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Research Article

Univalent Logharmonic Mappings in the Plane

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This paper surveys recent advances on univalent logharmonic mappings defined on a simply or multiply connected domain. Topics discussed include mapping theorems, logharmonic automorphisms, univalent logharmonic extensions onto the unit disc or the annulus, univalent logharmonic exterior mappings, and univalent logharmonic ring mappings. Logharmonic polynomials are also discussed, along with several important subclasses of logharmonic mappings.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. Denote by $H(D)$ (resp., by $M(D)$) the linear space of all analytic (resp., meromorphic) functions in D, and let $B(D)$ be the set of all functions $a \in H(D)$ satisfying $|a(z)| < 1$, $z \in D$. A nonconstant function f is logharmonic in D if f is the solution of the nonlinear elliptic differential equation

$$\overline{f_{\bar{z}}} = a\frac{\overline{f}}{f} f_z, \tag{1.1}$$

$a \in B(D)$. The function a is called the second dilatation of f. In contrast to the linear space $H(D)$ consisting of analytic functions, translations in the image do not preserve logharmonicity, and the inverse of a logharmonic function is not necessarily logharmonic. If f_1 and f_2 are two logharmonic functions with respect to $a \in B(D)$, then $f_1 \cdot f_2$ is logharmonic with respect to the same a. If, in addition, $0 \notin f_2(D)$, then f_1/f_2 is also logharmonic. The composition $f \circ \phi$ of a logharmonic mapping f with a conformal premapping ϕ is also logharmonic with respect to $a \circ \phi$. However, the composition $\phi \circ f$ of a conformal postmapping ϕ with a logharmonic mapping f is in general not logharmonic. If f is a logharmonic mapping in D, then f is a nonconstant locally quasiregular mapping, and, therefore, it is continuous, open, and light. It follows that f can be represented as a composition of two functions $f = A \circ \chi$, where χ is a locally quasiconformal homeomorphism in D and $A \in H(\chi(D))$. As an immediate consequence, the maximum principle, the identity principle, and the argument principle all still hold for logharmonic mappings.

The study of logharmonic mappings was initiated in the main by Abdulhadi, Bshouty, and Hengartner in the last century, and the basic theory of logharmonic mappings was developed in [1–8].

A local representation for logharmonic mappings was given by Abdulhadi and Bshouty in [1]. In particular, they obtained the following result.

Theorem 1.1. *Let f be a logharmonic mapping in D with respect to $a \in B(D)$. Suppose that $f(z_0) = 0$ and $B(z_0, \rho)\backslash\{z_0\} \subset D\backslash Z(f)$, where $B(z_0, \rho) = \{z : |z - z_0| < \rho\}$ and $Z(f) = \{z \in D : f(z) = 0\}$. Then f admits the representation*

$$f(z) = (z - z_0)|z - z_0|^{2\beta n} h(z)\overline{g(z)}, \quad z \in B(z_0, \rho), \tag{1.2}$$

where $n \in \mathbb{N}$, $\beta = n\overline{a(z_0)}(1 + a(z_0))/(1 - |a(z_0)|^2)$ and, therefore, $\mathrm{Re}(\beta) > -n/2$. The functions h and g are in $H(B(z_0, \rho))$, with $h(z_0) \neq 0$ and $g(z_0) = 1$.

As a direct consequence of Theorem 1.1, we have the following global representation for logharmonic mappings.

Corollary 1.2. *Let D be a simply connected domain in $\mathbb{C}$ and f a logharmonic mapping in D. If f has exactly p zeros $\{z_k\}_{k=1}^{p}$ in D (counting multiplicities), then f admits a global representation given by*

$$f(z) = \left[\prod_{k=1}^{p}(z - z_k)|z - z_k|^{2\beta_k}\right] h(z)\overline{g(z)}, \tag{1.3}$$

where $\beta_k = \overline{a(z_k)}(1 + a(z_k))/(1 - |a(z_k)|^2)$ and, therefore, $\mathrm{Re}(\beta_k) > -1/2$. The functions h and g are in $H(D)$, and $0 \notin h \cdot g(D)$.

For the converse, Abdulhadi and Hengartner [2] proved the following theorem.

Theorem 1.3. *Suppose that $f(z) = h(z)\overline{g(z)}$ is defined in a domain D, where h and g are in $H(D)$, such that $f(D)$ does not lie on a logarithmic spiral. Then either $f = \overline{g}$ or f is a solution of*

$$\overline{f_{\overline{z}}(z)} = a\frac{\overline{f(z)}}{f(z)} f_z(z), \quad a \in M(D), \ |a| \neq 1. \tag{1.4}$$

Remark 1.4. The converse of Theorem 1.3 does not hold. Indeed, consider the partial differential equation $\overline{f_{\overline{z}}} = (1/3)(\overline{f}/f)f_z$. Then $f_1(z) = z^6\overline{z}^2$ and $f_2(z) = z|z|$ are solutions of this equation. The function f_1 can be written in the form $h\overline{g}$ while f_2 could not.

Remark 1.5. The function $g_w(z) = f(z) - w,\ w \in \mathbb{C}$, cannot be written in the form $h\overline{g}$ unless $w = 0$ or f is a constant. However, it is a solution of the second Beltrami equation

$$\overline{\left(\frac{\partial g_w(z)}{\partial \overline{z}}\right)} = \mu_w(z, g_w)\frac{\partial g_w(z)}{\partial z}, \tag{1.5}$$

where

$$\mu_w(z, g_w) = a(z)\frac{\overline{g_w(z) + w}}{g_w(z) + w}. \tag{1.6}$$

Hence, $|\mu_w| \equiv |a|$ in D and is independent of w.

Corollary 1.6. *The image $f(D)$ of a nonconstant function $f(z) = h(z)\overline{g(z)}$ lies on a logarithmic spiral if and only if f is a solution of* (1.1) *with* $|a| \equiv 1$.

In the theory of quasiconformal mappings, it is proved that, for any measurable function μ with $|\mu| < 1$, the solution of Beltrami equation $f_{\overline{z}} = \mu f_z$ can be factorized in the form $f = \varphi \circ F$, where F is a univalent quasiconformal mapping and φ is an analytic function (see [9]). For sense-preserving harmonic mappings, the answer is negative. In [10], Duren and Hengartner gave a necessary and sufficient condition on sense-preserving harmonic mappings f for the existence of such a factorization. Moreover, for logharmonic mappings, such a factorization need not exist. For example, the function $f(z) = z^2/|1-z|^4$ is a sense-preserving logharmonic mapping with respect to $a(z) = z$, and f has no decomposition of the desired form (see [11]). The following factorization theorem was proved in [11].

