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Research Article

On a System of Nonlinear Variational Inclusions with $H_{h,\eta}$-Monotone Operators

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This paper is concerned mainly with the existence and iterative approximation of solutions for a system of nonlinear variational inclusions involving the strongly $H_{h,\eta}$-monotone operators in Hilbert spaces. The results presented in this paper extend, improve, and unify many known results in the literature.

1. Introduction

Recently, a few authors introduced and studied several classes of systems of nonlinear variational inequalities and inclusions in Hilbert spaces and established the existence of solutions or the approximate solutions for these systems of nonlinear variational inequalities and inclusions [1–19]. Using projection methods, Liu et al. [12], Verma [14–17], Rhoades and Verma [18], and Wu et al. [19] suggested some iterative algorithms for approximating the solutions of several classes of systems of variational inequalities involving relaxed monotone operators, strongly monotone operators, relaxed cocoercive operators, and pseudocontractive operators, respectively. Applying the resolvent operator techniques, Liu et al. [11] and Nie et al. [13] discussed the existence and uniqueness of solutions and suggested iterative algorithms for a system of general quasivariational-like inequalities and a system of nonlinear variational inequalities, respectively, and gave the convergence analysis for the iterative algorithms. Utilizing the resolvent operator method associated with (H, η)-monotone operators, Fang et al. [5] investigated the existence and uniqueness of solutions for a class of system of variational inclusions in Hilbert spaces, constructed an iterative algorithm for approximating the solution of this system of variational inclusions, and discussed the convergence of iterative sequences generated by the algorithm.

Motivated and inspired by the results [1–19], in this paper, we introduce two new classes of strictly $H_{h,\eta}$-monotone and strongly $H_{h,\eta}$-monotone operators and investigate the existence and Lipschitz continuity of the resolvent operators with respect to the strictly $H_{h,\eta}$-monotone and strongly $H_{h,\eta}$-monotone operators. We introduce also a class of system of nonlinear variational inclusions involving strongly $H_{h,\eta}$-monotone operators in Hilbert spaces and suggest two new iterative algorithms for approximating solutions of the system of nonlinear variational inclusions by using the resolvent operator technique associated with $H_{h,\eta}$-monotone operators. The convergence criteria of iterative sequences generated by the algorithms are established. The results presented in this paper extend, improve, and unify some known results in the literature.

This paper is organized as follows. In Section 2, we recall and introduce some definitions, notation, and a lemma. In Section 3, we study properties of the resolvent operators with respect to strictly $H_{h,\eta}$-monotone and strongly $H_{h,\eta}$-monotone operators, respectively, in Hilbert spaces. In Section 4, we introduce a new class of system of nonlinear variational inclusions in Hilbert spaces and use the resolvent operator technique with respect to strongly $H_{h,\eta}$-monotone operators to investigate the existence and uniqueness of solution and suggest two new iterative algorithms for the system of nonlinear variational inclusions. The convergence criteria of the sequences generated by the iterative algorithms are also given under certain conditions.

2. Preliminaries and Lemmas

In this section, we recall and introduce some notation, definitions, and a lemma, which will be used in this paper. Let H be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. I stands for the identity mapping on H and $\mathbb{R}=(-\infty,+\infty)$. Let $T: H \to 2^H$ be a set-valued operator. The graph of T, denoted by Graph (T), is defined as follows:

$$\operatorname{Graph}(T)=\{(x,y): x\in H,\ y\in T(x)\}. \tag{2.1}$$

Definition 2.1. Let $\eta: H\times H \to H$ and $M: H \to 2^H$ be operators. The operator M is said to be

(1) monotone if

$$\langle u-v,\ x-y\rangle \geq 0, \quad \forall x,y\in H,\ u\in M(x),\ v\in M(y); \tag{2.2}$$

(2) η-monotone if

$$\langle u-v,\ \eta(x,y)\rangle \geq 0, \quad \forall x,y\in H,\ u\in M(x),\ v\in M(y); \tag{2.3}$$

(3) strictly monotone if

$$\langle u-v,\ x-y\rangle > 0, \quad \forall x,y\in H,\ x\neq y,\ u\in M(x), v\in M(y); \tag{2.4}$$

(4) strictly η-monotone if

$$\langle u-v, \eta(x,y)\rangle > 0, \quad \forall x,y \in H,\, x \neq y,\, u \in M(x),\, v \in M(y); \tag{2.5}$$

(5) strongly monotone if there exists a constant $r > 0$ satisfying

$$\langle u-v, x-y\rangle \geq r\|x-y\|^2, \quad \forall x,y \in H,\, u \in M(x),\, v \in M(y); \tag{2.6}$$

(6) strongly η-monotone if there exists a constant $r > 0$ satisfying

$$\langle u-v, \eta(x,y)\rangle \geq r\|x-y\|^2, \quad \forall x,y \in H,\, u \in M(x),\, v \in M(y); \tag{2.7}$$

(7) maximal monotone (resp., maximally strictly monotone, maximally strongly monotone) if M is monotone (resp., strictly monotone, strongly monotone) and $(I+\lambda M)(H) = H$ for any $\lambda > 0$;

(8) maximal η-monotone (resp., maximally strictly η-monotone, maximally strongly η-monotone) if M is η-monotone (resp., strictly η-monotone, strongly η-monotone) and $(I+\lambda M)(H) = H$ for any $\lambda > 0$;

(9) H_h-monotone (resp., strictly H_h-monotone, strongly H_h-monotone) if M is monotone (resp., strictly monotone, strongly monotone) and $(h+\lambda M)(H) = H$ for any $\lambda > 0$;

(10) $H_{h,\eta}$-monotone (resp., strictly $H_{h,\eta}$-monotone, strongly $H_{h,\eta}$-monotone) if M is η-monotone (resp., strictly η-monotone, strongly η-monotone) and $(h+\lambda M)(H) = H$ for any $\lambda > 0$.

Definition 2.2. Let $g : H \to H$ and $\eta : H \times H \to H$ be operators. The operator g is called

(1) Lipschitz continuous if there exists a constant $r > 0$ satisfying

$$\|g(x)-g(y)\| \leq r\|x-y\|, \quad \forall x,y \in H; \tag{2.8}$$

(2) monotone if

$$\langle g(x)-g(y), x-y\rangle \geq 0, \quad \forall x,y \in H; \tag{2.9}$$

(3) η-monotone if

$$\langle g(x)-g(y), \eta(x,y)\rangle \geq 0, \quad \forall x,y \in H; \tag{2.10}$$

(4) strongly monotone if there exists a constant $r > 0$ satisfying

$$\langle g(x)-g(y), x-y\rangle \geq r\|x-y\|^2, \quad \forall x,y \in H; \tag{2.11}$$

(5) strongly η-monotone if there exists a constant $r > 0$ satisfying

$$\langle g(x) - g(y), \eta(x, y) \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H; \tag{2.12}$$

(6) relaxed Lipschitz if there exists a constant $r > 0$ satisfying

$$\langle g(x) - g(y), x - y \rangle \leq -r\|x - y\|^2, \quad \forall x, y \in H. \tag{2.13}$$

Definition 2.3. Let $N : H \times H \to H$ and $a, g : H \to H$ be operators. The operator N is said to be

(1) strongly monotone with respect to a and g in the first argument if there exists a constant $r > 0$ satisfying

$$\langle N(a(x), z) - N(a(y), z), g(x) - g(y) \rangle \geq r\|x - y\|^2, \quad \forall x, y, z \in H; \tag{2.14}$$

(2) Lipschitz continuous in the first argument if there exists a constant $r > 0$ satisfying

$$\|N(x, w) - N(y, w)\| \leq r\|x - y\|, \quad \forall x, y, w \in H. \tag{2.15}$$

Similarly, we could define the strong monotonicity of N with respect to a and g in the second argument and the Lipschitz continuity of N in the second argument.

Remark 2.4. For $h = I$, the definition of the H_h-monotone (resp., strictly H_h-monotone, strongly H_h-monotone) operator reduces to the definition of the maximal monotone (resp., maximally strictly monotone, maximally strongly monotone) operator.

