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Research Article

Schwarz-Pick Estimates for Holomorphic Mappings with Values in Homogeneous Ball

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Let B_X be the unit ball in a complex Banach space X. Assume B_X is homogeneous. The generalization of the Schwarz-Pick estimates of partial derivatives of arbitrary order is established for holomorphic mappings from the unit ball B^n to B_X associated with the Carathéodory metric, which extend the corresponding Chen and Liu, Dai et al. results.

1. Introduction

By the classical Pick's invariant form of Schwarz's lemma, a holomorphic function $f(z)$ which is bounded by one in the unit disk $D \subset \mathbb{C}$ satisfies the following inequlity

$$\left|f'(z)\right| \le \frac{1-\left|f(z)\right|^2}{1-|z|^2} \tag{1.1}$$

at each point z of D. Ruscheweyh in [1] firstly obtained best-possible estimates of higher order derivatives of bounded holomorphic functions on the unit disk in 1985. Recently, a lot of attention (see Ghatage et al. [2], MacCluer et al. [3], Avkhadiev and Wirths [4], Ghatage and Zheng [5], Dai and Pan [6]) has been paid to the Schwarz-Pick estimates of high-order derivative estimates in one complex variable. The best result is given as follows:

$$\left|f^{(k)}(z)\right| \le \frac{k!\left(1-\left|f(z)\right|^2\right)}{\left(1-|z|^2\right)^k}(1+|z|)^{k-1}, \quad z \in D,\ k \ge 1. \tag{1.2}$$

It is natural to consider an extension of the above Schwarz-Pick estimates to higher dimensions. Anderson et al. [7] gave Schwarz-Pick estimates of derivatives of arbitrary order of functions in the Schur-Agler class on the unit polydisk and the unit ball of $\mathbb{C}^n$, respectively. Recently, Chen and Liu in [8] obtained estimates of high-order derivatives for all the bounded holomorphic functions on the unit ball of $\mathbb{C}^n$. Later, Dai et al. in [9, 10] generalized the high order Schwarz-Pick estimates for holomorphic mappings between unit balls in complex Hilbert space. Their main result is expressed as follows.

Theorem A. *Suppose $f(z)$ is holomorphic mapping from B^n to B^m. Then for any multiindex $k \geq 1$ and $\beta \in \mathbb{C}^n \setminus \{0\}$,*

$$H_{f(z)}\left(D^k(f,z,\beta), D^k(f,z,\beta)\right) \leq k!\left(1 + \frac{(|\langle \beta, z\rangle|)}{((1-|z|^2)|\beta|^2 + |\langle \beta, z\rangle|^2)^{1/2}}\right)^{k-1} \left(H_z(\beta,\beta)\right)^k, \tag{1.3}$$

where $D^k(f,z,\beta) = \sum_{|\alpha|=k}(k!/\alpha!)(\partial f^k(z)/\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_N^{\alpha_N})\beta^\alpha$ and $H_z(\beta,\beta)$ is the Bergman metric on B^n.

In this paper, we will extend Theorem A to holomorphic mappings from the unit ball B^n to B_X associated with the Carathéodory metric. In particular, when $B_X = B^m$, our result coincides with Theorem A. Furthermore, our result shows that the high-order Schwarz-Pick estimates on the unit ball do depend on the geometric property of the image domain B_X.

Throughout this paper, the symbol X is used to denote a complex Banach space with norm $\|\cdot\|$, and $B_X = \{z \in X : \|z\| < 1\}$ is the unit ball in X. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w_j}$, where the symbol $'$ stands for the transpose of vector or matrix. The unit ball of $\mathbb{C}^n$ is always written by B^n. It is well known that if f is a holomorphic mapping from B_X into X, then the following well-known expansion

$$f(y) = \sum_{n=0}^{\infty}\frac{1}{n!}D^n f(x)\left((y-x)^n\right) \tag{1.4}$$

holds for all y in some neighborhood of $x \in B_X$, where $D^n f(x)$ means the nth Fréchet derivative of f at the point x, and

$$D^n f(x)\left((y-x)^n\right) = D^n f(x)(y-x, y-x, \ldots, y-x). \tag{1.5}$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^n X$ into X. For a domain $\Omega \in X$, a mapping $f : \Omega \to X$ is called to be biholomorphic if $f(\Omega)$ is a domain; the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. Let $\mathrm{Aut}(\Omega)$ denote the set of biholomorphic mappings of Ω onto itself. Ω is said to be homogeneous, if for each pair of points $x, y \in \Omega$, there is an $f \in \mathrm{Aut}(\Omega)$ such that $f(x) = y$.

