

Research Article

A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds

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We study slant submanifolds of a cosymplectic manifold. It is shown that a totally umbilical slant submanifold M of a cosymplectic manifold \overline{M} is either an anti-invariant submanifold or a 1-dimensional submanifold. We show that every totally umbilical proper slant submanifold of a cosymplectic manifold is totally geodesic.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by Chen as a natural generalization of both holomorphic and totally real submanifolds [1, 2]. Since then, many research papers have appeared concerning the existence of these submanifolds as well as on the geometry of the existent slant submanifolds in different known spaces (cf. [3, 4]). The slant submanifolds of an almost contact metric manifold were defined and studied by Lotta [4]. Later on, these submanifolds were studied by Cabrerizo et al. in the setting of Sasakian manifolds [3].

Recently, Şahin proved that a totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [5]. Our aim in the present paper is to investigate slant submanifolds in contact manifolds. Thus, we study slant submanifolds of a cosymplectic manifold. We have shown that a totally umbilical slant submanifold M of a cosymplectic manifold \overline{M} is either an anti-invariant submanifold or the $\dim M = 1$ or the mean curvature vector $H \in \Gamma(\mu)$, and then we have obtained an interesting result for a totally umbilical proper slant submanifold of a cosymplectic manifold.

2. Preliminaries

Let \overline{M} be a $(2n + 1)$ -dimensional manifold with $(1, 1)$ tensor field ϕ satisfying [6]:

$$\phi^2 = -I + \eta \otimes \xi, \quad (2.1)$$

where I is the identity transformation, ξ a vector field, and η a 1-form on \overline{M} satisfying $\phi\xi = \eta \circ \phi = 0$ and $\eta(\xi) = 1$. Then \overline{M} is said to have an almost contact structure. There always exists a Riemannian metric g on \overline{M} such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

for all vector fields X, Y , on \overline{M} . From (2.2), it is easy to observe that

$$g(\phi X, Y) + g(X, \phi Y) = 0. \quad (2.3)$$

The fundamental 2-form Φ is defined as: $\Phi(X, Y) = g(X, \phi Y)$. If $[\phi, \phi] + d\eta \otimes \xi = 0$, then the almost contact structure is said to be normal, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. If $\Phi = d\eta$, the almost contact structure is a contact structure. A normal almost contact structure such that Φ is closed and $d\eta = 0$ is called *cosymplectic structure*. It is well known [7] that the cosymplectic structure is characterized by

$$(\overline{\nabla}_X \phi)Y = 0, \quad (\overline{\nabla}_X \eta)Y = 0, \quad (2.4)$$

for all vector fields X, Y , on \overline{M} , where $\overline{\nabla}$ is the Levi-Civita connection of g . From the formula $\overline{\nabla}_X \phi = 0$, it follows that $\overline{\nabla}_X \xi = 0$.

Let M be submanifold of an almost contact metric manifold \overline{M} with induced metric g and let ∇ and ∇^\perp be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M , respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$ -module of smooth sections of a vector bundle TM over M , then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.5)$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.6)$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively for the immersion of M into \overline{M} . They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \quad (2.7)$$

where g denotes the Riemannian metric on \overline{M} as well as the one induced on M [8].

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX . Similarly for any $N \in \Gamma(T^\perp M)$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and fN is the normal component of ϕN . If we denote the orthogonal complementary distribution of $F(TM)$ in $T^\perp M$ by μ , then we have the direct sum

$$T^\perp M = F(TM) \oplus \mu. \tag{2.10}$$

We can see that μ is an invariant subbundle with respect to ϕ . Furthermore, the covariant derivatives of the tensor fields P and F are defined as

$$\begin{aligned} (\bar{\nabla}_X P)Y &= \nabla_X PY - P\nabla_X Y, \\ (\bar{\nabla}_X F)Y &= \nabla_X^\perp FY - F\nabla_X Y, \end{aligned} \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold \bar{M} is called *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is given by

$$H = \sum_{i=1}^m h(e_i, e_i), \tag{2.13}$$

where m is the dimension of M and $\{e_1, e_2, \dots, e_m\}$ is the local orthonormal frame on M . A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is *minimal* if $H = 0$ on M .

3. Slant Submanifolds

Throughout the section, we assume that M is a slant submanifold of a cosymplectic manifold \bar{M} . We always consider such submanifold tangent to the structure vector field ξ . For each

nonzero vector X tangent to M at x , we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and $T_x M$, known as the *Wirtinger angle* of X . If the Wirtinger angle $\theta(X)$ is constant, that is, independent of the choice of $x \in M$ and $X \in T_x M - \{\xi\}$, then M is said to be a *slant submanifold* [4]. In this case the constant angle θ is called *slant angle* of the slant submanifold. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM is decomposed as

$$TM = \mathfrak{D} \oplus \langle \xi \rangle, \quad (3.1)$$

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field ξ and \mathfrak{D} is the complementary distribution of $\langle \xi \rangle$ in TM , known as the *slant distribution*.

