Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 648635, 21 pages
doi:10.1155/2012/648635

Research Article

Existence and Multiplicity of Solutions for Some Fractional Boundary Value Problem via Critical Point Theory

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Received 18 October 2011; Accepted 27 November 2011

Academic Editor: Kanishka Perera

We study the existence and multiplicity of solutions for the following fractional boundary value problem: $(d/dt)((1/2)_0D_t^{-\beta}(u'(t))+(1/2)_tD_T^{-\beta}(u'(t)))+\nabla F(t,u(t))=0$, a.e. $t\in[0,T]$, $u(0)=u(T)=0$, where $F(t,\cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. Several examples are presented to illustrate our results.

1. Introduction and Main Results

Consider the fractional boundary value problem (BVP for short) of the following form:

$$\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right)+\nabla F(t,u(t))=0,\quad \text{a.e. } t\in[0,T],$$
$$u(0)=u(T)=0, \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\beta<1$, respectively, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumptions.

(A) $F(t,x)$ is measurable in t for every $x\in\mathbb{R}^N$ and continuously differentiable in x for a.e. $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$, such that

$$|F(t,x)|\le a(|x|)b(t),\qquad |\nabla F(t,x)|\le a(|x|)b(t), \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. In particular, if $\beta = 0$, BVP (1.1) reduces to the standard second-order boundary value problem of the following form:

$$\begin{gathered} u''(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0. \end{gathered} \tag{1.3}$$

Differential equations with fractional order are generalization of ordinary differential equations to noninteger order. This generalization is not mere mathematical curiosities but rather has interesting applications in many areas of science and engineering such as in viscoelasticity, electrical circuits, and neuron modeling. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modeled by differential equations with integer derivatives. Such differential equations got the attention of many researchers and considerable work has been done in this regard, see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Samko et al. [4], and the papers [5–20] and the references therein.

Recently, there are many papers dealing with the existence of solutions (or positive solutions) of nonlinear initial (or singular and nonsingular boundary) value problems of fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems [12–14], Leray-Schauder theory [15, 16], lower and upper solution method, monotone iterative method [17–19], Adomian decomposition method [20], etc.), see [12–20] and the references therein.

Variational methods are very powerful techniques in nonlinear analysis and are extensively used in many disciplines of pure and applied mathematics including ordinary and partial differential equations, mathematical physics, gauge theory, and geometrical analysis. The existence and multiplicity of solutions for Hamilton systems, Schrödinger equations, and Dirac equations have been studied extensively via critical point theory, see [21–34].

In [32], Jiao and Zhou obtained the existence of solutions for BVP (1.1) by Mountain Pass theorem under the Ambrosetti-Rabinowitz condition (denoted by A.R. condition). Under the usual A.R. condition, it is easy to show that the energy functional associated with the system has the Mountain Pass geometry and satisfies the (PS) condition. However, the A.R. condition is so strong that many potential functions cannot satisfy it, then the problem becomes more delicate and complicated.

In this paper, in order to establish the existence and multiplicity of solutions for BVP (1.1) under distinct hypotheses on potential function by critical point theory, we introduce some functional space E^α, where $\alpha \in (1/2, 1]$, and divide the problem into the following three cases.

1.1. The Superquadratic Case

For the superquadratic case, we make the following assumptions.

(A1) $\lim_{|x|\to 0} F(t,x)/|x|^2 = 0$, $\liminf_{|x|\to\infty} F(t,x)/|x|^2 \geq L > \pi^2/|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)$ uniformly for some $L > 0$ and a.e. $t \in [0,T]$.

(A2) $\limsup_{|x|\to+\infty} F(t,x)/|x|^r \leq M < +\infty$ uniformly for some $M > 0$ and a.e. $t \in [0,T]$.

(A3) $\liminf_{|x|\to+\infty}((\nabla F(t,x),x)-2F(t,x))/|x|^{\mu} \geq Q > 0$ uniformly for some $Q > 0$ and a.e. $t \in [0,T]$,

where $r > 2$ and $\mu > r-2$. We state our first existence result as follows.

Theorem 1.1. *Assume that (A1)–(A3) hold and that $F(t,x)$ satisfies the condition (A). Then BVP* (1.1) *has at least one solution on E^{α}.*

1.2. The Asymptotically Quadratic Case

For the asymptotically quadratic case, we assume the following.

(A2′) $\limsup_{|x|\to+\infty} F(t,x)/|x|^2 \leq M < +\infty$ uniformly for some $M > 0$ and a.e. $t \in [0,T]$.

(A4) There exists $\tau(t) \in L^1(0,T;\mathbb{R}^+)$ such that $(\nabla F(t,x),x)-2F(t,x) \geq \tau(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

(A5) $\lim_{|x|\to+\infty}[(\nabla F(t,x),x)-2F(t,x)] = +\infty$ for a.e. $t \in [0,T]$.

Our second and third main results read as follows.

Theorem 1.2. *Assume that $F(t,x)$ satisfies (A), (A1), (A2′), (A4), and (A5). Then BVP* (1.1) *has at least one solution on E^{α}.*

Theorem 1.3. *Assume that $F(t,x)$ satisfies (A), (A1), (A2′), and the following conditions:*

(A4′) *there exists $\tau(t) \in L^1(0,T;\mathbb{R}^+)$ such that $(\nabla F(t,x),x)-2F(t,x) \leq \tau(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;*

(A5′) $\lim_{|x|\to+\infty}[(\nabla F(t,x),x)-2F(t,x)] = -\infty$ *for a.e. $t \in [0,T]$.*

Then BVP (1.1) *has at least one solution on E^{α}.*

1.3. The Subquadratic Case

For the subquadratic case, we give the following multiplicity result.