Theorem 1.7. *Let f be a nonconstant logharmonic mapping defined in a domain $D \subset \mathbb{C}$, and let a be its second dilatation function. Then f can be factorized in the form $f = F \circ \varphi$, for some analytic function φ and some univalent logharmonic mapping F if and only if*

(a) $|a(z)| \neq 1$ *in D,*

(b) $f(z_1) = f(z_2)$ *implies* $a(z_1) = a(z_2)$.

Under these conditions, the representation is unique up to a conformal mapping; any other representation $f = F_1 \circ \varphi_1$ has the form $F_1 = F \circ \psi^{-1}$ and $\varphi_1 = \psi \circ \varphi$ for some conformal mapping defined in $\varphi(D)$.

Consider now the logharmonic mapping $f(z) = ze^{1/z}\overline{e^{-1/z}}$. The point $z = 0$ is an isolated singularity of f, and f is continuous at this point. However, f does not admit a logharmonic-continuation to $\mathbb{C}$. A further restriction is needed.

Theorem 1.8 (see [2] (logharmonic-continuation across an isolated singularity)). *Let D_r be the point disc $D_r = \{z : 0 < |z| < r\}$, and let $f = h\overline{g}$ defined in D_r be a logharmonic mapping with respect to $a \in B(D)$ satisfying* $\lim_{z\to 0} f(z) = 0$. *Then f admits a logharmonic-continuation across the origin and has the representation*

$$f(z) = z^{n_0}\overline{z}^{m_0}h_0(z)\overline{g_0(z)}, \tag{1.7}$$

where n_0 *and* m_0 *are nonnegative integers,* $0 \le m_0 < n_0$*, and* h_0 *and* g_0 *are analytic functions on* $|z| < r$ *satisfying* $h_0(0)g_0(0) \neq 0$.

Liouville's theorem does not hold for entire logharmonic functions. The function $f(z) = \exp(z)\exp(-\overline{z})$ is a nonconstant bounded logharmonic in $\mathbb{C}$. Its dilatation is $a(z) \equiv -1$. However, the following modified version of Liouville's theorem was given in [2].

Theorem 1.9 (modified Liouville's theorem). *Let* $f = h\overline{g}$ *be a bounded logharmonic function in* $\mathbb{C}$*. Then either the image* $f(\mathbb{C})$ *is a circle centered at the origin with dilatation function* $a(z) \equiv -1$ or f *is a constant.*

Let $f(z) = h(z)\overline{g(z)}$ be a logharmonic mapping defined in a domain D with respect to $a \in B(D)$ satisfying $|a(z)| \not\equiv 1$. Let

(1) $S_G(D) = \{z \in D : |a(z)| > 1\}$,

(2) $S_L(D) = \{z \in D : |a(z)| < 1\}$,

(3) $S_E(D) = \{z \in D : |a(z)| = 1\}$,

(4) $NZ(f-w,D)$ be the cardinality of $Z(f-w,D)$, that is, the number of zeros of $f-w$ in D, multiplicity is not counted,

(5) $VZ(f-w,G)$ be the number of zeros of $f-w$ in $S_G(D)$, multiplicity counted.

The following argument principle for logharmonic mappings in D is shown in [2].

Theorem 1.10 (generalized argument principle for logharmonic mappings). *Let* D *be a Jordan domain, and let* $f = h\overline{g}$ *be a logharmonic mapping defined in the closure* $\overline{D}$ *with respect to* $a \in B(D)$ *satisfying* $|a(z)| \not\equiv 1$*. Fix* $w \in \mathbb{C}$ *such that* $Z(f-w,\overline{D}) \cap (\partial D \cup S_E(D))$ *is empty. Then*

$$VZ(f-w,S_L(D)) - VZ(f-w,S_G(D)) = \frac{1}{2\pi}\oint_{\partial D} d\arg(f-w). \tag{1.8}$$

As a consequence of the argument principle, the following result is obtained.

Theorem 1.11. *Let* f_n *be a sequence of logharmonic mappings defined in* U *with respect to a given* $a_n \in B(U)$*, where* U *is the unit disc. Suppose that* a_n *converges locally uniformly to* $a \in B(U)$ *and that* f_n *converges locally uniformly to a logharmonic mapping* f *with respect to* a*. If* $w_0 \notin f_n(U)$ *for all* $n \in \mathbb{N}$*, then* $w_0 \notin f(U)$*.*

In Section 2, a survey is given on univalent logharmonic mappings defined in a simply connected domain D of $\mathbb{C}$. Section 3 deals with univalent logharmonic mappings defined on multiply connected domains, while Section 4 considers logharmonic polynomials. The final section of the survey discusses several important subclasses of logharmonic mappings.

2. Univalent Logharmonic Mappings in a Simply Connected Domain

2.1. Motivation

Let Ω be a domain in the complex plane $\mathbb{C}$, and let S be a nonparametric minimal surface lying over Ω. Then S can be represented by a function $s = G(u,v)$, $w = u + iv \in \Omega$, and there is a univalent orientation-preserving harmonic mapping $w = F(z)$ from an appropriate domain D of $\mathbb{C}$ onto Ω which determines S in the following sense. The mapping F is a solution of the system of linear elliptic partial differential equation

$$\overline{F_{\overline{z}}} = AF_z, \tag{2.1}$$

where $A \in H(D)$. Since F is orientation preserving, it follows that $|A(z)| < 1$ in D. The function A is the second dilatation of F. The value $(1 + |A(z)|)/(1 - |A(z)|)$ is the quotient of the maximum value and the minimum value of the differential $|dF(z)|$ when dz varies on the unit circle (see, e.g., [12, 13]). The representation of the minimal surface S is given by three real-valued harmonic functions (see, e.g., [13, 14]),

$$u(z) = \mathrm{Re}(F(z)), \qquad v(z) = \mathrm{Im}(F(z)), \qquad s(z) = \mathrm{Im}\int^{z} \sqrt{A}F_z\, dz. \tag{2.2}$$

Since $(s_z)^2 = -\overline{F_{\overline{z}}}F_z = -A(F_z)^2$ in D, it follows that $\sqrt{A}$ belongs to $H(D)$. In particular, each zero of A is of even order. Since the Riemannian metric of S is $ds^2 = |F_z|^2(1+|A|)^2|dz|^2$, it follows that $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ are isothermal parameters for S. Moreover, the exterior unit normal vector $\vec{n}(z) = (n_1(z), n_2(z), n_3(z))$, $n_3(z) \geq 0$, to the minimal surface S (known as the Gauss mapping) depends only on the second dilatation function A of F. More precisely,

$$\vec{n} = \left(2\,\mathrm{Im}\left(\sqrt{A}\right), 2\,\mathrm{Re}\left(\sqrt{A}\right), \frac{1 - A}{1 + A}\right). \tag{2.3}$$

The inverse of the stereographic projection of the Gauss mapping $\vec{n}$, $i/\sqrt{A(z)}$, is called the Weierstrass parameter.