In multiindex notation, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$.

Let $K(z,z)$ be the Bergman kernel function. Then the Bergman metric $H_z(\beta,\beta)$ can be defined as

$$H_z(\beta,\beta) = \sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k, \tag{1.6}$$

where $z \in \Omega$, $u = (u_1, u_2, \ldots, u_n) \in \mathbb{C}^n$. It is well known that $H_z(\beta,\beta) = (1-|z|^2+|\langle \beta, z\rangle|^2)/(1-|z|^2)^2$ in [9].

Let $F_c^{B_X}(z,\xi)$ be the infinitesimal form of Carathéodory metric of domain B_X. By the definition of the Carathéodory metric [11], we have for any $\xi \in X$,

$$F_c^{B_X}(z,\xi) = \sup\{|Df(z)\xi| : f \in H(B_X, B_X), f(z) = 0\}, \tag{1.7}$$

where $H(B_X, B_X)$ denotes the family of holomorphic mappings which map B_X into B_X.

2. Some Lemmas

In order to prove the main results, we need the following lemmas. Let B_X be the unit ball in a complex Banach space X, and B_X is homogeneous.

Lemma 2.1 (see [11]). *If $f \in H(B_X, B_X)$, then*

$$F_c^{B_X}(f(z), Df(z)\xi) \le F_c^{B_X}(z,\xi), \quad z \in B_X,\ \xi \in X. \tag{2.1}$$

In particular, when f is biholomorphic mapping, then $F_c^{B_X}(f(z), Df(z)\xi) = F_c^{B_X}(z,\xi)$.

Lemma 2.2 (see [12]). *Consider the following:*

$$F_c^{B_X}(0,\xi) = \|\xi\|, \quad \xi \in X. \tag{2.2}$$

Lemma 2.3. *Let $f \in H(D, B_X)$. Then f can be written with the following n-variable power series given by*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in D. \tag{2.3}$$

Then the following holds

$$F_c^{B_X}(a_0, a_k) \le 1 \tag{2.4}$$

for any integer $k \ge 0$.

Proof. For the fixed k, we define

$$f_k(z) = \sum_{j=1}^{k} \frac{f\left(e^{i(2\pi j/k)}z\right)}{k}. \tag{2.5}$$

Then $f_k \in H(D, B_X)$. It is clear that

$$\frac{1}{k}\sum_{j=1}^{k} e^{i(2\pi j l/k)} = \begin{cases} 1, & \text{if } l \equiv 0 \ (\mathrm{mod} k), \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

From the power series expansion of the holomorphic function f, we get

$$\begin{aligned} f_k(z) &= \frac{1}{k}\left(\sum_{j=1}^{k}\left(a_0 + \sum_{l=1}^{\infty} e^{i(2\pi j l/k)} \sum_{|\alpha|=l} a_\alpha z^\alpha\right)\right) \\ &= a_0 + \sum_{l=1}^{\infty} a_{lk} z^{lk}. \end{aligned} \tag{2.7}$$

In terms of the homogeneity of B_X, we can take $\Psi \in \mathrm{Aut}(B_X)$ and $\Psi(a_0) = 0$, then $\Psi \circ f_k \in H(D, B_X)$. This implies that

$$\begin{aligned} \Psi \circ f_k(z) &= \Psi\left(a_0 + \sum_{l=1}^{\infty} a_{lk} z^{lk}\right) \\ &= \Psi(a_0) + D\Psi(a_0)\left(\sum_{l=1}^{\infty} a_{lk} z^{lk}\right) + D^2\Psi(a_0)\left(\sum_{l=1}^{\infty} a_{lk} z^{lk}\right) + \cdots \\ &= D\Psi(a_0)(a_k)z^k + D\Psi(a_0)(a_{2k})z^{2k} + D\Psi(a_0)(a_{3k})z^{3k} + \cdots. \end{aligned} \tag{2.8}$$

By making use of the orthogonality, we obtain

$$D\Psi(a_0)(a_\alpha)z^\alpha = \frac{1}{2\pi}\int_0^{2\pi} (\Psi \circ f_k)\left(ze^{i\theta}\right) e^{-i\alpha\theta} d\theta. \tag{2.9}$$

Hence,

$$\|D\Psi(a_0)(a_\alpha)z^\alpha\| \leq \frac{1}{2\pi}\int_0^{2\pi} \left\|(\Psi \circ f_k)\left(ze^{i\theta}\right)e^{-i\alpha\theta}\right\| d\theta \leq 1. \tag{2.10}$$

This implies the following inequality

$$\|D\Psi(a_0)(a_\alpha)\| |z|^\alpha \leq 1 \tag{2.11}$$

holds for any $z \in D$. Thus,

$$\|D\Psi(a_0)(a_\alpha z^\alpha)\| \leq 1 \tag{2.12}$$

holds for any $z \in \overline{D}$. It means that $\|D\Psi(a_0)(a_\alpha)\| \leq 1$.