Theorem 1.4. *Assume that $F(t,x)$ satisfies the following assumption:*

(A6) $F(t,x) := a(t)|x|^{\gamma}$*, where $a(t) \in L^{\infty}(0,T;\mathbb{R}^+)$ and $1 < \gamma < 2$ is a constant.*

Then BVP (1.1) has infinitely many solutions on E^{α}.

2. Preliminaries

In this section, we recall some background materials in fractional differential equation and critical point theory. The properties of space E^{α} are also listed for the convenience of readers.

Definition 2.1 (see [1]). Let $f(t)$ be a function defined on $[a,b]$ and $q>0$. The left and right Riemann-Liouville fractional integrals of order q for function $f(t)$ denoted by ${}_aD_t^{-q}f(t)$ and ${}_tD_b^{-q}f(t)$, respectively, are defined by

$$\begin{aligned}{}_aD_t^{-q}f(t)&=\frac{1}{\Gamma(q)}\int_a^t(t-s)^{q-1}f(s)ds,\\ {}_tD_b^{-q}f(t)&=\frac{1}{\Gamma(q)}\int_t^b(t-s)^{q-1}f(s)ds,\end{aligned}\tag{2.1}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where Γ is the gamma function.

Definition 2.2 (see [1]). Let $f(t)$ be a function defined on $[a,b]$ and $q>0$. The left and right Riemann-Liouville fractional derivatives of order q for function $f(t)$ denoted by ${}_aD_t^{q}f(t)$ and ${}_tD_b^{q}f(t)$, respectively, are defined by

$$\begin{aligned}{}_aD_t^{q}f(t)&=\frac{d^n}{dt^n}\,{}_aD_t^{q-n}f(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\left(\int_a^t(t-s)^{n-q-1}f(s)ds\right),\\ {}_tD_b^{q}f(t)&=(-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{q-n}f(t)=\frac{1}{\Gamma(n-q)}(-1)^n\frac{d^n}{dt^n}\left(\int_t^b(s-t)^{n-q-1}f(s)ds\right),\end{aligned}\tag{2.2}$$

where $t\in[a,b]$, $n-1\le q<n$ and $n\in\mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives. In particular, they are defined for the function belonging to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)$ $(k=1,\ldots)$ is the space of functions f such that $f\in C^{k-1}([a,b],\mathbb{R}^N)$ and $f^{(k-1)}\in AC([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N)=AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3 (see [1]). Let $q\ge 0$ and $n\in\mathbb{N}$. If $q\in[n-1,n)$ and $f(t)\in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivative of order q for function $f(t)$ denoted by ${}_a^cD_t^qf(t)$ and ${}_t^cD_b^qf(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^qf(t)$ and ${}_t^cD_b^qf(t)$ are represented by

$${}_a^cD_t^qf(t)={}_aD_t^{q-n}f^{(n)}(t)=\frac{1}{\Gamma(n-q)}\left(\int_a^t(t-s)^{n-q-1}f^{(n)}(s)ds\right),\tag{2.3}$$

$${}_t^cD_b^qf(t)=(-1)^n\,{}_tD_b^{q-n}f^{(n)}(t)=\frac{(-1)^n}{\Gamma(n-q)}\left(\int_t^b(s-t)^{n-q-1}f^{(n)}(s)ds\right),\tag{2.4}$$

respectively, where $t\in[a,b]$.

Property 2.4 (see [1]). The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, that is,

$$ {}_aD_t^{-q_1}\left({}_aD_t^{-q_2}f(t)\right)={}_aD_t^{-q_1-q_2}f(t), \qquad {}_tD_b^{-q_1}\left({}_tD_b^{-q_2}f(t)\right)={}_tD_b^{-q_1-q_2}f(t), \quad \forall q_1, q_2>0. \tag{2.5} $$

Definition 2.5 (see [32]). Define $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R}^N)$ with respect to the norm

$$ \|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T \left|{}_0^cD_t^{\alpha}u(t)\right|^p dt\right)^{1/p}, \quad \forall u\in E_0^{\alpha,p}, \tag{2.6} $$

where $C_0^{\infty}([0,T],\mathbb{R}^N)$ denotes the set of all functions $u\in C^{\infty}([0,T],\mathbb{R}^N)$ with $u(0)=u(T)=0$. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u\in L^p(0,T;\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha}u\in L^p(0,T;\mathbb{R}^N)$ and $u(0)=u(T)=0$.

Proposition 2.6 (see [32]). *Let $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Proposition 2.7 (see [32]). *Let $0<\alpha\le 1$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$, one has*

$$ \|u\|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}\left\|{}_0^cD_t^{\alpha}u\right\|_{L^p}. \tag{2.7} $$

Moreover, if $\alpha>1/p$ and $1/p+1/q=1$, then

$$ \|u\|_{\infty} \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{1/q}}\left\|{}_0^cD_t^{\alpha}u\right\|_{L^p}. \tag{2.8} $$

According to (2.8), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$ \|u\|_{\alpha,p} = \left\|{}_0^cD_t^{\alpha}u\right\|_{L^p} = \left(\int_0^T \left|{}_0^cD_t^{\alpha}u(t)\right|^p dt\right)^{1/p}. \tag{2.9} $$

Proposition 2.8 (see [32]). *Define $0<\alpha\le 1$ and $1<p<\infty$. Assume that $\alpha>1/p$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, that is, $u_k \rightharpoonup u$. Then $u_k\to u$ in $C([0,T],\mathbb{R}^N)$, that is, $\|u-u_k\|_{\infty}\to 0$, as $k\to\infty$.*

Making use of Property 2.4 and Definition 2.3, for any $u\in AC([0,T],\mathbb{R}^N)$, BVP (1.1) is equivalent to the following problem:

$$ \begin{gathered} \frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{\alpha-1}\left({}_0^cD_t^{\alpha}u(t)\right) - \frac{1}{2}\,{}_tD_T^{\alpha-1}\left({}_t^cD_T^{\alpha}u(t)\right)\right) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t\in[0,T], \\ u(0)=u(T)=0, \end{gathered} \tag{2.10} $$

where $\alpha = 1-\beta/2\in(1/2,1]$.