The following question arises: What are the domains D? If φ is univalent and analytic and if F is univalent and harmonic, then the composition $F \circ \varphi$ (whenever well defined) is a univalent harmonic mapping but $\varphi \circ F$ need not be harmonic. Hence, if F represents a minimal surface over Ω (in the sense of relation (2.2)), then $F(\varphi)$ represents the same minimal surface but in other isothermal parameters.

Suppose that Ω is a proper simply connected domain in $\mathbb{C}$. Then, we may choose for D any proper simply connected domain in $\mathbb{C}$. In particular, $D = U$ or $D = \Omega$ are appropriate choices.

Consider now the left half-plane $D = \{z : \mathrm{Re}(z) < 0\}$, and let F be a univalent harmonic and orientation-preserving map defined in D satisfying the relation

$$F(z + \alpha i) = F(z) + \beta \quad \forall z \in D, \tag{2.4}$$

where α and β are real constants. Applying the transformation $(2\pi/\beta)F(2\alpha/2\pi)$, it may be assumed without loss of generality that $\alpha = \beta = 2\pi$, that is,

$$F(z + 2\pi i) = F(z) + 2\pi i \quad \forall z \in D. \tag{2.5}$$

Whenever $\lim_{x\to-\infty} \mathrm{Re}(F(z)) = c$ for some $c \in [-\infty, \infty)$, we will write $\mathrm{Re}(F(-\infty)) = c$. Similarly, $A(-\infty) = c$ means that $\lim_{x\to-\infty} A(z) = c$.

Let UHP denote the class of all univalent harmonic orientation-preserving mappings defined on the left half-plane $D = \{z : \mathrm{Re}(z) < 0\}$ satisfying

$$\begin{gathered} F(z+2\pi i) = F(z) + 2\pi i \quad \forall z \in D, \\ \mathrm{Re}(F(-\infty)) = -\infty. \end{gathered} \tag{2.6}$$

It follows that the second dilatation function A is periodic, that is, $A(z+2\pi i) = A(z) + 2\pi i$ in D, and therefore the Gauss map is also periodic. Observe that $A(-\infty)$ exists. Furthermore, it was shown in [6] that mappings in the class UHP admit the representation

$$F(z) = z + \beta x + H(z) + \overline{G(z)}, \tag{2.7}$$

where

(a) H and G are in $H(D)$ such that

 (i) $G(-\infty) = 0$ and $H(-\infty)$ exists and finite in $\mathbb{C}$,

 (ii) $H(z+2\pi i) = H(z)$ and $G(z+2\pi i) = G(z)$ for all $z \in D$;

(b)

$$\begin{gathered} \left| \frac{G'(z) + \overline{\beta}}{1 + \beta + H'(z)} \right| < 1 \quad \text{on } D, \\ \beta = \frac{\overline{A(-\infty)}(1 - A(-\infty))}{1 - |A(-\infty)|^2}, \text{ and hence } \mathrm{Re}(\beta) > -1. \end{gathered} \tag{2.8}$$

Define

$$f(z) = e^{F(\log(z))}, \quad z \in U. \tag{2.9}$$

Then f is a univalent logharmonic mapping in U with respect to $a(z) = A(\log(z))$ and hence $a \in B(U)$. Observe that the family of all univalent logharmonic and orientation-preserving mappings f defined in U satisfying $f(0) = 0$ is isomorphic to the class UHP. It was shown in [4, 7] that it is easier to work with logharmonic mappings even if the differential equation becomes nonlinear.

2.2. Univalent Logharmonic Mappings

Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$, and suppose that f is a univalent logharmonic mapping defined in D. If $0 \notin f(D)$, then $\log(f(z))$ is a univalent and harmonic mapping in D. This mapping has been extensively studied in [15–18]. If $f(0) = 0$ and f is a univalent logharmonic mapping defined in D, then the representation (1.2) of f becomes

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \tag{2.10}$$

for every $z \in U$, where

(a) $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$, and so $\mathrm{Re}(\beta) > -1/2$,

(b) h and g are in $H(U)$ satisfying $g(0) = 1$ and $0 \notin h \cdot g(U)$.

It follows that f is locally quasiconformal. The analogue of Caratheodory's Kernel Theorem might fail for univalent logharmonic mappings. Indeed, each function

$$f_r(z) = \frac{z}{(1-z)^2} \exp\left(-2r\left(\mathrm{Re}\int_0^z \frac{(1+z)}{(1+rz)(1-z)}dz\right)\right), \quad 0 < r < 1, \tag{2.11}$$

which is univalent and logharmonic with respect to $a_r(z) = -rz$, satisfies the normalization $f_r(0) = 0$, $(f_r)_z(0) = 1$, and maps the unit disc U onto the slit domain $\mathbb{C} \setminus (-\infty, -p_r)$. The tip p_r of the omitted slit varies monotonically from $-1/4$ to -1 as r varies from 0 to 1. The limit function $\lim_{r\to 1} f_r(z) = f_1(z) = (z(1-\overline{z}))/(1-z)$ is univalent and logharmonic and maps U onto U. It has the boundary value $f(e^{it}) = -1$ for $0 < |t| \le \pi$, and the cluster set of f_1 at the point 1 is the unit circle.

Let D be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. The following characterization theorem was proved in [1].

Theorem 2.1. *Let f be a univalent mapping defined in D such that $f(z_0) = 0$. Then f is of the form $h\overline{g}$ if and only if f is a logharmonic mapping with respect to $a \in B(D)$ satisfying $a(z_0) = m/(1+m)$, $m \in \mathbb{N} \cup \{0\}$.*

Univalent logharmonic mappings have the following properties.

Theorem 2.2 (see [1]). *Let D be a simply connected domain in $\mathbb{C}$ and f a univalent logharmonic mapping defined in D with respect to $a \in B(D)$.*

(a) *Then $f_z(z) \neq 0$ for all $z \in D$ whenever $f(z) \neq 0$.*

(b) *If $f(z_0) = 0$, then $\lim_{z\to z_0}(z - z_0)f_z(z)/f(z)$ exists and is in $\mathbb{C} \setminus \{0\}$. Therefore, $(z - z_0)f_z(z)/f(z)$ is a nonvanishing function in $H(D)$.*

(c) *Let α be a complex number such that $\mathrm{Re}(\alpha) > -1/2$. Then $F = f|f|^{2\alpha}$ is a univalent logharmonic mapping with respect to*

$$a^* = \frac{1+\overline{\alpha}}{1+\alpha}\frac{a + (\overline{\alpha}/(1+\overline{\alpha}))}{1 + a(\alpha/(1+\alpha))} \in B(D). \tag{2.12}$$

There are few logharmonic mappings that are univalent on the whole complex plane $\mathbb{C}$. Indeed, Abdulhadi and Bshouty [1] showed the following.