Remark 2.5. Notice that M is maximal monotone (resp., maximally strictly monotone, maximally strongly monotone) if and only if M is monotone (resp., strictly monotone, strongly monotone) and there is no other monotone (resp., strictly monotone, strongly monotone) operator whose graph contains strictly the graph Graph(M) of M.

Lemma 2.6 (see [20]). *Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$\alpha_{n+1} \leq (1 - \lambda_n)\alpha_n + \beta_n\lambda_n + \gamma_n, \quad \forall n \geq 0, \tag{2.16}$$

where $\{\lambda_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty}\alpha_n = 0$.

3. The Properties of Strictly $H_{h,\eta}$-Monotone Operators and Strongly $H_{h,\eta}$-Monotone Operators

In this section, we discuss some properties of the set-valued strictly $H_{h,\eta}$-monotone operators and set-valued strongly $H_{h,\eta}$-monotone operators, respectively, dealing with a η-monotone operator h in Hilbert spaces.

Theorem 3.1. *Let H be a real Hilbert space and let $h : H \to H$, $\eta : H \times H \to H$ and $M : H \to 2^H$ be operators such that h is η-monotone and M is strictly $H_{h,\eta}$-monotone. Then*

(i) *M is maximally strictly η-monotone;*

(ii) *$(h + \lambda M)^{-1} : H \to H$ is a single-valued operator for each $\lambda > 0$.*

Proof. (i) Since M is strictly $H_{h,\eta}$-monotone, it follows that M is strictly η-monotone. Suppose that there exists a strictly η-monotone set-valued operator $A : H \to 2^H$ satisfying $\mathrm{Graph}(A) \supsetneq \mathrm{Graph}(M)$, that is, there exists $(u_0, x_0) \in \mathrm{Graph}\,(A) \backslash \mathrm{Graph}\,(M)$ such that

$$\langle x_0 - y,\ \eta(u_0, v)\rangle > 0, \quad \forall (v, y) \in \mathrm{Graph}(M) \text{ with } v \neq u_0. \tag{3.1}$$

Notice that $h(u_0) + \lambda x_0 \in H = (h + \lambda M)(H)$ for any $\lambda > 0$. Thus there exists $(v_0, y_0) \in \mathrm{Graph}(M)$ satisfying

$$h(v_0) + \lambda y_0 = h(u_0) + \lambda x_0, \quad \forall \lambda > 0. \tag{3.2}$$

Suppose that $u_0 = v_0$. Equation (3.2) means that $x_0 = y_0$. Therefore,

$$\mathrm{Graph}(M) \ni (v_0, y_0) = (u_0, x_0) \in \mathrm{Graph}(A) \setminus \mathrm{Graph}(M), \tag{3.3}$$

which is impossible. Suppose that $u_0 \neq v_0$. It follows from (3.1), (3.2), and the η-monotonicity of h and strict η-monotonicity of A that

$$0 < \langle x_0 - y_0,\ \eta(u_0, v_0)\rangle = -\lambda^{-1}\langle h(u_0) - h(v_0), \eta(u_0, v_0)\rangle \leq 0, \tag{3.4}$$

which is a contradiction. Hence M is maximally strictly η-monotone.

(ii) Suppose that there exists some $u \in H$ such that $(h + \lambda M)^{-1}(u)$ contains at least two different elements x and y. Since M is strictly $H_{h,\eta}$-monotone, h is η-monotone, and $\lambda^{-1}(u - h(x)) \in M(x)$, $\lambda^{-1}(u - h(y)) \in M(y)$, it follows that

$$\begin{aligned} 0 &< \left\langle \lambda^{-1}(u - h(x)) - \lambda^{-1}(u - h(y)), \eta(x, y) \right\rangle \\ &= -\lambda^{-1}\langle h(x) - h(y), \eta(x, y)\rangle \leq 0, \end{aligned} \tag{3.5}$$

which is a contradiction. Consequently, the operator $(h + \lambda M)^{-1}$ is single valued. This completes the proof. □

Definition 3.2. Let H be a real Hilbert space and let $h : H \to H$, $\eta : H \times H \to H$ and $M : H \to 2^H$ be operators such that h is η-monotone and M is strictly $H_{h,\eta}$-monotone. Then for each $\lambda > 0$, the resolvent operator $J^h_{M,\lambda} : H \to H$ is defined by

$$J^{h,\eta}_{M,\lambda}(x) = (h + \lambda M)^{-1}(x), \quad \forall x \in H. \tag{3.6}$$

Theorem 3.3. *Let H be a real Hilbert space and let $\eta : H \times H \to H$ be Lipschitz continuous with constant L_η. Assume that $h : H \to H$ is η-monotone and $M : H \to 2^H$ is strongly $H_{h,\eta}$-monotone with constant r. Then for every $\lambda > 0$, the resolvent operator $J^{h,\eta}_{M,\lambda} : H \to H$ is Lipschitz continuous with constant $L_\eta/\lambda r$.*

Proof. Since M is strongly $H_{h,\eta}$-monotone with constant r, it follows that M is strictly $H_{h,\eta}$-monotone. Let x, y be in H. In view of $\lambda^{-1}(x - h(J^{h,\eta}_{M,\lambda}(x))) \in M(J^{h,\eta}_{M,\lambda}(x))$ and $\lambda^{-1}(y - h(J^{h,\eta}_{M,\lambda}(y))) \in M(J^{h,\eta}_{M,\lambda}(y))$ and the strong monotonicity of M, we deduce that

$$\begin{aligned}
&\lambda^{-1}\left\langle x-y, \eta\left(J^{h,\eta}_{M,\lambda}(x), J^{h,\eta}_{M,\lambda}(y)\right)\right\rangle \\
&\quad - \lambda^{-1}\left\langle h\left(J^{h,\eta}_{M,\lambda}(x)\right) - h\left(J^{h,\eta}_{M,\lambda}(y)\right), \eta\left(J^{h,\eta}_{M,\lambda}(x), J^{h,\eta}_{M,\lambda}(y)\right)\right\rangle \\
&\qquad = \left\langle \lambda^{-1}\left(x - h\left(J^{h,\eta}_{M,\lambda}(x)\right)\right) - \lambda^{-1}\left(y - h\left(J^{h,\eta}_{M,\lambda}(y)\right)\right), \eta\left(J^{h,\eta}_{M,\lambda}(x), J^{h,\eta}_{M,\lambda}(y)\right)\right\rangle \\
&\qquad \ge r\left\| J^{h,\eta}_{M,\lambda}(x) - J^{h,\eta}_{M,\lambda}(y)\right\|^2.
\end{aligned} \tag{3.7}$$

This leads to

$$\begin{aligned}
&L_\eta \|x-y\| \left\| J^{h,\eta}_{M,\lambda}(x) - J^{h,\eta}_{M,\lambda}(y)\right\| \\
&\quad \ge \left\langle x-y, \eta\left(J^{h,\eta}_{M,\lambda}(x), J^{h,\eta}_{M,\lambda}(y)\right)\right\rangle \\
&\quad \ge \left\langle h\left(J^{h,\eta}_{M,\lambda}(x)\right) - h\left(J^{h,\eta}_{M,\lambda}(y)\right), \eta\left(J^{h,\eta}_{M,\lambda}(x), J^{h,\eta}_{M,\lambda}(y)\right)\right\rangle \\
&\qquad + \lambda r\left\| J^{h,\eta}_{M,\lambda}(x) - J^{h,\eta}_{M,\lambda}(y)\right\|^2 \\
&\quad \ge \lambda r\left\| J^{h,\eta}_{M,\lambda}(x) - J^{h,\eta}_{M,\lambda}(y)\right\|^2,
\end{aligned} \tag{3.8}$$

which yields that

$$\left\| J^{h,\eta}_{M,\lambda}(x) - J^{h,\eta}_{M,\lambda}(y)\right\| \le \frac{L_\eta}{\lambda r}\|x-y\|. \tag{3.9}$$