In the following, we will treat BVP (2.10) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$. The variational structure of BVP (2.10) on the space E^α has been established.

Lemma 2.9 (see [32]). *Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$L(t,x,y,z) = -\frac{1}{2}(y,z) - F(t,x), \tag{2.11}$$

where $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the assumption (A).

If $1/2 < \alpha \le 1$, then the functional defined by

$$\varphi(u) = \int_0^T L\left(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right) dt \tag{2.12}$$

is continuously differentiable on E^α, and $\forall u, v \in E^\alpha$, we have

$$\begin{aligned}\left\langle \varphi'(u), v\right\rangle = &\int_0^T \left(D_x L\left(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right), v(t)\right) dt \\ &+ \int_0^T \left(D_y L\left(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right), {}_0^cD_t^\alpha v(t)\right) dt \\ &+ \int_0^T \left(D_z L\left(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right), {}_t^cD_T^\alpha v(t)\right) dt.\end{aligned} \tag{2.13}$$

Definition 2.10 (see [32]). A function $u \in AC([0,T], \mathbb{R}^N)$ is called a solution of BVP (2.10) if

(i) $D^\alpha(u(t))$ is derivative for almost every $t \in [0,T]$,

(ii) u satisfies (2.10),

where $D^\alpha(u(t)) := (1/2)\, {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - (1/2)\, {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))$.

Lemma 2.11 (see [32]). *Let $1/2 < \alpha \le 1$ and φ be defined by* (2.12)*. If assumption (A) is satisfied and $u \in E^\alpha$ is a solution of corresponding Euler equation $\varphi'(u) = 0$, then u is a solution of BVP* (2.10) *which corresponding to the solution of BVP* (1.1).

By Lemma 2.11, it means that the solutions for BVP (1.1) correspond to the critical points of the functional φ. We need the following estimate and known results for the sequel.

Proposition 2.12 (see [32]). *If $1/2 < \alpha \le 1$, then for any $u \in E^\alpha$, one has*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T \left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right) dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.14}$$

Lemma 2.13 (see [23]). *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ is differentiable. One says that Φ satisfies the (PS) condition if any sequence $\{u_k\}$ in X such that $\{\Phi(u_k)\}$ is bounded and $\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.*

Lemma 2.14 (Mountain Pass theorem [24]). *Let X be a real Banach space and $\Phi : X \to \mathbb{R}$ is differentiable and satisfies the (PS) condition. Suppose that*

(i) $\Phi(0) = 0$,

(ii) *there exist $\rho > 0$ and $\sigma > 0$ such that $\Phi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,*

(iii) *there exists z_1 in X with $\|z_1\| \geq \rho$ such that $\Phi(z_1) < \sigma$.*

Then Φ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{g \in \Omega} \max_{z \in g([0,1])} \Phi(z), \tag{2.15}$$

where $\overline{\Omega} = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = z_1\}$.

Lemma 2.15 (Clark theorem [24]). *Let X be a real Banach space, $\Phi \in C^1(X, \mathbb{R})$ with Φ even, bounded below, and satisfying the (PS) condition. Suppose $\Phi(0) = 0$, there is a set $K \subset X$ such that K is homeomorphic to S^{m-1}, $m \in \mathbb{N}$, by an odd map, and $\sup_K \Phi < 0$. Then Φ possesses at least m distinct pairs of critical points.*

3. Proof of the Theorems

For $u \in E^\alpha$, where

$$E^\alpha := \left\{ u \in L^2\left(0, T; \mathbb{R}^N\right) : {}_0^cD_t^\alpha u \in L^2\left(0, T; \mathbb{R}^N\right) \right\} \tag{3.1}$$

is a reflexive Banach space with the norm defined by

$$\begin{aligned} \|u\|_\alpha &= \left\| {}_0^cD_t^\alpha u \right\|_{L_2}, \\ \|u\|_\infty &:= \max_{t \in [0,T]} |u(t)|. \end{aligned} \tag{3.2}$$

It follows from Lemma 2.9 that the functional φ on E^α given by

$$\varphi(u) = \int_0^T \left[-\frac{1}{2} \left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t) \right) - F(t, u(t)) \right] dt \tag{3.3}$$

is continuously differentiable on E^α. Moreover, we have

$$\begin{aligned} \langle \varphi'(u), v \rangle = &- \int_0^T \frac{1}{2} \left[\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t) \right) + \left({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t) \right) \right] dt \\ &- \int_0^T (\nabla F(t, u(t)), v(t)) dt. \end{aligned} \tag{3.4}$$

Recall that a sequence $\{u_n\} \in E^\alpha$ is said to be a (C) sequence of φ if $\varphi(u_n)$ is bounded and $(1 + \|u_n\|_\alpha)\|\varphi(u_n)\|_\alpha \to 0$ as $n \to \infty$. The functional φ satisfies condition (C) if every (C) sequence of φ has a convergent subsequence. This condition is due to Cerami [21].

3.1. Proof of Theorem 1.1

We will first establish the following lemma and then give the proof of Theorem 1.1.

Lemma 3.1. *Assume (A), (A2), and (A3) hold, then the functional φ satisfies condition (C).*

Proof of Lemma 3.1. Let $\{u_n\} \subset E^\alpha$ be a (C) sequence of φ, that is, $\varphi(u_n)$ is bounded and $(1 + \|u_n\|_\alpha)\|\varphi'(u_n)\|_\alpha \to 0$ as $n \to \infty$. Then there exists M_0 such that

$$\left|\varphi(u_n)\right| \leq M_0, \qquad (1 + \|u_n\|_\alpha)\left\|\varphi'(u_n)\right\|_\alpha \leq M_0, \tag{3.5}$$

for all $n \in \mathbb{N}$.