Theorem 2.3. *A function f is a univalent logharmonic mapping defined in $\mathbb{C}$ with respect to $a \in U$ if and only if*

$$f(z) = \mathrm{const}\cdot(z - z_0)|z - z_0|^{2\beta}, \quad \beta = \frac{\overline{a}(1+a)}{\left(1-|a|^2\right)}, \ z_0 \in \mathbb{C}. \tag{2.13}$$

Now let D be a simply connected proper domain in $\mathbb{C}$ and f a univalent logharmonic function in D with respect to $a \in B(D)$. Denote by φ a conformal mapping from the unit disc U onto D. Then $f \circ \varphi$ is univalent logharmonic in U with respect to $a^* = a \circ \varphi \in B(U)$. Therefore, we may assume that $D = U$ and $f(0) = 0$.

Analogous to the analytic case, we denote

$$S_{Lh} = \Big\{ f(z) = z|z|^{2\beta} h\overline{g} : f \text{ is a univalent logharmonic mapping defined in } U \\ \text{with } h(0) = g(0) = 1 \Big\}. \tag{2.14}$$

Now $1^{2\beta} = 1$, and S_{Lh} is not compact with respect to the topology of normal convergence. Indeed, the sequence $f_n(z) = z|z|^{(1-n)/n}$ is in S_{Lh}, and it converges uniformly to $f(z) = z|z|^{-1}$ not in S_{Lh}. Our next result deals with the subclass S^0_{Lh} of S_{Lh} defined by $S^0_{Lh} = \{f \in S_{Lh} : a(0) = 0 \text{ (resp., } \beta = 0)\}$. The following result was proved in [1].

Theorem 2.4. *S^0_{Lh} is compact in the topology of normal (locally uniform) convergence.*

Remark 2.5. In contrast to univalent harmonic mappings, S_{Lh} is not a normal family. Indeed,

$$f_n(z) = \frac{z}{(1-z)^2}\left|\frac{z}{(1-z)^2}\right|^{2n} \tag{2.15}$$

is not locally uniformly bounded for n sufficiently large.

The following interesting distortion theorem is due to Abdulhadi and Bshouty [1], and it was used in the proof of the mapping theorem.

Theorem 2.6. *If $f \in S^0_{Lh}$, then $|f(z)| \geq |z|/4(1+|z|)^2$. In particular, the disc $\{w : |w| < 1/16\}$ is in $f(U)$.*

2.3. Mapping Theorem

We look for an analogue of the Riemann Mapping Theorem. Let $\Omega \neq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$, and let $a \in B(U)$ be given. Fix $z_0 \in U$ and $w_0 \in \Omega$. We are interested in the existence of a univalent logharmonic function f from U into Ω with respect to the given function a and normalized by $f(z_0) = w_0$ and $f_z(z_0) > 0$. If $|a| \leq k < 1$ for all $z \in U$, then the univalent logharmonic mappings are quasiconformal, and therefore the problem is solvable.

Suppose that we want to find a univalent logharmonic mapping f with $a(z) = -z$, normalized by $f(0) = 0$ and $f_z(0) > 0$ such that f maps U onto $\Omega = \mathbb{C} \setminus (-\infty, -1]$. Assume that such a function exists. Then, using Theorem 5.1 ($\alpha = 0$), it follows that f must be of the form

$$f = \text{const} \cdot \frac{z(1-\overline{z})}{(1-z)}. \tag{2.16}$$

Observe that f is univalent in U, but maps U onto a disc, and not onto a slit domain. In other words, there is no univalent logharmonic mapping defined in U with respect to $a(z) = -z$

satisfying $f(0) = 0$, $f_z(0) > 0$, and $f(U) = \Omega$. However, the following mapping theorem was proved in [1].

Theorem 2.7. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ containing the origin, and whose boundary is locally connected. Let $a \in B(U)$ be given. Then there is a univalent logharmonic function defined in U with the following properties.*

(i) f *is a solution of* (1.1).

(ii) $f(U) \subset \Omega$, *normalized at the origin by* $f(z) = cz|z|^{2\beta}(1 + o(1))$, *where* $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ *and* $c > 0$.

(iii) $\lim_{z \to e^{it}} f(z) = \widehat{f}(e^{it})$ *exists and is in* $\partial\Omega$ *for each* $t \in \partial U \setminus E$, *where* E *is a countable set.*

(iv) *For each* $e^{it_0} \in \partial U$, $f_*(e^{it_0}) = \operatorname{ess\,lim}_{t \uparrow t_0} \widehat{f}(e^{it})$ *and* $f^*(e^{it_0}) = \operatorname{ess\,lim}_{t \downarrow t_0} \widehat{f}(e^{it})$ *exist and are in* $\partial\Omega$.

(v) *For* $e^{it_0} \in E$, *the cluster set of* f *at* e^{it_0} *lies on a helix joining the point* $f^*(e^{it_0})$ *to the point* $f_*(e^{it_0})$.

Remark 2.8. In the case where $\|a\| = \sup_{z \in U}|a(z)| < 1$, properties (ii) and (iii) imply that $f(U) = \Omega$.

Remark 2.9. If $e^{it_0} \in E$ and $f_*(e^{it_0}) = f^*(e^{it_0})$, then the cluster set at e^{it_0} is a circle. Suppose that $A = f_*(e^{it_0}) \neq f^*(e^{it_0}) = B$, then there are infinitely many helices joining A and B. But the cluster set of f at e^{it_0} lies on one of them. For example, the cluster set of

$$f(z) = z\frac{(1 - \overline{z})}{(1 - z)} \exp\left(-2 \arg \frac{1 - iz}{1 - z}\right) \tag{2.17}$$

at $z = 1$ lies on the helix, $\gamma(\tau) = \exp[-\tau + i(\pi/2 + \tau)]$ joining the points $f^*(1) = -e^{-\pi/2}$ and $f_*(1) = -e^{3\pi/2}$, where the cluster set of f at $z = -i$ is the straight line segment from $f^*(-i) = -e^{-\pi/2}$ and $f_*(-i) = -e^{3\pi/2}$.

The uniqueness of the mapping theorem was proved in [6] for the special case Ω is a strictly starlike and bounded domain; that is, every ray starting at the origin intersects $\partial\Omega$ at exactly one point.