This completes the proof. □

Theorem 3.4. *Let H be a real Hilbert space and let $\eta : H \times H \to H$ be Lipschitz continuous with constant L_η. Assume that $h : H \to H$ is strongly η-monotone with constant s and $M : H \to 2^H$ is strongly $H_{h,\eta}$-monotone with constant r. Then for every $\lambda > 0$, the resolvent operator $J^{h,\eta}_{M,\lambda} : H \to H$ is Lipschitz continuous with constant $L_\eta/(s + \lambda r)$.*

Proof. As in the proof of Theorem 3.3, by the strong monotonicity of h and M, we infer that for any $x, y \in H$

$$\begin{aligned} L_\eta \|x-y\| \left\| J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right\| &\geq \|x-y\| \left\| \eta\left(J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right) \right\| \\ &\geq \left\langle x-y, \eta\left(J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right) \right\rangle \\ &\geq \left\langle h\left(J_{M,\lambda}^{h,\eta}(x) \right) - h\left(J_{M,\lambda}^{h,\eta}(y) \right), \eta\left(J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right) \right\rangle \\ &\quad + \lambda r \left\| J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right\|^2 \\ &\geq (s+\lambda r) \left\| J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right\|^2, \end{aligned} \tag{3.10}$$

which means that

$$\left\| J_{M,\lambda}^{h,\eta}(x) - J_{M,\lambda}^{h,\eta}(y) \right\| \leq \frac{L_\eta}{s+\lambda r} \|x-y\|. \tag{3.11}$$

This completes the proof. □

The following example shows that Theorem 3.3 is different from Lemma 2.2 in [5].

Example 3.5. Let $H = \mathbb{R}$ with the usual norm. Define single-valued and set-valued operators $h : H \to H$, $\eta : H \times H \to H$ and $M : H \to 2^H$ by

$$\begin{gathered} h(x) = 3x^3 - 2, \quad \forall x \in H, \qquad \eta(x,y) = 2(x-y), \quad \forall x, y \in H, \\ M(x) = \begin{cases} \{3x+1\} & \text{for } x < 0, \\ [1,5] & \text{for } x = 0, \\ \{5+4x\} & \text{for } x > 0. \end{cases} \end{gathered} \tag{3.12}$$

It is clear that η is Lipschitz continuous with constant 2 and

$$\langle h(x) - h(y), \eta(x,y) \rangle = 6(x-y)^2 \left(x^2 + xy + y^2 \right) \geq 0, \quad \forall x, y \in H, \tag{3.13}$$

which yields that h is η-monotone. On the other hand, for any $r > 0$, there exist $(x_r, y_r) = (\sqrt{r}/10, \sqrt{r}/20) \in H \times H$ satisfying

$$\langle h(x_r) - h(y_r), \eta(x_r, y_r) \rangle = \frac{21r^2}{80000} < \frac{r^2}{400} = r(x_r - y_r)^2, \tag{3.14}$$

that is, h is not strongly η-monotone. Now we claim that M is strongly $H_{h,\eta}$-monotone. Let x, y be in H with $y > x$. We have to consider the following five cases.

Case 1. Suppose that $x < y < 0$. It follows that

$$\langle M(x) - M(y), \eta(x,y)\rangle = 6(x-y)^2 \geq 2(x-y)^2. \tag{3.15}$$

Case 2. Suppose that $x < 0 = y$. For any $z \in [1,5]$, we get that

$$\begin{aligned}\langle M(x) - z, \eta(x,y)\rangle &= 2(3x+1-z)(y-x)\\ &= 2(y-x)^2 + 2x(1+2x-z) = 2(x-y)^2.\end{aligned} \tag{3.16}$$

Case 3. Suppose that $x < 0 < y$. Clearly, we have

$$\begin{aligned}\langle M(x) - M(y), \eta(x,y)\rangle &= 2(3x-4-4y)(x-y)\\ &= 8(x-y)^2 + 2(4+x)(y-x) \geq 2(x-y)^2.\end{aligned} \tag{3.17}$$

Case 4. Suppose that $x = 0 < y$. For each $z \in [0,1]$, we deduce that

$$\begin{aligned}\langle z - M(y), \eta(x,y)\rangle &= 2(z-5-4y)(x-y)\\ &\geq 8(x-y)^2 + 2(5-z-x) \geq 2(x-y)^2.\end{aligned} \tag{3.18}$$

Case 5. Suppose that $0 < x < y$. It is easy to verify that

$$\langle M(x) - M(y), \eta(x,y)\rangle = 8(x-y)^2 \geq 2(x-y)^2. \tag{3.19}$$

Hence M is strongly η-monotone with constant 2.

Let $\lambda > 0$. In order to show that $(h+\lambda M)(H) = H$, we need only to verify that $(h+\lambda M)(H) \supseteq H$. Assume that $t \in [\lambda-2, 5\lambda-2] \subseteq H$. We know that $t \in (h+\lambda M)(0) = [\lambda-2, 5\lambda-2]$. Assume that $t \in (-\infty, \lambda-2) \subseteq H$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 3x^3 + 3\lambda x + \lambda - 2 - t, \quad \forall x \in \mathbb{R}. \tag{3.20}$$

Notice that the function f is continuous and

$$f(0) f\left(-\frac{\lambda-2-t}{\lambda}\right) = (\lambda-2-t)\left[-3\left(\frac{\lambda-2-t}{\lambda}\right)^3 - 2(\lambda-2-t)^2\right] < 0. \tag{3.21}$$

It follows that there exists $x_0 \in (-(\lambda-2-t)/\lambda, 0)$ such that $f(x_0) = 0$, that is,

$$t = 3x_0^3 + 3\lambda x_0 + \lambda - 2 = (h+\lambda M)(x_0). \tag{3.22}$$

Assume that $t \in (5\lambda - 2, +\infty) \subseteq H$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = 3x^3 + 4\lambda x + 5\lambda - 2 - t, \quad \forall x \in \mathbb{R}. \tag{3.23}$$

Obviously, g is continuous and

$$g(0)g\left(\frac{t-5\lambda+2}{\lambda}\right) = 3(5\lambda - 2 - t)\left[\left(\frac{t-5\lambda+2}{\lambda}\right)^3 + t - 5\lambda + 2\right] < 0, \tag{3.24}$$

which implies that there exists $x_0 \in (0, (t - 5\lambda + 2)/\lambda)$ with $g(x_0) = 0$. Consequently,

$$t = 3x_0^3 + 4\lambda x_0 + 5\lambda - 2 = (h + \lambda M)(x_0). \tag{3.25}$$

Therefore, $H \subseteq (h+\lambda M)(H)$. Thus M is strongly $H_{h,\eta}$-monotone. It follows from Theorem 3.3 that the resolvent operator $J_{M,\lambda}^{h,\eta}$ is Lipschitz continuous with constant $1/\lambda$. However, we cannot invoke Lemma 2.2 in [5] to prove the Lipschitz continuity of the resolvent operator $J_{M,\lambda}^{h,\eta}$ because h is not strongly η-monotone.

4. A System of Nonlinear Variational Inclusions

In this section, we investigate a new class of system of nonlinear variational inclusions involving strongly $H_{h,\eta}$-monotone operators in Hilbert spaces and suggest two new iterative algorithms for approximating solutions of the system of nonlinear variational inclusions by using the resolvent operator technique.

Let $a_i, b_i, f_i, g_i, h_i : H \to H$, $\eta_i, N_i : H \times H \to H$ be single-valued operators and $M_i : H \to 2^H$ set-valued operators and $p_i \in H$ for $i \in \{1,2\}$. We now consider the following system of nonlinear variational inclusions:

find $(x, y) \in H \times H$ such that

$$\begin{aligned} p_1 &\in N_1(a_1(x), b_1(y)) + M_1(f_1(x) - g_1(x)), \\ p_2 &\in N_2(a_2(x), b_2(y)) + M_2(f_2(y) - g_2(y)). \end{aligned} \tag{4.1}$$

It is easy to see that the system of nonlinear variational inclusions (4.1) includes a lot of variational inequalities, quasivariational inequalities, variational-like inequalities, variational inclusions, and systems of variational inequalities, quasivariational inequalities, variational-like inequalities, and variational inclusions in [1–16] and the references therein as special cases.