By (A2), there exist positive constants B_1 and M_1 such that

$$F(t,x) \leq B_1|x|^r, \tag{3.6}$$

for all $|x| \geq M_1$ and a.e. $t \in [0,T]$.

It follows from (A) that

$$|F(t,x)| \leq \max_{s\in[0,M_1]} a(s)b(t), \tag{3.7}$$

for all $|x| \leq M_1$ and a.e. $t \in [0,T]$. Therefore, we obtain

$$F(t,x) \leq B_1|x|^r + \max_{s\in[0,M_1]} a(s)b(t), \tag{3.8}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Combining (2.14) and (3.8), we get

$$\begin{aligned}\frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 &\leq \varphi(u_n) + \int_0^T F(t,u_n(t))dt \\ &\leq M_0 + \max_{s\in[0,M_1]} a(s)\int_0^T b(t)dt + B_1\int_0^T |u_n(t)|^r dt.\end{aligned} \tag{3.9}$$

On the other hand, by (A3), there exist $\eta > 0$ and $M_2 > 0$ such that

$$(\nabla F(t,x),x) - 2F(t,x) \geq \eta|x|^\mu, \tag{3.10}$$

for a.e. $t \in [0,T]$ and $|x| \geq M_2$.

By (A), we have

$$|(\nabla F(t,x),x) - 2F(t,x)| \leq (2 + M_2)\max_{s\in[0,M_2]} a(s)b(t), \tag{3.11}$$

for all $|x| \leq M_2$ and a.e. $t \in [0,T]$.

Therefore, we obtain

$$(\nabla F(t,x),x)-2F(t,x)\geq \eta|x|^{\mu}-\eta M_2^{\mu}-(2+M_2)\max_{s\in[0,M_2]}a(s)b(t), \tag{3.12}$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$.

It follows from (3.5) and (3.12) that

$$\begin{aligned} 3M_0 &\geq 2\varphi(u_n)-\langle\varphi'(u_n),u_n\rangle \\ &= 2\int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^{\alpha}u_n(t),\,{}_t^cD_T^{\alpha}u_n(t)\right)-F(t,u_n(t))\right]dt \\ &\quad -\int_0^T\left[-\left({}_0^cD_t^{\alpha}u_n(t),{}_t^cD_T^{\alpha}u_n(t)\right)-(\nabla F(t,u_n(t)),u_n(t))\right]dt \\ &= \int_0^T[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt \\ &\geq \eta\int_0^T|u_n(t)|^{\mu}dt-(2+M_2)\max_{s\in[0,M_2]}a(s)\int_0^T b(t)dt-\eta M_2^{\mu}T, \end{aligned} \tag{3.13}$$

thus, $\int_0^T|u_n(t)|^{\mu}dt$ is bounded.

If $\mu>r$, then

$$\int_0^T|u_n(t)|^r dt\leq T^{(\mu-r)/\mu}\left(\int_0^T|u_n(t)|^{\mu}dt\right)^{r/\mu}, \tag{3.14}$$

which, combining (3.9), implies that $\|u_n\|_{\alpha}$ is bounded.

If $\mu\leq r$, then

$$\int_0^T|u_n(t)|^r dt\leq\|u_n\|_{\infty}^{r-\mu}\int_0^T|u_n(t)|^{\mu}dt\leq C_1^{r-\mu}\|u_n\|_{\alpha}^{r-\mu}\int_0^T|u_n(t)|^{\mu}dt, \tag{3.15}$$

where

$$C_1:=\frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}, \tag{3.16}$$

by (2.8).

Since $\mu>r-2$, it follows from (3.9) that $\|u_n\|_{\alpha}$ is bounded too. Thus $\|u_n\|_{\alpha}$ is bounded in E^{α}.

By Proposition 2.8, the sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n\rightharpoonup u\quad\text{weakly in } E^{\alpha},\qquad u_n\longrightarrow u\quad\text{strongly in } C\left([0,T],\mathbb{R}^N\right). \tag{3.17}$$

Then we obtain $u_n \to u$ in E^α by use of the same argument of Theorem 5.2 in [32]. The proof of Lemma 3.1 is completed. □

Proof of Theorem 1.1. By (A1), there exist $\epsilon_1 \in (0, |\cos(\pi\alpha)|)$ and $\delta > 0$ such that

$$F(t,x) \leq (|\cos(\pi\alpha)| - \epsilon_1)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}|x|^2, \tag{3.18}$$

for a.e. $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta$.

Let

$$\rho = \frac{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}}{T^{\alpha-1/2}}\delta, \qquad \sigma = \frac{\epsilon_1\rho^2}{2} > 0. \tag{3.19}$$

Then it follows from (2.8) that

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}}\|u\|_\alpha = \delta, \tag{3.20}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$.

Therefore, we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right) - F(t,u(t))\right]dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - (|\cos(\pi\alpha)| - \epsilon_1)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}\int_0^T |u(t)|^2 dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{|\cos(\pi\alpha)| - \epsilon_1}{2}\|u\|_\alpha^2 \\
&= \frac{\epsilon_1}{2}\|u\|_\alpha^2 \\
&= \sigma,
\end{aligned} \tag{3.21}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. This implies that (ii) in Lemma 2.14 is satisfied.

It is obvious from the definition of φ and (A1) that $\varphi(0) = 0$, and therefore, it suffices to show that φ satisfies (iii) in Lemma 2.14.

By (A1), there exist $\epsilon_2 > 0$ and $M_3 > 0$ such that

$$F(t,x) > \left(\frac{\pi^2}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)} + \epsilon_2\right)|x|^2, \tag{3.22}$$

for all $|x| \geq M_3$ and a.e. $t \in [0,T]$.