Theorem 2.10 (uniqueness in the mapping theorem). *Let $a \in B(U)$ be given such that $\|a\| = \sup_{z \in U}|a(z)| < 1$. Let Ω be a strictly starlike and bounded domain. Then there exists a unique univalent logharmonic function $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ with respect to a such that $f(U) = \Omega$ and $h(0) > 0$.*

2.4. Logharmonic Automorphisms

We consider univalent logharmonic mappings from U onto U. With no loss of generality, it is assumed that $f(0) = 0$ and $h(0) > 0$. Otherwise, we consider an appropriate Möbius transformation of the preimage. Let $\mathrm{AUT}_{Lh}(U)$ denote the class of such mappings. The following two theorems established in [8] characterize completely mappings in $\mathrm{AUT}_{Lh}(U)$.

Theorem 2.11. *Let h and g be two nonvanishing analytic functions in U. Then $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is in $AUT_{Lh}(U)$ satisfying $h(0) > 0$ and $g(0) = 1$ if and only if $g = 1/h$, $\mathrm{Re}(\beta) > -1/2$, and $\mathrm{Re}(zh'(z))/h(z) > -1/2$ in U.*

We now associate to each $f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}$ in $\mathrm{AUT}_{Lh}(U)$ with the mapping $\varphi(z) = z(h(z))^2 \in S^*$.

Theorem 2.12. (a) *For each $\varphi \in S^*$ and for each β, $\mathrm{Re}(\beta) > -1/2$, there is one and only one $f \in AUT_{Lh}(U)$ such that $f(z)/(\varphi(z)|z|^{2\beta}) > 0$ for every $z \in U$ and $h(0) = 1$.*

(b) *For each $a \in B(U)$, there is a unique solution of* (1.1) *which is in $AUT_{Lh}(U)$.*

Remark 2.13. Part (a) of Theorem 2.12 is quite surprising. Indeed, consider $\varphi(z) = z/(1-z)^2$ and $\beta = 0$. Then $\arg(f(e^{it})) = \arg(\varphi(e^{it})) = \pm\pi$, almost everywhere; however, $f(U) = U$. To be more precise, the corresponding mapping is $f(z) = z(1-\overline{z})/(1-z)$ satisfying $f(e^{it}) = -1$ for all $0 < |t| \le \pi$, where the cluster set of f at the point 1 is the unit circle.

2.5. Univalent Logharmonic Mappings Extensions onto the Unit Disc

In 1926 Kneser [19] obtained the following result.

Theorem 2.14. *Let Ω be a bounded simply connected Jordan domain, and let f^* be an orientation-preserving homeomorphism from the unit disc circle ∂U onto $\partial\Omega$. Then, if $f(U) = \Omega$, the solution of the Dirichlet problem* (*the Poisson integral*) *is univalent on the unit disc U.*

Since $f(U)$ always contains Ω and lies in the convex hull of Ω, Kneser used Theorem 2.14 to obtain the following solution to a problem posed by Rado in [20].

Theorem 2.15. *Let f^* be a homeomorphism from ∂U onto $\partial\Omega$, where Ω is a bounded convex domain. Then the Dirichlet solution f is univalent on U.*

In 1945, Choquet [21] independently gave another proof of Theorem 2.15, and he pointed out that it holds whenever Ω is not a convex domain.

We will use the following definition.

Definition 2.16. Let D be the unit disc U or the annulus $A(r,1)$, $r \in (0,1)$, and suppose that f^* is a continuous function defined on ∂D. One says that f is a logharmonic solution of the Dirichlet problem if

(a) f is a solution of the form (1.1),

(b) f is continuous in D,

(c) $f|_{\partial D} \equiv f^*$.

The next two theorems proved in [6] deal with the solutions of the Dirichlet problem for logharmonic mappings of the form (2.10).

Theorem 2.17. *Let f^* be a nonvanishing continuous complex-valued function defined on ∂U. Then there exist h and g analytic in U which are independent of β, such that*

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}, \quad \mathrm{Re}(\beta) > -\frac{1}{2} \tag{2.18}$$

is a logharmonic solution of the Dirichlet problem (i.e., $f(e^{it}) \equiv f^*(e^{it})$*). Furthermore, if* $g(0) = 1$*, then* h *and* g *are uniquely determined.*

Theorem 2.18. *Let* f^* *be an orientation-preserving homeomorphism from* ∂U *onto* ∂U*, that is,* $f(e^{it}) = e^{i\lambda(t)}$*, where* λ *is continuous and strictly monotonically increasing on* $[0, 2\pi)$*. Furthermore, suppose that* $\lambda(2\pi) = \lambda(0) + 2\pi$*. Then, for a given* β *with* $\mathrm{Re}(\beta) > -1/2$*, the logharmonic solution of the Dirichlet problem which is of the form* $f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}$ *is univalent in* U*.*

2.6. Boundary Behavior

Let f be a univalent logharmonic mapping in the unit disc U with respect to $a \in B(U)$. If $|a(z)| \leq k < 1$ for all $z \in U$, then f is a quasiconformal map, and its boundary behavior is the same as for conformal mappings. However, if $|a|$ approaches one as z tends to the boundary, then the boundary behavior of f is quite different. It may happen that the boundary values are constant on an interval of ∂U, or that there are jumps as the following example shows.

Example 2.19. The function $f(z) = z(1-\overline{z})/(1-z)$ is a univalent logharmonic mapping in the unit disc U with respect to $a(z) = -z$, such that $f(U) = U$. It follows that $f(e^{it}) = -1$ for all $0 < |t| \leq \pi$ and that the cluster set of f at the point 1 is the unit circle.

The following theorem was stated in [1].

Theorem 2.20. *Let* Ω *be a simply connected domain of* $\mathbb{C}$ *whose boundary* $\partial\Omega$ *is locally connected, and* $a \in B(U)$*. Let* f *be a univalent logharmonic mapping from* U *onto* Ω *satisfying* $f(0) = 0$*. Then the nonrestricted limit* $f^*(e^{it})$ *of* f *at* e^{it} *exists on* $\partial U \setminus E$*, where* E *is a countable set. If* $e^{it} \in E$*, then* f^* *jumps at* e^{it}*, and the cluster set at* e^{it} *is a subinterval of a logarithmic spiral.*

The next theorem [22] shows that the boundary values of f depend strongly on the values of $a(e^{it})$.