Now we use the resolvent operator technique to establish the equivalence between the existence of solutions for the system of nonlinear variational inclusions (4.1) and the existence of fixed points for the single-valued operator Q defined by (4.3) below.

Lemma 4.1. *Let H be a real Hilbert space. Let $t_i \in (0,1)$ and $\rho_i > 0$ be constants, $a_i, b_i, f_i, g_i, h_i : H \to H, \eta_i, N_i : H \times H \to H$ and $M_i : H \to 2^H$ be single-valued and set-valued operators, respectively, such that h_i is η_i-monotone and M_i is strictly H_{h_i,η_i}-monotone for $i \in \{1,2\}$. Then the following statements are pairwise equivalent.*

(a) *The system of nonlinear variational inclusions* (4.1) *has a solution* $(x, y) \in H \times H$.

(b) *There exists* $(x, y) \in H \times H$ *satisfying*

$$\begin{aligned} f_1(x) - g_1(x) &= J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(x) - g_1(x)\right) + \rho_1 p_1 - \rho_1 N_1\left(a_1(x), b_1(y)\right)\right), \\ f_2(y) - g_2(y) &= J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2\left(f_2(y) - g_2(y)\right) + \rho_2 p_2 - \rho_2 N_2\left(a_2(x), b_2(y)\right)\right). \end{aligned} \tag{4.2}$$

(c) *The single-valued operator* $Q : H \times H \to H \times H$ *defined by*

$$Q(u, v) = (G_1(u, v), G_2(u, v)), \quad \forall (u, v) \in H \times H, \tag{4.3}$$

where

$$\begin{aligned} G_1(u, v) = {} & (1 - t_1)u \\ & + t_1\Big[u - \left(f_1(u) - g_1(u)\right) \\ & \qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(u) - g_1(u)\right) + \rho_1 p_1 - \rho_1 N_1\left(a_1(u), b_1(v)\right)\right)\Big], \quad \forall (u, v) \in H \times H, \\ G_2(u, v) = {} & (1 - t_2)v \\ & + t_2\Big[v - \left(f_2(v) - g_2(v)\right) \\ & \qquad + J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2\left(f_2(v) - g_2(v)\right) + \rho_2 p_2 - \rho_2 N_2\left(a_2(u), b_2(v)\right)\right)\Big], \quad \forall (u, v) \in H \times H \end{aligned} \tag{4.4}$$

has a fixed point $(x, y) \in H \times H$.

Proof. It follows from Theorem 3.1, (4.3), and (4.4) that $(x, y) \in H \times H$ is a solution of the system of nonlinear variational inclusions (4.1) if and only if

$$\begin{aligned} & p_1 \in N_1\left(a_1(x), b_1(y)\right) + M_1\left(f_1(x) - g_1(x)\right), \\ & p_2 \in N_2\left(a_2(x), b_2(y)\right) + M_2\left(f_2(y) - g_2(y)\right) \\ & \quad \Longleftrightarrow h_1\left(f_1(x) - g_1(x)\right) + \rho_1 p_1 - \rho_1 N_1\left(a_1(x), b_1(y)\right) \in \left(h_1 + \rho_1 M_1\right)\left(f_1(x) - g_1(x)\right), \\ & \quad \Longleftrightarrow h_2\left(f_2(y) - g_2(y)\right) + \rho_2 p_2 - \rho_2 N_2\left(a_2(x), b_2(y)\right) \in \left(h_2 + \rho_2 M_2\right)\left(f_2(y) - g_2(y)\right), \end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow f_1(x) - g_1(x) = J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1(f_1(x) - g_1(x)) + \rho_1 p_1 - \rho_1 N_1(a_1(x), b_1(y))\right),\\
&\Longleftrightarrow f_2(y) - g_2(y) = J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2(f_2(y) - g_2(y)) + \rho_2 p_2 - \rho_2 N_2(a_2(x), b_2(y))\right),\\
&\Longleftrightarrow x = (1-t_1)x\\
&\qquad\qquad + t_1\Big[x - (f_1(x) - g_1(x))\\
&\qquad\qquad\qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1(f_1(x) - g_1(x)) + \rho_1 p_1 - \rho_1 N_1(a_1(x), b_1(y))\right)\Big]\\
&\qquad = G_1(x,y),\\
&\Longleftrightarrow y = (1-t_2)y\\
&\qquad\qquad + t_2\Big[y - (f_2(y) - g_2(y))\\
&\qquad\qquad\qquad + J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2(f_2(y) - g_2(y)) + \rho_2 p_2 - \rho_2 N_2(a_2(x), b_2(y))\right)\Big]\\
&\qquad = G_2(x,y),\\
&\Longleftrightarrow Q(x,y) = (G_1(x,y), G_2(x,y)) = (x,y).
\end{aligned}
\tag{4.5}
$$

This completes the proof. □

Based on Lemma 4.1, we suggest the following new iterative algorithms for the system of nonlinear variational inclusions (4.1).

Algorithm 4.2 (The Mann iteration method with errors). For any $x_0, y_0 \in H$, compute $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0} \subset H$ by

$$
\begin{aligned}
x_{n+1} &= (1-\sigma_n)x_n\\
&\quad + \sigma_n\Big[x_n - (f_1(x_n) - g_1(x_n))\\
&\qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1(f_1(x_n) - g_1(x_n))\rho_1 p_1 - \rho_1 N_1(a_1(x_n), b_1(y_n))\right)\Big] + \xi_n, \quad \forall n \geq 0,\\
y_{n+1} &= (1-\delta_n)y_n\\
&\quad + \delta_n\Big[y_n - (f_2(y_n) - g_2(y_n))\\
&\qquad + J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2(f_2(y_n) - g_2(y_n))\rho_2 p_2 - \rho_2 N_2(a_2(x_n), b_2(y_n))\right)\Big] + \xi_n, \quad \forall n \geq 0,
\end{aligned}
\tag{4.6}
$$

where $\{\sigma_n\}_{n\geq 0}$ and $\{\delta_n\}_{n\geq 0}$ are sequences in $(0,1)$, $\{\xi_n\}_{n\geq 0}$ and $\{\zeta_n\}_{n\geq 0}$ are sequences in H introduced to take into account possible in inexact computation with $\sum_{n=0}^{\infty}(\|\xi_n\| + \|\zeta_n\|) < +\infty$.

Algorithm 4.3 (The implicit Mann iteration method with errors). For any $x_0, y_0 \in H$, compute $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0} \subset H$ by

$$\begin{aligned}
x_{n+1} &= (1-\sigma_n)x_n \\
&\quad + \sigma_n\Big[x_{n+1} - \big(f_1(x_{n+1}) - g_1(x_{n+1})\big) \\
&\qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\big(h_1(f_1(x_{n+1}) - g_1(x_{n+1})) + \rho_1 p_1 - \rho_1 N_1(a_1(x_{n+1}), b_1(y_{n+1}))\big)\Big] + \xi_n, \quad \forall n \geq 0, \\
y_{n+1} &= (1-\delta_n)y_n \\
&\quad + \delta_n\Big[y_{n+1} - \big(f_2(y_{n+1}) - g_2(y_{n+1})\big) \\
&\qquad + J_{M_2,\rho_2}^{h_2,\eta_2}\big(h_2(f_2(y_{n+1}) - g_2(y_{n+1})) + \rho_2 p_2 - \rho_2 N_2(a_2(x_{n+1}), b_2(y_{n+1}))\big)\Big] + \zeta_n, \quad \forall n \geq 0,
\end{aligned} \tag{4.7}$$

where $\{\sigma_n\}_{n\geq 0}$ and $\{\delta_n\}_{n\geq 0}$ are sequences in $(0,1)$, and $\{\xi_n\}_{n\geq 0}$ and $\{\zeta_n\}_{n\geq 0}$ are sequences in H introduced to take into account possible in inexact computation with $\sum_{n=0}^{\infty}(\|\xi_n\| + \|\zeta_n\|) < +\infty$.

Now we investigate those conditions under which the approximate solutions generated by Algorithms 4.2 and 4.3 converge strongly to the exact solutions of the system of nonlinear variational inclusions (4.1).