It follows from (A) that

$$|F(t,x)| \le \max_{s\in[0,M_3]} a(s)b(t), \tag{3.23}$$

for all $|x| \le M_3$ and a.e. $t \in [0,T]$.

Therefore, we obtain

$$F(t,x) \ge \left(\frac{\pi^2}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)} + \epsilon_2\right)\left(|x|^2 - M_3^2\right) - \max_{s\in[0,M_3]} a(s)b(t), \tag{3.24}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Choosing $u_0 = ((T/\pi)\sin(\pi t/T), 0, \ldots, 0) \in E^\alpha$, then

$$\|u_0\|_{L_2}^2 = \frac{T^3}{2\pi^2}, \qquad \|u_0\|_\alpha^2 \le \frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}. \tag{3.25}$$

For $\varsigma > 0$ and noting that (3.24) and (3.25), we have

$$\begin{aligned}
\varphi(\varsigma u_0) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha \varsigma u_0(t), {}_t^cD_T^\alpha \varsigma u_0(t)\right) - F(t, \varsigma u_0(t))\right] dt \\
&\le \frac{\varsigma^2}{2|\cos(\pi\alpha)|}\|u_0\|_\alpha^2 - \left(\frac{\varsigma^2\pi^2}{|\cos(\pi\alpha)|T^{2\alpha}\Gamma^2(2-\alpha)(3-2\alpha)} + \varsigma^2\epsilon_2\right)\int_0^T |u_0(t)|^2 dt + C_2 \\
&\le \frac{\varsigma^2}{2|\cos(\pi\alpha)|}\cdot\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)} - \frac{\varsigma^2\pi^2}{|\cos(\pi\alpha)|T^{2\alpha}\Gamma^2(2-\alpha)(3-2\alpha)}\cdot\frac{T^3}{2\pi^2} \\
&\quad - \frac{\varsigma^2\epsilon_2 T^3}{2\pi^2} + C_2 \\
&\longrightarrow -\infty,
\end{aligned} \tag{3.26}$$

as $\varsigma \to \infty$, where C_2 is a positive constant. Then there exists a sufficiently large ς_0 such that $\varphi(\varsigma_0 u_0) \le 0$. Hence (iii) holds.

Finally, noting that $\varphi(0) = 0$ while for critical point u, $\varphi(u) \ge \sigma > 0$. Hence u is a nontrivial solution of BVP (1.1), and this completes the proof. □

3.2. Proof of Theorem 1.2

The following lemmata are needed in the proof of Theorem 1.2.

Lemma 3.2. *Assume (A5), then for any $\varepsilon > 0$, there exists a subset $E_\varepsilon \subset [0,T]$ with* $\operatorname{meas}([0,T] \setminus E_\varepsilon) < \varepsilon$ *such that*

$$\lim_{|x|\to\infty} [(\nabla F(t,x),x) - 2F(t,x)] = +\infty, \tag{3.27}$$

uniformly for $t \in E_\varepsilon$.

Proof of Lemma 3.2. The proof is similar to that of Lemma 2 in [29] and is omitted. □

Lemma 3.3. *Assume (A), (A2′), (A4), and (A5), then the functional φ satisfies condition (C).*

Proof of Lemma 3.3. Suppose that $\{u_n\} \subset E^\alpha$ is a (C) sequence of φ, that is, $\varphi(u_n)$ is bounded and $(1 + \|u_n\|_\alpha)\|\varphi'(u_n)\|_\alpha \to 0$ as $n \to \infty$. Then we have

$$\liminf_{n\to\infty}\left[\langle \varphi'(u_n), u_n\rangle - 2\varphi(u_n)\right] > -\infty, \tag{3.28}$$

which implies that

$$\limsup_{n\to\infty} \int_0^T [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt < +\infty. \tag{3.29}$$

We only need to show that $\{u_n\}$ is bounded in E^α. If $\{u_n\}$ is unbounded, we may assume, without loss of generality, that $\|u_n\|_\alpha \to \infty$ as $n \to \infty$. Put $z_n = u_n/\|u_n\|_\alpha$, we then have $\|z_n\|_\alpha = 1$. Going to a sequence if necessary, we assume that $z_n \rightharpoonup z$ weakly in E^α, $z_n \to z$ strongly in $C([0,T],\mathbb{R}^N)$ and $L^2(0,T;\mathbb{R}^N)$.

By (A2), it follows that there exist constants $B_2 > 0$ and $M_4 > 0$ such that

$$F(t,x) \le B_2|x|^2, \tag{3.30}$$

for all $|x| \ge M_4$ and a.e. $t \in [0,T]$.

By assumption (A), it follows that

$$|F(t,x)| \le \max_{s\in[0,M_4]} a(s)b(t), \tag{3.31}$$

for all $|x| \le M_4$ and a.e. $t \in [0,T]$. Therefore, we obtain

$$F(t,x) \le B_2|x|^2 + \max_{s\in[0,M_4]} a(s)b(t) \tag{3.32}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Therefore, we have

$$\begin{aligned}\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right) - F(t,u(t))\right]dt \\ &\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - B_2\int_0^T |u|^2 dt - \max_{s\in[0,M_4]} a(s) \int_0^T b(t)dt,\end{aligned} \tag{3.33}$$

from which, it follows that

$$\frac{\varphi(u_n)}{\|u_n\|_\alpha^2} \geq \frac{|\cos(\pi\alpha)|}{2} - B_2\|z_n\|_{L_2}^2 - \frac{1}{\|u_n\|_\alpha^2} \max_{s\in[0,M_4]} a(s) \int_0^T b(t)dt. \tag{3.34}$$

Passing to the limit in the last inequality, we get

$$\frac{|\cos(\pi\alpha)|}{2} - B_2\|z\|_{L_2}^2 \leq 0, \tag{3.35}$$

which yields $z \neq 0$. Therefore, there exists a subset $E \subset [0,T]$ with $\text{meas}(E) > 0$ such that $z(t) \neq 0$ on E.