Theorem 2.21. *Let* Ω *be a simply connected domain of* $\mathbb{C}$ *whose boundary* $\partial\Omega$ *is locally connected and* $a \in B(U)$*. Suppose that the function* a *has an analytic extension across an open subinterval* $I = \{e^{it} : \sigma < t < \sigma + 2\pi\}$ *of the unit circle* ∂U*, such that* $|a(z)| \equiv 1$ *in* I*. Let* f *be a univalent logharmonic mapping with respect to* a *which maps* U *onto* Ω *and satisfies* $f(0) = 0$*. Then the following relations hold in* I*.*

(a) *Let* $\sigma < t < \sigma + 2\pi$ *and* $\arg(f(z))$ *be a continuous function on the set* $\Upsilon := |z : 1/2 < |z| < 1,\ \sigma < \arg(z) < \tau\}$*. If* $\sigma < t < t + h < \tau$*, then*

$$\log\left(f^*\left(e^{i(t+h)}\right)\right) - \overline{a(e^{i(t+h)})\log(f^*(e^{i(t+h)}))} - \log\left(f^*\left(e^{it}\right)\right) + \overline{a(e^{it})\log(f^*(e^{it}))} + \overline{\int_t^{t+h} \log(f^*(e^{i\phi}))da(e^{i\phi})} \equiv 0. \tag{2.19}$$

(b) *If* f^* *is continuous at* e^{it}*, then*

$$\lim_{t\downarrow 0} \mathrm{Im}\sqrt{a(e^{it})}\frac{f^*(e^{i(t+h)})/f^*(e^{i(t-h)}) - 1}{h} = 0. \tag{2.20}$$

(c) *If f^* jumps at e^{it}, which must and can happen only when $f^*(I)$ lies on a segment of a logarithmic spiral, for $q \in f^*(I)$, then*

$$\arg\left(\log \frac{f^*\left(e^{i(t+0)}\right)}{q}\right) = -\frac{1}{2}\arg\left(a\left(e^{it}\right)\right) \bmod \pi. \tag{2.21}$$

(d) *If f^* is not constant on a subinterval of I, then the right limit*

$$\lim_{t \downarrow 0} \arg\left(\frac{f^*\left(e^{i(t+h)}\right)}{f^*\left(e^{i(t-h)}\right)} - 1\right) = -\frac{1}{2}\arg\left(a\left(e^{it}\right)\right) \bmod \pi \tag{2.22}$$

exists everywhere on I.

2.7. A Constructive Method

In this section, a method is introduced for constructing univalent logharmonic mappings from the unit disc onto a strictly starlike domain Ω, which has been successfully applied to conformal mappings (see, e.g., [23–25]), as well as for univalent harmonic mappings (see, e.g., [26, 27]).

Let Ω be a strictly starlike domain of $\mathbb{C}$. Then $\partial\Omega$ can be expressed in the parametric form

$$w(t) = R(t)e^{it}, \quad 0 \le t \le 2\pi, \tag{2.23}$$

where R is a positive continuous function on $[0, 2\pi]$. The following notations will be used:

$$\begin{gathered} \|f\|_\infty = \sup\{|f(z)|; z \in U\}, \\ \|\Omega\|_\infty = \sup\{|w|; w \in \Omega\}, \\ d(\partial\Omega) = \text{distance from the origin to } \partial\Omega. \end{gathered} \tag{2.24}$$

For all $w \in \mathbb{C}$, define

$$\lambda_\Omega(w) = \begin{cases} \dfrac{|w|}{R(t)}, & 0 \ne w = |w|e^{it}, \\ 0, & w = 0. \end{cases} \tag{2.25}$$

Then

$$\begin{gathered} \lambda_\Omega(w) < 1 \Longleftrightarrow w \in \Omega, \\ \lambda_\Omega(w) = 1 \Longleftrightarrow w \in \partial\Omega, \\ \lambda_\Omega(w) > 1 \Longleftrightarrow w \in \mathbb{C} \setminus \overline{\Omega}, \\ \lambda_\Omega(w) = 0 \Longleftrightarrow w = 0. \end{gathered} \tag{2.26}$$

For any complex-valued function f in U, define

$$\mu_\Omega(f) = \sup\{\lambda_\Omega(w) : w \in f(U)\}. \tag{2.27}$$

The following properties are due to Bshouty et al. [26].

Lemma 2.22. (a) $\mu_\Omega(f) \le 1 \Leftrightarrow f(U) \subset \Omega$,
(b) $\mu_\Omega(tf) = t\mu_\Omega(f)$ *for all* $t \ge 0$,
(c) $\mu_\Omega(f) \le \|f\|_\infty / d(\partial\Omega)$,
(d) $\|f\|_\infty \le \mu_\Omega(f)\|\Omega\|_\infty$,
(e) $\mu_\Omega(f_1 + f_2) \le (\mu_\Omega(f_1) + \mu_\Omega(f_2))(\|f\|_\infty / d(\partial\Omega))$.

The next lemma shows that μ_Ω is lower semicontinuous with respect to the point-wise convergence; this was proved in [28].

Lemma 2.23. *Let Ω be a strictly starlike domain of $\mathbb{C}$, and let f_n be a sequence of mappings from U into $\mathbb{C}$ which converges pointwise to f. Then $\lim_{n\to\infty} \inf(\mu_\Omega(f_n)) \ge \mu_\Omega(f)$. Strict inequality can hold even in the case of locally uniform convergence.*

Let Ω be a fixed strictly starlike domain of $\mathbb{C}$, and let $a \in H(U)$, $a(0) = 0$, $|a| \le k < 1$ be a given (second) dilatation function. Denote by N the set of all logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ with respect to the given dilatation function which are normalized by $g(0) = h(0) = 1$. Observe that $\beta = 0$ since it is assumed that $a(0) = 0$. Hengartner and Nadeau [27] solved the following optimization problem.

Theorem 2.24. *Let Ω be a strictly starlike domain of $\mathbb{C}$, and let $a \in H(U)$, $a(0) = 0$, $|a| \le k < 1$ be given. Denote by $F(z) = zH(z)\overline{G(z)}$ the univalent logharmonic mapping satisfying $F(U) = \Omega$, $G(0) = 1$, and $H(0) > 0$. Then there exists a unique $f^* \in N$ such that $\mu_\Omega(f^*) \le \mu_\Omega(f)$ for all $f \in N$ and $f^* = F/H(0)$.*

Theorem 2.24 allows us to solve the following mathematical program:

$$\min M, \qquad \lambda(f(z)) \le M \qquad \forall z \in U,\ \forall f \in N. \tag{2.28}$$

For $f \in N$, $f(z) = zh(z)\overline{g(z)}$, where $h(z) = \exp(\sum_{k=1}^{\infty} a_k z^k)$ and

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{s} ds + \sum_{k=1}^{\infty} k a_k \int_0^z a(s) s^{k-1} ds\right). \tag{2.29}$$

Furthermore, each $f \in N$ is an open mapping. Denote by V_n the set of all mappings $f \in N$ of the form

$$f(z) = z\exp\left(\overline{\int_0^z \frac{a(s)}{s} ds} + \sum_{k=1}^{\infty}\left[a_k z^k + k\overline{a_k}\overline{\int_0^z a(s) s^{k-1} ds}\right]\right) \tag{2.30}$$

and by f_n^* any solution of the optimization problem

$$\min\{\mu_\Omega(f); f \in V_n\}. \tag{2.31}$$

As a consequence of Theorem 2.24, Hengartner and Rostand [28] obtained the following result.