We recall the following result for a slant submanifold.

Theorem 3.1 (see [3]). *Let M be a submanifold of an almost contact metric manifold \overline{M} , such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \quad (3.2)$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequence of (3.2):

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (3.3)$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (3.4)$$

for any X, Y tangent to M .

Now, we prove the following.

Theorem 3.2. *Let M be a totally umbilical slant submanifold of a cosymplectic manifold \overline{M} . Then at least one of the following statements is true:*

- (i) M is an anti-invariant submanifold;
- (ii) M is a 1-dimensional submanifold;
- (iii) If M is a proper slant submanifold, then $H \in \Gamma(\mu)$,

where H is the mean curvature vector of the submanifold M .

Proof. Let M be a totally umbilical slant submanifold of a cosymplectic manifold \overline{M} , then for any $X, Y \in \Gamma(TM)$, we have

$$h(PX, PY) = g(PX, PY)H. \quad (3.5)$$

From (2.5) and (3.3), we deduce that

$$\overline{\nabla}_{PX} PX - \nabla_{PX} PX = \cos^2 \theta \{g(X, X) - \eta(X)\eta(X)\}H. \quad (3.6)$$

Using (2.8) and the fact that \overline{M} is cosymplectic we obtain that

$$\phi \overline{\nabla}_{PX} X - \overline{\nabla}_{PX} FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.7)$$

Then from (2.5) and (2.6), we get

$$\phi \nabla_{PX} X + \phi h(X, PX) + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.8)$$

Thus by (2.8) and (2.12), we obtain

$$P \nabla_{PX} X + F \nabla_{PX} X + g(PX, X) \phi H + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.9)$$

Equating the normal components, we get

$$F \nabla_{PX} X - \nabla_{PX}^\perp FX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.10)$$

On the other hand, from (3.4), we have

$$g(FX, FX) = \sin^2 \theta \{ g(X, X) - \eta(X) \eta(X) \}, \quad (3.11)$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation with respect to PX , we obtain

$$2g(\overline{\nabla}_{PX} FX, FX) = 2\sin^2 \theta g(\overline{\nabla}_{PX} X, X) - 2\sin^2 \theta \eta(X) g(\overline{\nabla}_{PX} X, \xi) - 2\sin^2 \theta \eta(X) g(X, \overline{\nabla}_{PX} \xi). \quad (3.12)$$

Using the property of metric connection $\overline{\nabla}$, the last two terms of the right-hand side are cancelling each other, thus we have

$$g(\overline{\nabla}_{PX} FX, FX) = \sin^2 \theta g(\overline{\nabla}_{PX} X, X). \quad (3.13)$$

Then by (2.5) and (2.6), we derive

$$g(\nabla_{PX}^\perp FX, FX) = \sin^2 \theta g(\nabla_{PX} X, X). \quad (3.14)$$

Now, taking the inner product in (3.10) with FX , for any $X \in \Gamma(TM)$, then

$$g(F \nabla_{PX} X, FX) - g(\nabla_{PX}^\perp FX, FX) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \quad (3.15)$$

Then from (3.4) and (3.14), we obtain

$$-\sin^2\theta\eta(X)\eta(\nabla_{PX}X) = \cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX), \quad (3.16)$$

or

$$-\sin^2\theta\eta(X)g(\nabla_{PX}X, \xi) = \cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \quad (3.17)$$

Using (2.5), we derive

$$-\sin^2\theta\eta(X)g(\bar{\nabla}_{PX}X, \xi) = \cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \quad (3.18)$$

Since $\bar{\nabla}$ is the metric connection, then the above equation can be written as

$$\sin^2\theta\eta(X)g(X, \bar{\nabla}_{PX}\xi) = \cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \quad (3.19)$$

As \bar{M} is cosymplectic thus using the fact that $\bar{\nabla}_{PX}\xi = 0$, the left hand side of the above equation vanishes identically, then

$$\cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX) = 0. \quad (3.20)$$

Thus from (3.20), it follows that either $\theta = \pi/2$ or $X = \xi$ or $H \in \Gamma(\mu)$, where μ is the invariant normal subbundle orthogonal to FTM . This completes the proof. \square