Theorem 4.4. *Let H be a real Hilbert space and let $a_i, b_i, f_i, g_i, h_i : H \to H$, $\eta_i : H \times H \to H$ be Lipschitz continuous with constants $L_{a_i}, L_{b_i}, L_{f_i}, L_{g_i}, L_{h_i}, L_{\eta_i}$, respectively, $h_i\eta_i$-monotone, f_i strongly monotone with constant α_i, g_i relaxed Lipschitz with constant β_i for $i \in \{1,2\}$. Let $M_i : H \to 2^H$ be strongly H_{h_i,η_i}-monotone with constant r_i for $i \in \{1,2\}$. Assume that $N_i : H \times H \to H$ is Lipschitz continuous with constants $L_{N_{i1}}$ and $L_{N_{i2}}$ in the first and second arguments, respectively, N_i is strongly monotone with constant γ_i with respect to a_i and $f_i - g_i$ in the first argument for $i \in \{1,2\}$. Let*

$$\begin{aligned}
K_1 &= \sqrt{1 - 2(\alpha_1 - \beta_1) + (L_{f_1} + L_{g_1})^2} + \frac{L_{\eta_1}}{\rho_1 r_1}\left[(L_{f_1} + L_{g_1})(1 + L_{h_1}) + \sqrt{(L_{f_1} + L_{g_1})^2 - 2\rho_1\gamma_1 + L_{N_{11}}^2 L_{a_1}^2}\right], \\
K_2 &= \sqrt{1 - 2(\alpha_2 - \beta_2) + (L_{f_2} + L_{g_2})^2} + \frac{L_{\eta_2}}{\rho_2 r_2}\left[(L_{f_2} + L_{g_2})(1 + L_{h_2}) + \sqrt{(L_{f_2} + L_{g_2})^2 - 2\rho_2\gamma_2 + L_{N_{22}}^2 L_{b_2}^2}\right], \\
V_1 &= \frac{1}{r_1} L_{\eta_1} L_{N_{12}} L_{b_1}, \qquad V_2 = \frac{1}{r_2} L_{\eta_2} L_{N_{21}} L_{a_2}.
\end{aligned} \tag{4.8}$$

If the following condition is fulfilled:

$$V_1 V_2 < (1 - K_1)(1 - K_2), \qquad \max\{K_1, K_2\} < 1, \tag{4.9}$$

then the system of nonlinear variational inclusions (4.1) *has a unique solution* $(x, y) \in H \times H$. *Moreover, if*

$$\sup\{1-(1-K_1)\sigma_n + V_2\delta_n,\ 1-(1-K_2)\delta_n + V_1\sigma_n : n \geq 0\} < 1, \tag{4.10}$$

then for each $x_0, y_0 \in H$, *the sequences* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *generated by Algorithm 4.2 converge strongly to* x *and* y, *respectively.*

Proof. First of all we prove that the operator Q defined by (4.3) is a contraction on the Banach space $H \times H$ with norm $\|(u,v)\|_1 = \|u\| + \|v\|$ for $(u,v) \in H \times H$. It follows from (4.9) that

$$\begin{aligned}
&V_1V_2 < (1-K_1)(1-K_2)\\
&\quad\Longleftrightarrow \frac{V_2}{1-K_1} < \frac{1-K_2}{V_1}\\
&\quad\Longleftrightarrow \text{there exists } t \neq 1 \text{ with } \frac{V_2}{1-K_1} < t < \frac{1-K_2}{V_1}\\
&\quad\Longleftrightarrow \text{there exist } t_1, t_2 \in (0,1) \text{ with}\\
&\quad t_1 = \begin{cases} t(1-t) & \text{for } t \in (0,1),\\ \dfrac{t}{1+t} & \text{for } t > 1,\end{cases}\\
&\quad t_2 = \begin{cases} 1-t & \text{for } t \in (0,1),\\ \dfrac{1}{1+t} & \text{for } t > 1\end{cases}
\end{aligned} \tag{4.11}$$

such that

$$\begin{aligned}
&V_2t_2 < (1-K_1)t_1, V_1t_1\\
&\quad < (1-K_2)t_2\\
&\quad\Longleftrightarrow \text{ there exist } t_1, t_2 \in (0,1) \text{ satisfying (4.11) and}
\end{aligned} \tag{4.12}$$

$$\max\{1-(1-K_1)t_1 + t_2V_2,\ 1-(1-K_2)t_2 + t_1V_1\} < 1. \tag{4.13}$$

Put $(u_1, v_1), (u_2, v_2) \in H \times H$. By the Lipschitz continuity and various monotonicity of $a_i, b_i, f_i, g_i, h_i, M_i, N_i$ for $i \in \{1,2\}$, (4.4) and (4.9), we deduce that

$$\begin{aligned}
&\|G_1(u_1,v_1) - G_1(u_2,v_2)\|\\
&\quad\leq (1-t_1)\|u_1-u_2\| + t_1\left\|u_1-u_2-\left[f_1(u_1)-g_1(u_1)-\left(f_1(u_2)-g_1(u_2)\right)\right]\right\|\\
&\qquad + t_1\left\|J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(u_1)-g_1(u_1)\right) + \rho_1p_1 - \rho_1N_1(a_1(u_1), b_1(v_1))\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad -J^{h_1,\eta_1}_{M_1,\rho_1}\left(h_1\left(f_1(u_2)-g_1(u_2)\right)+\rho_1 p_1-\rho_1 N_1(a_1(u_2),b_1(v_2))\right)\Big\| \\
&\le \left[1-t_1\left(1-\sqrt{1-2(\alpha_1-\beta_1)+\left(L_{f_1}+L_{g_1}\right)^2}\right)\right]\|u_1-u_2\| \\
&\quad +\frac{t_1 L_{\eta_1}}{\rho_1 r_1}\left[\left\|h_1\left(f_1(u_1)-g_1(u_1)\right)-h_1\left(f_1(u_2)-g_1(u_2)\right)\right\|\right. \\
&\qquad +\left\|f_1(u_1)-g_1(u_1)-\left(f_1(u_2)-g_1(u_2)\right)\right\| \\
&\qquad +\left\|f_1(u_1)-g_1(u_1)-\left(f_1(u_2)-g_1(u_2)\right)\right. \\
&\qquad \left.-\rho_1(N_1(a_1(u_1),b_1(v_1))-N_1(a_1(u_2),b_1(v_1)))\right\| \\
&\qquad \left.+\rho_1\|N_1(a_1(u_2),b_1(v_1))-N_1(a_1(u_2),b_1(v_2))\|\right] \\
&\le \left[1-t_1\left(1-\sqrt{1-2(\alpha_1-\beta_1)+\left(L_{f_1}+L_{g_1}\right)^2}\right)\right]\|u_1-u_2\| \\
&\quad +\frac{t_1 L_{\eta_1}}{\rho_1 r_1}\left[(1+L_{h_1})\left(L_{f_1}+L_{g_1}\right)\right. \\
&\qquad \left.+\sqrt{\left(L_{f_1}+L_{g_1}\right)^2-2\rho_1\gamma_1+\rho_1^2 L_{N_{11}}^2 L_{a_1}^2}\right]\|u_1-u_2\| \\
&\quad +\frac{t_1 L_{\eta_1}}{r_1} L_{N_{12}} L_{b_1}\|v_1-v_2\| \\
&= [1-(1-K_1)t_1]\|u_1-u_2\|+t_1 V_1\|v_1-v_2\|, \\
&\|G_2(u_1,v_1)-G_2(u_2,v_2)\| \\
&\le (1-t_2)\|v_1-v_2\|+t_2\left\|v_1-v_2-\left[f_2(v_1)-g_2(v_1)-\left(f_2(v_2)-g_2(v_2)\right)\right]\right\| \\
&\quad +t_2\Big\|J^{h_2,\eta_2}_{M_2,\rho_2}\left(h_2\left(f_2(v_1)-g_2(v_1)\right)+\rho_2 p_2-\rho_2 N_2(a_2(u_1),b_2(v_1))\right) \\
&\qquad -J^{h_2,\eta_2}_{M_2,\rho_2}\left(h_2\left(f_2(v_2)-g_2(v_2)\right)+\rho_2 p_2-\rho_2 N_2(a_2(u_2),b_2(v_2))\right)\Big\| \\
&\le \left[1-t_2\left(1-\sqrt{1-2(\alpha_2-\beta_2)+\left(L_{f_2}+L_{g_2}\right)^2}\right)\right]\|v_1-v_2\| \\
&\quad +\frac{t_2 L_{\eta_2}}{\rho_2 r_2}\left[\left\|h_2\left(f_2(v_1)-g_2(v_1)\right)-h_2\left(f_2(v_2)-g_2(v_2)\right)\right\|\right. \\
&\qquad +\left\|f_2(v_1)-g_2(v_1)-\left(f_2(v_2)-g_2(v_2)\right)\right\| \\
&\qquad +\left\|f_2(v_1)-g_2(v_1)-\left(f_2(v_2)-g_2(v_2)\right)\right. \\
&\qquad \left.-\rho_2(N_2(a_2(u_1),b_2(v_1))-N_2(a_2(u_1),b_2(v_2)))\right\| \\
&\qquad \left.+\rho_2\|N_2(a_2(u_1),b_2(v_2))-N_2(a_2(u_2),b_2(v_2))\|\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq \left[1 - t_2\left(1 - \sqrt{1 - 2(\alpha_2 - \beta_2) + \left(L_{f_2} + L_{g_2}\right)^2}\right)\right]\|v_1 - v_2\| \\
&\quad + \frac{t_2 L_{\eta_2}}{\rho_2 r_2}\left[(1 + L_{h_2})\left(L_{f_2} + L_{g_2}\right)\right. \\
&\qquad\qquad \left. + \sqrt{\left(L_{f_2} + L_{g_2}\right)^2 - 2\rho_2\gamma_2 + \rho_2^2 L_{N_{22}}^2 L_{b_2}^2}\right]\|v_1 - v_2\| \\
&\quad + \frac{t_2 L_{\eta_2}}{r_2} L_{N_{21}} L_{a_2}\|u_1 - u_2\| \\
&= [1 - (1 - K_2)t_2]\|v_1 - v_2\| + t_2 V_2\|u_1 - u_2\|.
\end{aligned} \tag{4.14}$$