By Lemmas 2.1 and 2.2, we obtain

$$F_c^{B_X}(a_0, a_\alpha) = F_c^{B_X}(0, D\Psi(a_0)(a_\alpha)) = \|D\Psi(a_0)(a_\alpha)\| \leq 1, \tag{2.13}$$

which is the desired result. □

3. Main Results

Theorem 3.1. *Let $f : D \to B_X$ be a holomorphic mapping. Then the following inequality*

$$F_c^{B_X}\left(f(z), f^{(k)}(z)\right) \leq k! \frac{(1+|z|)^{k-1}}{\left(1-|z|^2\right)^k} \tag{3.1}$$

holds for $k \geq 1$ and $z \in D$.

Proof. Let $g(\xi)$ be a holomorphic function on D defined by

$$g(\xi) = f\left(\frac{z+\xi}{1+\overline{z}\xi}\right), \quad \xi \in D. \tag{3.2}$$

Then g can be written as a power series as follows:

$$g(\xi) = \sum_{j=0}^{\infty} a_j \xi^j. \tag{3.3}$$

In order to obtain Theorem 3.1, we need to prove the following equality:

$$f^{(k)}(z) = \frac{k!}{1-|z|^2} \sum_{j=0}^{k} \binom{k-1}{j} a_{k-j} \overline{z}^{|j|}. \tag{3.4}$$

Let $0 < r < 1$ such that $D(z, r) \subset D$, the Cauchy integral formula shows that

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z} dw. \tag{3.5}$$

Thus,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_{|w|=r}\frac{f(w)}{(w-z)^{k+1}}dw. \tag{3.6}$$

Let $w = (z+\xi)/(1+\overline{z}\xi)$. Then

$$\frac{dw}{d\xi} = \frac{1-|z|^2}{(1+\overline{z}\xi)^2}, \quad w - z = \xi\frac{1-|z|^2}{(1+\overline{z}\xi)^2}. \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\begin{aligned} f^{(k)}(z) &= \frac{k!}{2\pi i\left(1-|z|^2\right)^k}\int_{|(z+\xi)/(1+z\bar{\xi})|=r}\frac{g(\xi)(1+\overline{z}\xi)^{k-1}}{\xi^{k+1}}d\xi \\ &= \frac{k!}{\left(1-|z|^2\right)^k}\sum_{j=0}^{k-1}\binom{k-1}{j}a_{k-j}\overline{z}^j, \end{aligned} \tag{3.8}$$

which prove the equality (3.4).

From Lemma 2.3, we have for any integer $k \geq 1$,

$$F_c^{B_X}(a_0, a_k) \leq 1. \tag{3.9}$$

This implies that

$$\begin{aligned} F_c^{B_X}\left(f(z), f^{(k)}(z)\right) &\leq F_c^{B_X}\left(a_0, \frac{k!}{\left(1-|z|^2\right)^k}\sum_{j=0}^{k-1}\binom{k-1}{j}a_{k-j}|z|^j\right) \\ &\leq \frac{k!}{\left(1-|z|^2\right)^k}(1+|z|)^{k-1} \end{aligned} \tag{3.10}$$

which completes the desired result. □

Remark 3.2. If $B_X = D$, then the inequality (3.1) reduces to

$$\left|f^{(k)}(z)\right| \leq k!\frac{1-\left|f(z)\right|^2}{\left(1-|z|^2\right)^k}(1+|z|)^{k-1} \tag{3.11}$$

which coincides with the Theorem 1.1 of Dai and Pan [6] in one complex variable.

Theorem 3.3. *Let $f : B^n \to B_X$ be a holomorphic mapping. Then the following inequality*

$$F_c^{B_X}\left(f(z), D^k(f,z,\beta)\right) \le k!\left(1+\frac{|\langle \beta, z\rangle|}{\left(\left(1-|z|^2\right)|\beta|^2+|\langle \beta, z\rangle|^2\right)^{1/2}}\right)^{k-1}\left[F_c^{B^n}(z,\beta)\right]^k \quad (3.12)$$

holds for $k \ge 1$, $\beta \in \mathbb{C}^n \setminus \{0\}$ and $z \in B^n$.

