - \tau(t)$. Then we have

$$\mu(t) = t + 2\pi, \qquad \gamma(t) = t + 4\pi,$$
$$\Gamma(t) = \frac{b(\mu(t))}{1 - \sigma'(\mu(t))} - \frac{c'(\gamma(t))}{1 - \tau'(\gamma(t))} = b(t + 2\pi) - c'(t + 4\pi) = (2.35 + 0.25\sin t)\epsilon. \tag{3.71}$$

Hence, (H_2) is satisfied.

It is easy to see that

$$T = 2\pi, \quad a^u = \frac{\epsilon}{2.1e}, \quad a^l = \frac{\epsilon}{4.2e}, \quad \overline{a} = \frac{\epsilon}{2.8e}, \quad b^u = 2.5\epsilon, \quad b^l = 2.2\epsilon, \quad \overline{b} = 2.35\epsilon,$$
$$\Gamma^u = 2.6\epsilon, \quad \Gamma^l = 2.1\epsilon, \quad \overline{\Gamma} = 2.35\epsilon, \quad c^u = \epsilon, \quad h^u = \frac{2\epsilon}{e}, \quad h^l = \frac{\epsilon}{e}, \quad \overline{h} = \frac{3\epsilon}{2e}. \tag{3.72}$$

Therefore, we have

$$\frac{h^l}{\Gamma^u} = \frac{1}{2.6e}, \qquad \frac{a^u}{\Gamma^l} = \frac{1}{4.41e},$$
$$1 + \ln\frac{h^l}{\Gamma^u} = -\ln 2.6. \tag{3.73}$$

So,

$$1 + \ln\frac{h^l}{\Gamma^u} < \frac{a^u}{\Gamma^l} < \frac{h^l}{\Gamma^u} < 1. \tag{3.74}$$

Hence, (H_3) is satisfied.

Also, it is easy to see that

$$\frac{a^l - c^u(1-\tau')^u M_0}{b^u} = \frac{\epsilon/(4.2e) - \epsilon M_0}{2.5\epsilon} = \frac{1}{10.5e} - \frac{M_0}{2.5} < 1,$$

$$q = 2\left|\ln\frac{h^l}{\Gamma^u}\right| = 2(\ln 2.6 + 1) > 1, \qquad p = \frac{2T\overline{a} + \left(\overline{\Gamma} + \overline{b}\right)Tq}{1 - T\left(\overline{\Gamma} + \overline{b}\right) - c^u} = \frac{\pi\epsilon/(0.7e) + 18.8(\ln 2.6 + 1)\pi\epsilon}{1 - 9.4\pi\epsilon - \epsilon},$$

$$R_0 = q + p,$$

$$M_0 = \frac{a^u + b^u R_0 + h^u e^{R_0}}{1 - c^u(1-\tau')^u} = \frac{\epsilon/(2.1e) + 2.5\epsilon R_0 + 2\epsilon e^{R_0 - 1}}{1 - \epsilon}. \tag{3.75}$$

Therefore, we obtain that

$$R_0 > q > \frac{a^l - c^u(1-\tau')^u M_0}{b^u}. \tag{3.76}$$

Noticing that $\lim_{\epsilon\to 0^+} R_0 = q$, we have $\lim_{\epsilon\to 0^+} M_0 = 0$. Therefore, for some sufficiently small $\epsilon > 0$, the following inequalities hold:

$$\max\left\{1 + \ln\frac{h^u}{b^l}, 0\right\} = \max\{-\ln 1.1, 0\} = 0 < \frac{a^l - c^u(1-\tau')^u M_0}{b^u},$$
$$T\left(\overline{\Gamma} + \overline{b}\right) + c^u = 9.4\pi\epsilon + \epsilon < 1. \tag{3.77}$$

By (3.76)-(3.77), (H_4) is also satisfied. Therefore, all necessary conditions of Theorem 3.6 are satisfied. By Theorem 3.6, (3.69) has at least two positive 2π-periodic solutions.

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