In light of (4.3) and (**??**)–(4.14), we infer that

$$\begin{aligned}
\|Q(u_1, v_1) - Q(u_2, v_2)\|_1 &= \|G_1(u_1, v_1) - G_1(u_2, v_2)\| + \|G_2(u_1, v_1) - G_2(u_2, v_2)\| \\
&\leq [1 - (1 - K_1)t_1 + t_2 V_2]\|u_1 - u_2\| \\
&\quad + [1 - (1 - K_2)t_2 + t_1 V_1]\|v_1 - v_2\| \\
&\leq \max\{1 - (1 - K_1)t_1 + t_2 V_2,\ 1 - (1 - K_2)t_2 + t_1 V_1\} \\
&\quad \times [\|u_1 - u_2\| + \|v_1 - v_2\|] \\
&= K\|(u_1, v_1) - (u_2, v_2)\|_1,
\end{aligned} \tag{4.15}$$

where

$$K = \max\{1 - (1 - K_1)t_1 + t_2 V_2,\ 1 - (1 - K_2)t_2 + t_1 V_1\} < 1, \tag{4.16}$$

that is, Q is a contraction on $(H \times H, \|\cdot\|_1)$. Therefore, the operator Q has a unique fixed point $(x, y) \in H \times H$, which is the unique solution of the system of nonlinear variational inclusions (4.1) by Lemma 4.1. It follows that

$$\begin{aligned}
x &= (1 - \sigma_n)x \\
&\quad + \sigma_n\left[x - \left(f_1(x) - g_1(x)\right)\right. \\
&\qquad \left. + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(x) - g_1(x)\right) + \rho_1 p_1 - \rho_1 N_1\left(a_1(x), b_1(y)\right)\right)\right], \quad \forall n \geq 0,
\end{aligned}$$

$$y = (1-\delta_n)y + \delta_n\Big[y - (f_2(y) - g_2(y)) + J_{M_2,\rho_2}^{h_2,\eta_2}(h_2(f_2(y) - g_2(y)) + \rho_2 p_2 - \rho_2 N_2(a_2(x), b_2(y)))\Big], \quad \forall n \geq 0. \tag{4.17}$$

In view of Algorithm 4.2 and (4.17), we deduce that for any $n \geq 0$

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\quad\leq (1-\sigma_n)\|x_n - x\| \\
&\qquad + \sigma_n\big\|x_n - x - [f_1(x_n) - g_1(x_n) - (f_1(x) - g_1(x))]\big\| \\
&\qquad + \sigma_n\Big\|J_{M_1,\rho_1}^{h_1,\eta_1}(h_1(f_1(x_n) - g_1(x_n)) + \rho_1 p_1 - \rho_1 N_1(a_1(x_n), b_1(y_n))) \\
&\qquad\qquad - J_{M_1,\rho_1}^{h_1,\eta_1}(h_1(f_1(x) - g_1(x)) + \rho_1 p_1 - \rho_1 N_1(a_1(x), b_1(y)))\Big\| + \|\xi_n\| \\
&\quad\leq \left[1 - \sigma_n\left(1 - \sqrt{1 - 2(\alpha_1 - \beta_1) + (L_{f_1} + L_{g_1})^2}\right)\right]\|x_n - x\| \\
&\qquad + \frac{\sigma_n L_{\eta_1}}{\rho_1 r_1}\big[\|h_1(f_1(x_n) - g_1(x_n)) - h_1(f_1(x) - g_1(x))\| \\
&\qquad\qquad + \|f_1(x_n) - g_1(x_n) - (f_1(x) - g_1(x))\| \\
&\qquad\qquad + \|f_1(x_n) - g_1(x_n) - (f_1(x) - g_1(x)) \\
&\qquad\qquad - \rho_1(N_1(a_1(x_n), b_1(y_n)) - N_1(a_1(x), b_1(y_n)))\| \\
&\qquad\qquad + \rho_1\|N_1(a_1(x), b_1(y_n)) - N_1(a_1(x), b_1(y))\|\big] + \|\xi_n\| \\
&\quad\leq \left[1 - \sigma_n\left(1 - \sqrt{1 - 2(\alpha_1 - \beta_1) + (L_{f_1} + L_{g_1})^2}\right)\right]\|x_n - x\| \\
&\qquad + \frac{\sigma_n L_{\eta_1}}{\rho_1 r_1}\Bigg[(1 + L_{h_1})(L_{f_1} + L_{g_1}) \\
&\qquad\qquad + \sqrt{(L_{f_1} + L_{g_1})^2 - 2\rho_1\gamma_1 + \rho_1^2 L_{N_{11}}^2 L_{a_1}^2}\Bigg]\|x_n - x\| \\
&\qquad + \frac{\sigma_n L_{\eta_1}}{r_1} L_{N_{12}} L_{b_1}\|y_n - y\| + \|\xi_n\| \\
&\quad= [1 - (1 - K_1)\sigma_n]\|x_n - x\| + \sigma_n V_1\|y_n - y\| + \|\xi_n\|, \\
&\|y_{n+1} - y\| \\
&\quad\leq (1-\delta_n)\|y_n - y\| + \delta_n\|y_n - y - [f_2(y_n) - g_2(y_n) - (f_2(y) - g_2(y))]\|
\end{aligned}$$

$$\begin{aligned}
&+\delta_n\left\| J^{h_2,\eta_2}_{M_2,\rho_2}(h_2(f_2(y_n)-g_2(y_n))+\rho_2 p_2-\rho_2 N_2(a_2(x_n),b_2(y_n)))\right.\\
&\qquad\left.-J^{h_2,\eta_2}_{M_2,\rho_2}(h_2(f_2(x)-g_2(x))+\rho_2 p_2-\rho_2 N_2(a_2(x),b_2(y)))\right\| + \|\zeta_n\|\\
&\le [1-(1-K_2)\delta_n]\|y_n-y\| + \delta_n V_2\|x_n-x\| + \|\zeta_n\|.
\end{aligned} \tag{4.18}$$