By virtue of Lemma 3.2, for $\varepsilon = (1/2)\,\text{meas}(E) > 0$, we can choose a subset $E_\varepsilon \subset [0,T]$ with $\text{meas}([0,T] \setminus E_\varepsilon) < \varepsilon$ such that

$$\lim_{|x|\to\infty} [(\nabla F(t,x), x) - 2F(t,x)] = +\infty, \tag{3.36}$$

uniformly for $t \in E_\varepsilon$.

We assert that $\text{meas}(E \bigcap E_\varepsilon) > 0$. If not, $\text{meas}(E \bigcap E_\varepsilon) = 0$.

Since $E = (E \bigcap E_\varepsilon) \bigcup (E \setminus E_\varepsilon)$, it follows that

$$\begin{aligned} 0 < \text{meas}(E) &= \text{meas}(E \cap E_\varepsilon) + \text{meas}(E \setminus E_\varepsilon) \\ &\leq \text{meas}([0,T] \setminus E_\varepsilon) \\ &< \varepsilon = \frac{1}{2}\,\text{meas}(E), \end{aligned} \tag{3.37}$$

which leads to a contradiction and establishes the assertion.

By (A4), we obtain thye following:

$$\begin{aligned} &\int_0^T [(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt \\ &\quad = \int_{E\bigcap E_\varepsilon} [(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt + \int_{[0,T]\setminus(E\bigcap E_\varepsilon)} [(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt \\ &\quad \geq \int_{E\bigcap E_\varepsilon} [(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt - \int_0^T |\tau(t)|dt. \end{aligned} \tag{3.38}$$

By (3.36), (3.38), and Fatou's lemma, it follows that

$$\lim_{n\to\infty} \int_0^T [(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt = +\infty, \tag{3.39}$$

which contradicts (3.29). This contradiction shows that $\|u_n\|_\alpha$ is bounded in E^α, and this completes the proof.

By virtue of Lemmas 3.2 and 3.3, the rest of the proof is similar to Theorem 1.1. Theorem 1.3 can be proved similarly. □

3.3. Proof of Theorem 1.4

The proof of Theorem 1.4 is divided into a sequence of lemma.

Lemma 3.4. *The functional φ is bounded below on E^α.*

Proof of Lemma 3.4. By (2.8) and (2.14), for every $u \in E^\alpha$, we have

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)dt - \int_0^T F(t,u(t))dt \\ &= -\int_0^T \frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)dt - \int_0^T a(t)|u(t)|^\gamma dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - a_0\|u\|_\infty^\gamma T \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - a_0 T C_1^\gamma \|u\|_\alpha^\gamma,\end{aligned} \tag{3.40}$$

where $a_0 = \text{ess sup}\{a(t) : t \in [0,T]\}$. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *The functional φ satisfies the (PS) condition.*

Proof of Lemma 3.5. Let $\{u_n\}$ be a Palais-Smale sequence in E^α, that is,

$$\varphi(u_n) \text{ is bounded and } \varphi'(u_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.41}$$

Suppose that $\{u_n\}$ is unbounded in E^α, that is, $\|u_n\|_\alpha \to +\infty$ as $n \to +\infty$. Since

$$\langle \varphi'(u_n), u_n\rangle - \gamma\varphi(u_n) = \left(-1 + \frac{\gamma}{2}\right)\int_0^T \left({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)\right)dt. \tag{3.42}$$

However, from (3.42), we have

$$-\gamma\varphi(u_n) \geq \left(1 - \frac{\gamma}{2}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \left\|\varphi'(u_n)\right\|\|u_n\|_\alpha, \tag{3.43}$$

thus $\|u_n\|_\alpha$ is a bounded sequence in E^α. Since E^α is a reflexive space, going, if necessary, to a subsequence, we can assume that $u_n \rightharpoonup u$ in E^α, thus we have

$$\begin{aligned}\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \langle \varphi'(u_n), u_n - u\rangle - \langle \varphi'(u), u_n - u\rangle \\ &\leq \left\|\varphi'(u_n)\right\|_\alpha \|u_n - u\|_\alpha - \langle \varphi'(u), u_n - u\rangle \longrightarrow 0,\end{aligned} \tag{3.44}$$

as $n \to \infty$. Moreover, according to (2.8) and Proposition 2.8, we have that $\{u_n\}$ is bounded in $C([0,T],\mathbb{R}^N)$ and $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.

Noting that

$$\begin{aligned}
&\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \\
&\quad = -\int_0^T \left({}_0^cD_t^\alpha(u_n(t) - u(t)),\, {}_0^cD_T^\alpha(u_n(t) - u(t))\right)dt \\
&\qquad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u(t)), u_n(t) - u(t))dt \\
&\quad \geq |\cos(\pi\alpha)|\|u_n - u\|_\alpha^2 - \left|\int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u(t)))dt\right| \|u_n - u\|_\infty.
\end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45), it is easy to verify that $\|u_n - u\|_\alpha \to 0$ as $n \to \infty$, and hence that $u_n \to u$ in E^α. Thus, $\{u_n\}$ admits a convergent subsequence. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *For any $m \in \mathbb{N}$, there exists a set $K \subset E^\alpha$ which is homeomorphic to S^{m-1} by an odd map, and* $\sup_k \varphi < 0$.