Theorem 2.25. *Let a be a polynomial such that $\|a\|_\infty \leq k < 1$ in U. Then the sequence f_n^* of solutions of*

$$\min \mu_\Omega(f), \quad f \in V_n, \tag{2.32}$$

converges locally uniformly to the univalent solution f^* of

$$\min \mu_\Omega(f), \quad f \in N. \tag{2.33}$$

The question remains how big could n be. It follows from Theorem 5.24 that $|a_n| \leq 2 + n^{-1}$ and $|b_n| \leq 2 - n^{-1}$. Suppose that Ω is a Jordan domain whose boundary $\partial\Omega$ is rectifiable and piecewise smooth. Hengartner and Nadeau [27] obtained the following additional estimate for the coefficients.

Theorem 2.26. *Let*

$$F(z) = z|z|^{2\beta} \exp\left(\sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}\right) \tag{2.34}$$

be a univalent logharmonic mapping from U onto Ω, and let L be the length of $F(|z| = r)$, $0 < r < 1$. Then

$$\begin{aligned} |a_n| &\leq \frac{\lim_{r\to 1}\inf L(r)}{2\pi d(\partial\Omega) n}, \\ |b_n| &\leq \frac{\lim_{r\to 1}\inf L(r)}{2\pi d(\partial\Omega) n}. \end{aligned} \tag{2.35}$$

Equality holds for the case $\Omega = U$ and $f(z) = z(1-\overline{z})/(1-z)$.

3. Univalent Logharmonic Mappings on Multiply Connected Domains

3.1. Univalent Logharmonic Exterior Mappings

This section considers univalent logharmonic and orientation-preserving mappings f defined on the exterior of the unit disc U, $\Delta = \{z : |z| > 1\}$, satisfying $f(\infty) = \infty$. These mappings are called univalent logharmonic exterior mappings. If f does not vanish on Δ, then $\Psi(z) = 1/f(1/z)$ is a univalent logharmonic mapping defined in U normalized by $\Psi(0) = 0$. Moreover, $F(\zeta) = \log f(e^\zeta)$ is a univalent harmonic mapping defined on the right half-plane $\{\zeta : \operatorname{Re}(\zeta) > 0\}$ satisfying the relation $F(\zeta + 2\pi i) = F(\zeta) + 2\pi i$ and F is a solution of the linear elliptic partial differential equation

$$\overline{F_{\overline{\zeta}}} = AF_{\zeta}, \tag{3.1}$$

where the second dilatation function $A(\zeta) = a(e^{\zeta})$, $a \in B(\Delta)$, satisfies $A(\zeta + 2\pi i) = A(\zeta)$ on $\{\zeta : \operatorname{Re}(\zeta) > 0\}$. Such mappings were studied in [9, 29–32]. Several authors have also studied harmonic mappings between Riemannian manifolds, and an excellent survey has been given in [33–37].

The next result proved in [4] is a global representation of univalent logharmonic exterior mappings.

Theorem 3.1. *Let f be a univalent logharmonic mapping defined on the exterior Δ of the closed unit disc $\overline{U}$ such that $f(\infty) = \infty$. Suppose that $f(p) = 0$ for some $p \in \Delta$, or if f does not vanish, let $p = 1$. Then there are two complex numbers β and γ, $\operatorname{Re}(\beta) > -1/2$, $\operatorname{Re}(\gamma) > -1/2$, and two nonvanishing analytic functions h and g on $\Delta \cup \{\infty\}$ such that $g(\infty) = 1$, and f is of the form*

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-\overline{p}z}\right)\left|\frac{z-p}{1-\overline{p}z}\right|^{2\gamma} h(z)\overline{g(z)} \tag{3.2}$$

for all $z \in \Delta$.

Remark 3.2. Observe that not each function of the form (3.2) is univalent. Indeed, the function

$$f(z) = \overline{z}|z|^{2}\frac{z-4}{1-4\overline{z}} \tag{3.3}$$

is not a univalent logharmonic mapping on Δ, but it can be written in the form (3.2) by putting $\beta = 1$, $\gamma = 0$, $p = 4$, $h(z) = 1/g(z) = (4z-1)/(4z)$.

Let f be a univalent logharmonic exterior mapping defined on the exterior Δ of the closed unit disc $\overline{U}$ such that $f(\infty) = \infty$. Applying an appropriate rotation to the preimage, we may assume that $p \geq 1$.

Definition 3.3. The class $\sum_{Lh}$ consists of all univalent logharmonic mappings defined on Δ which are of the form (3.2), where $p \geq 1$, $\operatorname{Re}(\beta) > -1/2$, $\operatorname{Re}(\gamma) > -1/2$, and h and g are analytic nonvanishing functions on $\Delta \cup \{\infty\}$, normalized by $g(\infty) = 1$ and $|h(\infty)| = 1$.

Let f be a univalent logharmonic mapping in Δ with $f(\infty) = \infty$. Then there is a real number α and a positive constant A such that $Af(e^{-\alpha}z)$ belongs to $\sum_{Lh}$. If f does not vanish on Δ, then the set of omitted values is a continuum. In other words, there is no univalent logharmonic mapping f defined on Δ satisfying $f(\infty) = \infty$ and $f(\Delta) = \mathbb{C} \setminus \{0\}$. Note that 0 is an exceptional point, since, for each $w_0 \in \mathbb{C} \setminus \{0\}$, there are univalent logharmonic mappings f such that $f(\Delta) = \mathbb{C} \setminus \{w_0\}$. Assume that $p > 1$, let $f \in \sum_{Lh}$, and let w_0 be an omitted value of f. Applying a rotation to the image $f(\Delta)$, we may assume that $w_0 = 1$, and we restrict ourselves to the subclass $\sum_{Lh}^{*} = \{f \in \sum_{Lh},\ p > 1,\ w_0 = 1 \notin f(\Delta)\}$.

In the next theorem, Abdulhadi and Hengartner [4] gave a complete characterization of all mappings in the class $\sum_{Lh}^{*}$.