By virtue of (4.18), we have

$$\begin{aligned}
\|x_{n+1}-x\| + \|y_{n+1}-y\| &\le (1-(1-K_1)\sigma_n)\|x_n-x\| + \sigma_n V_1\|y_n-y\|\\
&\quad + (1-(1-K_2)\delta_n)\|y_n-y\| + \delta_n V_2\|x_n-x\| + \|\xi_n\| + \|\zeta_n\|\\
&\le \max\{1-(1-K_1)\sigma_n + V_2\delta_n,\ 1-(1-K_2)\delta_n + V_1\sigma_n\}\\
&\quad \times [\|x_n-x\| + \|y_n-y\|] + \|\xi_n\| + \|\zeta_n\|\\
&\le K^*(\|x_n-x\| + \|y_n-y\|) + \|\xi_n\| + \|\zeta_n\|, \quad \forall n \ge 0,
\end{aligned} \tag{4.19}$$

where

$$K^* = \sup\{1-(1-K_1)\sigma_n + V_2\delta_n,\ 1-(1-K_2)\delta_n + V_1\sigma_n : n \ge 0\} < 1 \tag{4.20}$$

by (4.10). Lemma 2.6 and (4.19) ensure that the sequences $\{x_n\}_{n\ge0}$ and $\{x_n\}_{n\ge0}$ generated by Algorithm 4.2 converge strongly to x and y, respectively. This completes the proof. □

Theorem 4.5. *Let $H, a_i, b_i, f_i, g_i, h_i, \eta_i, M_i, N_i, K_i, V_i$ for $i \in \{1,2\}$ be as in Theorem 4.4. If* (4.9) *is satisfied, then the system of nonlinear variational inclusions* (4.1) *has a unique solution $(x,y) \in H \times H$. Moreover, if*

$$\sup\{K_1\sigma_n + V_2\delta_n,\ V_1\sigma_n + K_2\delta_n : n \ge 0\} < 1, \tag{4.21}$$

$$\sup\left\{\frac{1-\sigma_n}{1-K_1\sigma_n}, \frac{1-\delta_n}{1-K_2\delta_n} : n \ge 0\right\} + \sup\left\{\frac{V_1\sigma_n}{1-K_1\sigma_n}, \frac{V_2\delta_n}{1-K_2\delta_n} : n \ge 0\right\} < 1 \tag{4.22}$$

hold, then for each $x_0, y_0 \in H$, the sequences $\{x_n\}_{n\ge0}$ and $\{x_n\}_{n\ge0}$ generated by Algorithm 4.3 converge strongly to x and y, respectively.

Proof. It follows from Theorem 4.4 that the system of nonlinear variational inclusions (4.1) has a unique solution $(x,y) \in H \times H$. Now we prove that Algorithm 4.3 is well defined.

Let $n \geq 0$ be a fixed integer. For each $(x_n, y_n) \in H \times H$, define $S_1, S_2 : H \times H \rightarrow H$ and $S : H \times H \rightarrow H \times H$ by

$$\begin{aligned}
S_1(u,v) &= (1-\sigma_n)x_n \\
&\quad + \sigma_n\Big[u - \left(f_1(u) - g_1(u)\right) \\
&\qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(u) - g_1(u)\right) + \rho_1 p_1 - \rho_1 N_1(a_1(u), b_1(v))\right)\Big] + \xi_n, \quad \forall u, v \in H, \\
S_2(u,v) &= (1-\delta_n)y_n \\
&\quad + \delta_n\Big[u - \left(f_2(u) - g_2(u)\right) \\
&\qquad + J_{M_2,\rho_2}^{h_2,\eta_2}\left(h_2\left(f_2(u) - g_2(u)\right) + \rho_2 p_2 - \rho_2 N_2(a_2(u), b_2(v))\right)\Big] + \zeta_n, \quad \forall u, v \in H \\
&\qquad S(u,v) = (S_1(u,v), S_2(u,v)), \quad \forall u, v \in H.
\end{aligned} \tag{4.23}$$

As in the proof of Theorem 4.4, we infer that for any $(u_1, v_1), (u_2, v_2) \in H \times H$

$$\begin{aligned}
&\|S_1(u_1,v_1) - S_1(u_2,v_2)\| \\
&\quad \leq \sigma_n\Big\{\Big\|u_1 - u_2 - \left[f_1(u_1) - g_1(u_1) - \left(f_1(u_2) - g_1(u_2)\right)\right] \\
&\qquad + J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(u_1) - g_1(u_1)\right) + \rho_1 p_1 - \rho_1 N_1(a_1(u_1), b_1(v_1))\right) \\
&\qquad - J_{M_1,\rho_1}^{h_1,\eta_1}\left(h_1\left(f_1(u_2) - g_1(u_2)\right) + \rho_1 p_1 - \rho_1 N_1(a_1(u_2), b_1(v_2))\right)\Big\|\Big\} \\
&\quad \leq K_1\sigma_n\|u_1 - u_2\| + V_1\sigma_n\|v_1 - v_2\|, \\
&\qquad \|S_2(u_1,v_1) - S_2(u_2,v_2)\| \leq K_2\delta_n\|v_1 - v_2\| + V_2\delta_n\|u_1 - u_2\|,
\end{aligned} \tag{4.24}$$

which yield that

$$\begin{aligned}
\|S(u_1,v_1) - S(u_2,v_2)\|_1 &= \|S_1(u_1,v_1) - S_1(u_2,v_2)\| + \|S_2(u_1,v_1) - S_2(u_2,v_2)\| \\
&\leq (K_1\sigma_n + V_2\delta_n)\|u_1 - u_2\| + (V_1\sigma_n + K_2\delta_n)\|v_1 - v_2\| \\
&\leq \max\{K_1\sigma_n + V_2\delta_n,\ V_1\sigma_n + K_2\delta_n\}\|(u_1,v_1) - (u_2,v_2)\|_1.
\end{aligned} \tag{4.25}$$

(4.21) and (4.25) ensure that S is a contraction on $H \times H$. Thus S has a unique fixed point $(x_{n+1}, y_{n+1}) \in H \times H$. Consequently, for each $(x_n, y_n) \in H \times H$, there exists a unique element $(x_{n+1}, y_{n+1}) \in H \times H$ satisfying

$$\begin{aligned} x_{n+1} = {} & (1-\sigma_n)x_n \\ & + \sigma_n \Big[x_{n+1} - (f_1(x_{n+1}) - g_1(x_{n+1})) \\ & \quad + J^{h_1,\eta_1}_{M_1,\rho_1}(h_1(f_1(x_{n+1}) - g_1(x_{n+1})) + \rho_1 p_1 - \rho_1 N_1(a_1(x_{n+1}), b_1(y_{n+1}))) \Big] + \xi_n, \\ y_{n+1} = {} & (1-\delta_n)y_n \\ & + \delta_n \Big[y_{n+1} - (f_2(y_{n+1}) - g_2(y_{n+1})) \\ & \quad + J^{h_2,\eta_2}_{M_2,\rho_2}(h_2(f_2(y_{n+1}) - g_2(y_{n+1})) + \rho_2 p_2 - \rho_2 N_2(a_2(x_{n+1}), b_2(y_{n+1}))) \Big] + \zeta_n. \end{aligned} \tag{4.26}$$