Proof of Lemma 3.6. For every $m \in \mathbb{N}$, define

$$\begin{gathered}
u_i(t) = \left(\sin\frac{i\pi t}{T}, 0, \ldots, 0\right), \quad i = 1,2,\ldots,m, \\
E_m = \operatorname{span}\{u_1, \ldots, u_m\}, \\
K_{m,\beta} = \{u \in E_m : \|u\|_\alpha = \beta\},
\end{gathered} \tag{3.46}$$

where β is a positive number to be chosen later.

For any $u \in E_m$, there exist $\lambda_i \in \mathbb{R}$, $i = 1,2,\ldots,m$, such that

$$\begin{aligned}
u &= \sum_{i=1}^m \lambda_i u_i(t), \\
\|u\|_\alpha^2 &= \int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^2 dt \\
&= \int_0^T \left({}_0^cD_t^\alpha u(t),\, {}_0^cD_t^\alpha u(t)\right)dt \\
&= \int_0^T \left(\lambda_1\, {}_0^cD_t^\alpha u_1(t) + \cdots + \lambda_m\, {}_0^cD_t^\alpha u_m(t), \lambda_1\, {}_0^cD_t^\alpha u_1(t) + \cdots + \lambda_m\, {}_0^cD_t^\alpha u_m(t)\right)dt \\
&= \sum_{i=1}^m \sum_{j=1}^m a_{ij}\lambda_i\lambda_j = F(\lambda_1, \ldots, \lambda_m),
\end{aligned} \tag{3.47}$$

where $a_{ij} = \int_0^T ({}_0^cD_t^\alpha u_i(t), {}_0^cD_t^\alpha u_j(t))dt$ and $F(\lambda_1, \ldots, \lambda_m)$ is a real quadratic form.

Since

$$F(\lambda_1,\ldots,\lambda_m)=\left\|\sum_{i=1}^{m}\lambda_i u_i(t)\right\|_\alpha^2\geq 0,\quad \forall(\lambda_1,\ldots,\lambda_m)^T\in\mathbb{R}^m,$$
$$F(\lambda_1,\ldots,\lambda_m)=0\Longleftrightarrow\sum_{i=1}^{m}\lambda_i u_i(t)\equiv 0$$
$$\Longleftrightarrow \lambda_1=\lambda_2=\cdots=\lambda_m=0. \tag{3.48}$$

So, $F(\lambda_1,\ldots,\lambda_m)$ is a real positive definite quadratic form. Then there exist an invertible matrix $C\in\mathbb{R}^{m\times m}$ and $\mu_i\in\mathbb{R}$, $i=1,2,\ldots,m$, such that

$$(\lambda_1,\lambda_2,\ldots,\lambda_m)^T=C(\mu_1,\mu_2,\ldots,\mu_m)^T,$$
$$F(\lambda_1,\ldots,\lambda_m)=\sum_{i=1}^{m}\mu_i^2. \tag{3.49}$$

It is easy to prove that the odd mapping $\Psi: K_{m,\beta}\to S^{m-1}$ defined by

$$\Psi(u)=\beta^{-1}(\mu_1,\ldots,\mu_m) \tag{3.50}$$

is a homeomorphism between $K_{m,\beta}$ and S^{m-1}.

Since $E_m\subset E^\alpha$ is a finite dimensional space, there exists $\varepsilon(m)>0$ such that

$$\operatorname{meas}\left\{t\in[0,T]: a(t)|u(t)|^\gamma\geq\varepsilon\|u\|_\alpha^\gamma\right\}\geq\varepsilon,\quad \forall u\in E_m\setminus\{0\}. \tag{3.51}$$

Otherwise, for any positive integer n, there exists $u_n\in E_m\setminus\{0\}$ such that

$$\operatorname{meas}\left\{t\in[0,T]: a(t)|u_n(t)|^\gamma\geq\frac{1}{n}\|u_n\|_\alpha^\gamma\right\}<\frac{1}{n}. \tag{3.52}$$

Set $v_n(t):=u_n(t)/\|u_n\|_\alpha\in E_m\setminus\{0\}$, then $\|v_n\|_\alpha=1$ for all $n\in\mathbb{N}$ and

$$\operatorname{meas}\left\{t\in[0,T]: a(t)|v_n(t)|^\gamma\geq\frac{1}{n}\right\}<\frac{1}{n}. \tag{3.53}$$

Since $\dim E_m<\infty$, it follows from the compactness of the unit sphere of E_m that there exists a subsequence, denoted also by $\{v_n\}$, such that $\{v_n\}$ converges to some v_0 in E_m. It is obvious that $\|v_0\|_\alpha=1$.

By the equivalence of the norms on the finite dimensional space, we have $v_n\to v_0$ in $L^2(0,T;\mathbb{R}^N)$, that is,

$$\int_0^T|v_n-v_0|^2dt\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.54}$$

By (3.54) and Hölder inequality, we have

$$\begin{aligned}\int_0^T a(t)|v_n-v_0|^\gamma dt &\le \left(\int_0^T a(t)^{2/(2-\gamma)}dt\right)^{(2-\gamma)/2}\left(\int_0^T |v_n-v_0|^2 dt\right)^{\gamma/2}\\ &= \|a\|_{(2-\gamma)/2}\left(\int_0^T |v_n-v_0|^2 dt\right)^{\gamma/2} \longrightarrow 0, \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.55}$$

Thus, there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\{t\in[0,T] : a(t)|v_0(t)|^\gamma \ge \xi_1\} \ge \xi_2. \tag{3.56}$$

In fact, if not, we have

$$\operatorname{meas}\left\{t\in[0,T] : a(t)|v_0(t)|^\gamma \ge \frac{1}{n}\right\} = 0, \tag{3.57}$$

for all positive integer n.