Theorem 3.3. *Every totally umbilical proper slant submanifold M of a cosymplectic manifold \bar{M} is totally geodesic, provided $\nabla_X^\perp H \in \Gamma(\mu)$, for any $X \in TM$.*

Proof. As \bar{M} is cosymplectic, then we have

$$\bar{\nabla}_U\phi V = \phi\bar{\nabla}_U V, \quad (3.21)$$

for any $U, V \in \Gamma(T\bar{M})$. Using this fact and formulae (2.5) and (2.8) we obtain that

$$\bar{\nabla}_X P Y + \bar{\nabla}_X F Y = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y), \quad (3.22)$$

for any $X, Y \in \Gamma(TM)$. Then from (2.5), (2.6) and (2.12), we get

$$\nabla_X P Y + h(X, P Y) - A_{FY} X + \nabla_X^\perp F Y = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H. \quad (3.23)$$

Taking the inner product in (3.23) with ϕH and using the fact that $H \in \Gamma(\mu)$ (by Theorem 3.2), we obtain

$$g(h(X, PY), \phi H) + g(\nabla_X^\perp FY, \phi H) = g(X, Y)g(\phi H, \phi H). \quad (3.24)$$

Then from (2.2) and (2.12), we derive

$$g(X, PY)g(H, \phi H) + g(\nabla_X^\perp FY, \phi H) = g(X, Y)g(H, H). \quad (3.25)$$

That is,

$$g(\nabla_X^\perp FY, \phi H) = g(X, Y)\|H\|^2. \quad (3.26)$$

Now, we consider

$$\bar{\nabla}_X \phi H = \phi \bar{\nabla}_X H, \quad (3.27)$$

for any $X \in \Gamma(TM)$. From (2.6), we obtain

$$-A_{\phi H} X + \nabla_X^\perp \phi H = \phi(-A_H X + \nabla_X^\perp H). \quad (3.28)$$

Thus, on using (2.8), (2.9), we get

$$-A_{\phi H} X + \nabla_X^\perp \phi H = -PA_H X - FA_H X + t\nabla_X^\perp H + f\nabla_X^\perp H. \quad (3.29)$$

Taking the inner product with FY , for any $Y \in \Gamma(TM)$, then

$$g(\nabla_X^\perp \phi H, FY) = -g(FA_H X, FY) + g(f\nabla_X^\perp H, FY). \quad (3.30)$$

Since $f\nabla_X^\perp H \in \Gamma(\mu)$, then by (3.4) the above equation takes the form

$$g(\nabla_X^\perp \phi H, FY) = -\sin^2\theta\{g(A_H X, Y) - \eta(A_H X)\eta(Y)\}. \quad (3.31)$$

Using (2.6), (2.7), and (2.12), we get

$$g(\bar{\nabla}_X \phi H, FY) = -\sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\|H\|^2. \quad (3.32)$$

The above equation can be written as

$$g(\bar{\nabla}_X FY, \phi H) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\|H\|^2. \quad (3.33)$$

Again using the fact that $H \in \Gamma(\mu)$, then by (2.6), we obtain

$$g\left(\nabla_X^\perp FY, \phi H\right) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\|H\|^2. \quad (3.34)$$

From (3.26) and (3.34), we derive

$$\left\{\cos^2\theta g(X, Y) + \sin^2\theta\eta(X)\eta(Y)\right\}\|H\|^2 = 0. \quad (3.35)$$

Thus, (3.35) implies either $H = 0$ or $\theta = \tan^{-1}(\sqrt{-g(X, Y)/\eta(X)\eta(Y)})$, which is not possible, because the slant angle $\theta \in (0, \pi/2)$. Hence, M is totally geodesic in \overline{M} . \square

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References

- [1] B. Y. Chen, "Slant immersions," *Bulletin of the Australian Mathematical Society*, vol. 41, no. 1, pp. 135–147, 1990.
- [2] B.-Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.
- [3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, "Slant submanifolds in Sasakian manifolds," *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.
- [4] A. Lotta, "Slant submanifolds in contact geometry," *Bulletin Mathematical Society Roumanie*, vol. 39, pp. 183–198, 1996.
- [5] B. Şahin, "Every totally umbilical proper slant submanifold of a Kahler manifold is totally geodesic," *Results in Mathematics*, vol. 54, no. 1-2, pp. 167–172, 2009.
- [6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer-Verlag, New York, NY, USA, 1976.
- [7] G. D. Ludden, "Submanifolds of cosymplectic manifolds," *Journal of Differential Geometry*, vol. 4, pp. 237–244, 1970.
- [8] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, World Scientific Publishing, Singapore, Singapore, 1984.



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