That is, Algorithm 4.3 is well defined. Next we show the convergence of the sequences $\{x_n\}_{n\geq 0}$ and $\{x_n\}_{n\geq 0}$ generated by Algorithm 4.3. In light of (4.17) and (4.26), we conclude that for any $n \geq 0$

$$\begin{aligned} & \|x_{n+1} - x\| \\ & \quad \leq (1-\sigma_n)\|x_n - x\| \\ & \qquad + \sigma_n \| x_{n+1} - x - [f_1(x_{n+1}) - g_1(x_{n+1}) - (f_1(x) - g_1(x))] \| \\ & \qquad + \sigma_n \Big\| J^{h_1,\eta_1}_{M_1,\rho_1}(h_1(f_1(x_{n+1}) - g_1(x_{n+1})) \\ & \qquad\qquad + \rho_1 p_1 - \rho_1 N_1(a_1(x_{n+1}), b_1(y_{n+1}))) \\ & \qquad\qquad - J^{h_1,\eta_1}_{M_1,\rho_1}(h_1(f_1(x) - g_1(x)) + \rho_1 p_1 - \rho_1 N_1(a_1(x), b_1(y))) \Big\| + \|\xi_n\| \\ & \quad \leq (1-\sigma_n)\|x_n - x\| + K_1\sigma_n\|x_{n+1} - x\| + V_1\sigma_n \|y_{n+1} - y\| + \|\xi_n\|, \end{aligned} \tag{4.27}$$

which means that

$$\|x_{n+1} - x\| \leq \frac{1-\sigma_n}{1-K_1\sigma_n}\|x_n - x\| + \frac{V_1\sigma_n}{1-K_1\sigma_n}\|y_{n+1} - y\| + \frac{1}{1-K_1\sigma_n}\|\xi_n\|, \quad \forall n \geq 0. \tag{4.28}$$

Similarly, we have

$$\|y_{n+1} - y\| \leq \frac{1-\delta_n}{1-K_2\delta_n}\|y_n - y\| + \frac{V_2\delta_n}{1-K_2\delta_n}\|x_{n+1} - x\| + \frac{1}{1-K_2\delta_n}\|\zeta_n\|, \quad \forall n \geq 0. \tag{4.29}$$

In view of (4.28) and (4.29), we deduce that

$$\begin{aligned} &\|x_{n+1}-x\|+\|y_{n+1}-y\| \\ &\quad\le \max\left\{\frac{1-\sigma_n}{1-K_1\sigma_n},\frac{1-\delta_n}{1-K_2\delta_n}\right\}[\|x_n-x\|+\|y_n-y\|] \\ &\qquad+\max\left\{\frac{V_1\sigma_n}{1-K_1\sigma_n},\frac{V_2\delta_n}{1-K_2\delta_n}\right\}[\|x_{n+1}-x\|+\|y_{n+1}-y\|] \\ &\qquad+\max\left\{\frac{1}{1-K_1\sigma_n},\frac{1}{1-K_2\delta_n}\right\}[\|\xi_n\|+\|\zeta_n\|],\quad \forall n\ge 0, \end{aligned} \tag{4.30}$$

which implies that

$$\begin{aligned} &\|x_{n+1}-x\|+\|y_{n+1}-y\| \\ &\quad\le \frac{\max\{(1-\sigma_n)/(1-K_1\sigma_n),(1-\delta_n)/(1-K_2\delta_n)\}}{1-\max\{(V_1\sigma_n)/(1-K_1\sigma_n),(V_2\delta_n)/(1-K_2\delta_n)\}}[\|x_n-x\|+\|y_n-y\|] \\ &\qquad+\frac{\max\{1/(1-K_1\sigma_n),1/(1-K_2\delta_n)\}}{1-\max\{V_1\sigma_n/1-K_1\sigma_n,V_2\delta_n/1-K_2\delta_n\}}[\|\xi_n\|+\|\zeta_n\|] \\ &\quad\le \frac{a}{1-b}[\|x_n-x\|+\|y_n-y\|] \\ &\qquad+\frac{\max\{1/(1-K_1),1/(1-K_2)\}}{1-b}[\|\xi_n\|+\|\zeta_n\|],\quad \forall n\ge 0, \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} a&=\sup\left\{\frac{1-\sigma_n}{1-K_1\sigma_n},\frac{1-\delta_n}{1-K_2\delta_n}:n\ge 0\right\}, \\ b&=\sup\left\{\frac{V_1\sigma_n}{1-K_1\sigma_n},\frac{V_2\delta_n}{1-K_2\delta_n}:n\ge 0\right\}. \end{aligned} \tag{4.32}$$

Note that (4.22) yields that $a/(1-b)<1$. Consequently, Lemma 2.6 and (4.31) yield that the sequences $\{x_n\}_{n\ge 0}$ and $\{x_n\}_{n\ge 0}$ generated by Algorithm 4.3 converge strongly to x and y, respectively. This completes the proof. □

Remark 4.6. Lemma 4.1 and Theorems 4.4 and 4.5 extend, improve, and unify the corresponding results in [2–5, 11–19].

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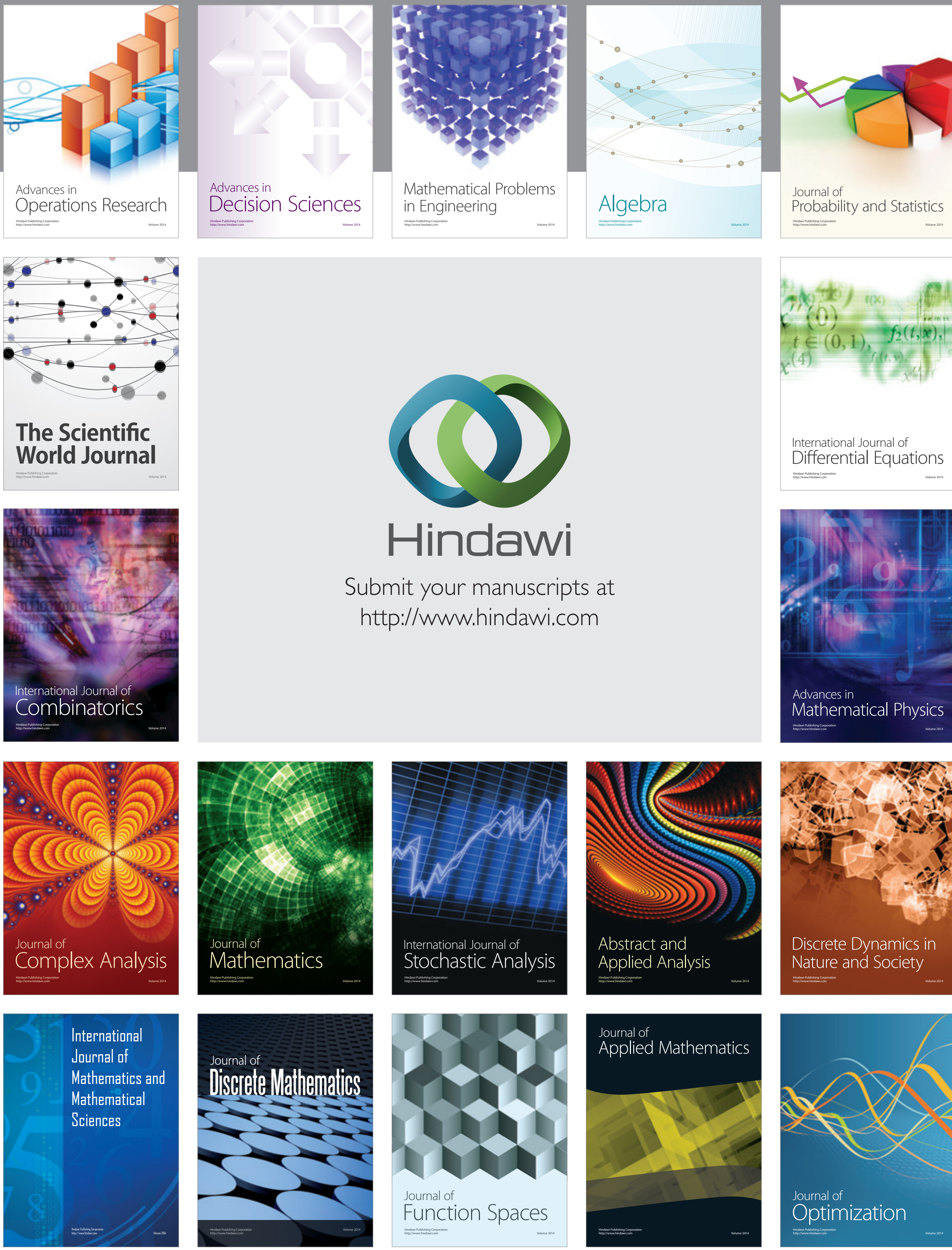
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