Proof. For any fixed $k \ge 1$, $\beta \in \partial B^n$, and $\xi \in B^n$. Define the following disk:

$$\Delta = \left\{\lambda \in \mathbb{C} : |\xi + \lambda\beta|^2 < 1\right\}. \quad (3.13)$$

Notice that $\langle \beta, \xi - \langle \xi, \beta\rangle\beta\rangle = 0$. Hence,

$$\begin{aligned} |\xi + \lambda\beta|^2 &= |(\lambda + \langle \xi, \beta\rangle)\beta + \xi - \langle \xi, \beta\rangle\beta|^2 \\ &= |\lambda + \langle \xi, \beta\rangle|^2 + |\xi - \langle \xi, \beta\rangle\beta|^2 < 1. \end{aligned} \quad (3.14)$$

That is,

$$|\lambda + \langle \xi, \beta\rangle| < \sqrt{1 - |\xi - \langle \xi, \beta\rangle\beta|^2} = \sqrt{1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2}. \quad (3.15)$$

Set $\sigma = \sqrt{1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2}$. For the fixed ξ and β, we define

$$g(\omega) = f(\xi + (\omega\sigma - \langle \xi, \beta\rangle)\beta), \quad \omega \in D. \quad (3.16)$$

Then $g(\omega)$ is holomorphic mapping from the unit disk D to the homogeneous domain Ω.

According to Theorem 3.1 to the functions g and $\omega' = (\langle \xi, \beta\rangle)/\sigma$, we have

$$F_c^{B_X}\left(g(\omega'), g^{(k)}(\omega')\right) \le k!\frac{(1+|\omega'|)^{k-1}}{\left(1-|\omega'|^2\right)^k}, \quad (3.17)$$

which holds for $k \ge 1$. Since $g(\omega') = f(\xi)$, and

$$|\omega'| = \frac{|\langle \beta, \xi\rangle|}{\sqrt{1-|\xi|^2+|\langle \xi, \beta\rangle|^2}}, \qquad 1 - |\omega'|^2 = \frac{1-|\xi|^2}{1-|\xi|^2+|\langle \xi, \beta\rangle|^2}. \quad (3.18)$$

In terms of the chain rule, we have

$$g^{(k)}(\omega') = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_N^{\alpha_N}}(\sigma\beta)^\alpha = \sigma^k\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_N^{\alpha_N}}\beta^\alpha = \sigma^k D^k(f,\xi,\beta). \quad (3.19)$$

Hence,

$$F_c^{B_X}\Big(f(\xi),\sigma^k D^k(f,\xi,\beta)\Big) \le k!\left(1+\frac{|\langle\beta,\xi\rangle|}{(1-|\xi|^2+|\langle\beta,\xi\rangle|^2)^{1/2}}\right)^{k-1}\left[\frac{(1-|\xi|^2)+|\langle\beta,\xi\rangle|^2}{\left(1-|\xi|^2\right)^2}\right]^k\sigma^k. \tag{3.20}$$

Note the definition of Carathéodory metric and $F_c^{B^n}(z,\beta) = (1-|z|^2+|\langle\beta,z\rangle|^2)/(1-|z|^2)^2$ in [11], we can get

$$F_c^{B_X}\Big(f(z),D^k(f,z,\beta)\Big) \le k!\left(1+\frac{|\langle\beta,z\rangle|}{(1-|z|^2+|\langle\beta,z\rangle|^2)^{1/2}}\right)^{k-1}\Big[F_c^{B^n}(z,\beta)\Big]^k. \tag{3.21}$$

This gives the proof of the case $z=\xi$ and $\beta\in\partial B_n$. For general vector $\beta\in\mathbb{C}^n\setminus\{0\}$, we may substitute $\beta/\|\beta\|$ for β. By the homogeneous of β from the above inequality, we can obtain the same result, which completes the proof of the Theorem 3.3. □

Remark 3.4. If $B_X = B^m$, then $H_{f(z)}(D^k(f,z,\beta),D^k(f,z,\beta)) = F_c^{B^m}(f(z),D^k(f,z,\beta))$ and $H_z(\beta,\beta) = F_c^{B^m}(z,\beta)$. Thus, the Theorem 3.3 reduces to Theorem A established by Dai et al. [9].

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