It implies that

$$0 \le \int_0^T a(t)|v_0|^{\gamma+2}dt < \frac{T}{n}\|v_0\|_\infty^2 \le \frac{C_1^2 T}{n}\|v_0\|_\alpha^2 \longrightarrow 0, \tag{3.58}$$

as $n \to \infty$. Hence $v_0 = 0$ which contradicts that $\|v_0\|_\alpha = 1$. Therefore, (3.56) holds.

Now let

$$\Omega_0 = \{t\in[0,T] : a(t)|v_0(t)|^\gamma \ge \xi_1\}, \qquad \Omega_n = \left\{t\in[0,T] : a(t)|v_n(t)|^\gamma < \frac{1}{n}\right\}, \tag{3.59}$$

and $\Omega_n^c = [0,T]\setminus\Omega_n = \{t\in[0,T] : a(t)|v_n(t)|^\gamma \ge 1/n\}$.

By (3.53) and (3.56), we have

$$\begin{aligned}\operatorname{meas}(\Omega_n\cap\Omega_0) &= \operatorname{meas}(\Omega_0\setminus(\Omega_n^c\cap\Omega_0))\\ &\ge \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c\cap\Omega_0)\\ &\ge \xi_2 - \frac{1}{n},\end{aligned} \tag{3.60}$$

for all positive integer n. Let n be large enough such that

$$\xi_2 - \frac{1}{n} \ge \frac{1}{2}\xi_2, \qquad \frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n} \ge \frac{1}{2^\gamma}\xi_1, \tag{3.61}$$

then we have

$$\begin{aligned}\int_0^T a(t)|v_n - v_0|^\gamma dt &\geq \int_{\Omega_n\cap\Omega_0} a(t)|v_n - v_0|^\gamma dt \\ &\geq \frac{1}{2^{\gamma-1}}\int_{\Omega_n\cap\Omega_0} a(t)|v_0|^\gamma dt - \int_{\Omega_n\cap\Omega_0} a(t)|v_n|^\gamma dt \\ &\geq \left(\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}\right)\operatorname{meas}(\Omega_n\cap\Omega_0) \\ &\geq \frac{\xi_1}{2^\gamma}\cdot\frac{\xi_2}{2} = \frac{\xi_1\xi_2}{2^{\gamma+1}} > 0,\end{aligned} \tag{3.62}$$

for all large n, which is a contradiction to (3.55). Therefore, (3.51) holds.

For any $u \in K_{m,\beta}$, we have

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)dt - \int_0^T F(t,u(t))dt \\ &\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \int_0^T a(t)|u(t)|^\gamma dt \\ &\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \varepsilon\|u\|_\alpha^\gamma \operatorname{meas}(\Omega_u) \\ &\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \varepsilon^2\|u\|_\alpha^\gamma,\end{aligned} \tag{3.63}$$

by (3.51), where $\Omega_u := \{t \in [0,T] : a(t)|u(t)|^\gamma \geq \varepsilon\|u\|_\alpha^\gamma\}$.

Choosing $\beta = (|\cos(\pi\alpha)|\varepsilon^2)^{1/(2-\gamma)}$, we conclude $\sup_{K_{m,\beta}}\varphi < -\varepsilon^2\beta^\gamma/2 < 0$ which completes the proof. □

Now from the assertion of Lemma 2.15, we know that φ has at least m distinct pairs of critical points for every $m \in \mathbb{N}$, therefore, BVP (1.1) possesses infinitely many solutions on E^α. The proof of Theorem 1.4 is completed.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. In BVP (1.1), let

$$F(t,x) = \ln\left(1 + 2|x|^2\right)|x|^2 . \tag{4.1}$$

These show that all conditions of Theorem 1.1 are satisfied, where

$$r = 2.5, \qquad \mu = 2. \tag{4.2}$$

By Theorem 1.1, BVP (1.1) has at least one solution $u \in E^\alpha$.

Example 4.2. In BVP (1.1), let $T = 2\pi$ and $F(t,x) = \kappa f(x)(2+\sin t)\arctan|x|^2$, where $\kappa > 0$ and $f(x)$ will be specified below.

Let $f(x) = |x|^2 + \ln(1+|x|^2)$. Noting that $0 \le \ln(1+|x|^2) \le |x|^2$, we see that (A) and (A2′) hold. It is also easy to see that (A1) holds for

$$\kappa > \frac{(2\pi)^{1-2\alpha}}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)(3-2\alpha)}. \tag{4.3}$$

Furthermore, we have

$$(\nabla f(x), x) - 2f(x) = \frac{2|x|^2}{1+|x|^2} - 2\ln\left(1+|x|^2\right) \longrightarrow -\infty, \tag{4.4}$$

as $|x| \to +\infty$. Therefore, we have

$$\begin{aligned}(\nabla F(t,x), x) - 2F(t,x) &= \kappa\frac{2|x|^2}{1+|x|^4} f(x)(2+\sin t) + \kappa\left[(\nabla f(x), x) - 2f(x)\right](2+\sin t)\arctan|x|^2 \\ &\longrightarrow -\infty,\end{aligned} \tag{4.5}$$

uniformly for all $t \in [0, 2\pi]$ as $|x| \to +\infty$. Thus (A4′) and (A5′) hold. By virtue of Theorem 1.3, we conclude that BVP (1.1) has at least one solution on E^α.

If $f(x) = |x|^2 - \ln(1+|x|^2)$, then exactly the same conclusions as above hold true by Theorem 1.2.

Example 4.3. In BVP (1.1), let $F(t,x) = a(t)|x|^{3/2}$ where

$$a(t) = \begin{cases} T, & t = 0 \\ 2t, & 0 < t \le \dfrac{T}{2} \\ -2(t-T), & \dfrac{T}{2} < t < T \\ T, & t = T. \end{cases} \tag{4.6}$$

By Theorem 1.4, BVP (1.1) has infinite solutions on E^α.

Acknowledgment

This paper is partially supported by the NNSF (no. 11171351) of China.

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