Theorem 3.4. *A mapping f belongs to $\sum_{Lh}^{*}$ and $f(\Delta) = \mathbb{C} \setminus \{1\}$ if and only if f is of the form*

$$f(z) = \overline{z}|z|^{2\beta}\left(\frac{z-p}{1-p\overline{z}}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma}, \quad p > 1, \tag{3.4}$$

where β and γ satisfy the inequality

$$\left|\frac{\beta(1+\overline{\gamma}) - \gamma\left(1+\overline{\beta}\right)}{1+\gamma+\overline{\gamma}} - \frac{1}{p^2-1}\right| \leq \frac{p}{p^2-1}. \tag{3.5}$$

3.2. *Univalent Logharmonic Ring Mappings*

In this section we investigate the family A_r of all univalent logharmonic mappings f which map an annulus $A(r,1) = \{z : r < |z| < 1\}$, $0 < r < 1$, onto an annulus $A(R,1)$ for some $R \in [0,1)$ satisfying the condition

$$\frac{1}{2\pi}\int_{|z|=\rho} d\arg f\left(\rho e^{it}\right) = 1 \tag{3.6}$$

for all $\rho \in (r,1)$. The last condition says that the outer boundary corresponds to the outer boundary. We call an element $f \in A_r$ a univalent logharmonic ring mapping.

If $a \equiv 0$, then $R = r$ and $f(z) = e^{i\alpha}z$, $\alpha \in R$, are the only mappings in A_r. In the case of univalent harmonic mappings from $A(r,1)$ onto $A(R,1)$, it may be possible that $R = 0$; for example, $f(z) = (1-r^2)^{-1}(z-(r^2/\overline{z}))$ has this property. However, Nitsche [38] has shown that there is an $R_0(r) < 1$ such that there is no univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$ whenever $R_0 < R < 1$.

There is no univalent logharmonic mappings from $A(r,1)$, $0 < r < 1$, onto $A(0,1)$. This is a direct consequence of Theorem 3.5. But, on the other hand, for R there is neither a positive lower bound nor a uniform upper bound strictly less than one. Indeed, $f(z) = z|z|^{2\beta}$, $\mathrm{Re}(\beta) > -1/2$, is univalent on $A(r,1)$, and its image is $A(r^{1+2\,\mathrm{Re}(\beta)},1)$.

Unlike the case of univalent harmonic mappings, univalent logharmonic mappings need not have a continuous extension onto the closure of $A(r,1)$. Indeed, $f(z) = z(\overline{z}-1)/(z-1)$ is a univalent logharmonic ring mapping from $A(1/2,1)$ onto itself whose cluster sets on the outer boundary are $C(f,e^{it}) = \{-1\}$, if $z = e^{it}$, $0 < t < 2\pi$, and $C(f,1) = \{w : |w| = 1\}$.

Let $S^*(r,1)$ be the set of all univalent analytic functions φ on $A(r,1)$ with the properties

(i) $p(z) = z\varphi'(z)/\varphi(z) \in H(A(r,1))$,

(ii) $\mathrm{Re}(p(z)) > 0$ on $A(r,1)$.

Theorems 3.5 and 3.6 [5] give a complete characterization of univalent logharmonic mappings in A_r.

Theorem 3.5. *A function f belongs to A_r if and only if*

$$f(z) = z|z|^{2\beta}\frac{h(z)}{\overline{h(z)}}, \tag{3.7}$$

where

(a) $h \in H(A(r,1))$ *and* $0 \notin h(A(r,1))$,

(b) $\mathrm{Re}(zh'(z)/h(z)) > -1/2$ *on* $A(r,1)$,

(c) $(1/2\pi)\int_{|z|=\rho} d\arg f(\rho e^{it}) = 0$, $r < \rho < 1$,

(d) $\mathrm{Re}(\beta) > -1/2$.

In particular, functions belonging to A_r map concentric circles onto concentric circles.

Theorem 3.6. *A function f is in A_r if and only if it is of the form*

$$f(z) = \left(\frac{\varphi(z)}{|\varphi(z)|}|z|^{2\gamma}\right), \tag{3.8}$$

where $\mathrm{Re}(\varphi) > 0$ *and* $\varphi \in S^*(r,1)$.

Next we fix the second dilatation function $a \in H(A(r,1))$, $|a(z)| < 1$ for all $z \in A(r,1)$. The following existence and uniqueness theorem was established in [5].

Theorem 3.7. *For a given $a \in H(A(r,1))$, $|a(z)| < 1$ for all $z \in A(r,1)$, and, for a given $z_0 \in A(r,1)$, there exists one and only one univalent solution f of* (1.1) *in A_r such that $f(z_0) > 0$.*

Remark 3.8. Theorem 3.7 is not true for univalent harmonic ring mappings (see [32, Theorem 7.3].)

3.3. *Univalent Logharmonic Mappings Extensions onto the Annulus*

The next two theorems proved in [6] deal with the solution of the Dirichlet problem for ring domains.

Theorem 3.9. *Let f^* be a nonvanishing continuous function defined on the boundary $\partial A(r,1)$ of the annulus $A(r,1)$. Then there exists, for each β, $\mathrm{Re}(\beta) > -1/2$, a unique mapping f of the form* (2.10), *which is continuous on the closure of $A(r,1)$ and satisfies $f = f^*$ on $\partial A(r,1)$.*

Theorem 3.10. *Let $f^*(e^{it}) = e^{i\lambda(t)}$ and $f^*(re^{it}) = \mathrm{R}e^{i\mu(t)}$, $0 < R < 1$, be a given continuous function on $\partial A(r,1)$, $0 < r < 1$, satisfying*

(a) $d\lambda(t) \geq 0$ *and* $d\mu(t) \geq 0$ *on* $[0,2\pi]$,

(b) $\int_0^{2\pi} d\lambda(t) = \int_0^{2\pi} d\mu(t) = 2\pi$.

Then the logharmonic solution of the Dirichlet problem with respect to $f^*$and $A(r,1)$ is a univalent mapping from $A(r,1)$ onto $A(R,1)$.

4. Logharmonic Polynomials

Denote by p_n an analytic polynomial of degree n. A logharmonic polynomial is a function of the form $f = p_n\overline{p_m}$. In contrast to the analytic case, there are nonconstant logharmonic polynomials which are not p-valent for every $p > 0$. For example, the function $f(z) = z\overline{z}$

is a logharmonic polynomial in $\mathbb{C}$ with respect to $a = -1$. Moreover, the function $f(z) = (z-1)(\overline{z}+1)$ is a logharmonic polynomial in $\mathbb{C}$ with respect to $a(z) = (z+1)/(z-1)$. This polynomial is two-valent and omits the half-plane $\mathrm{Re}(w) < -1$. On the other hand, they inherit the property $\lim_{z\to\infty} f(z) = \infty$ of analytic polynomials. This follows from the fact that $|f| = |p_n\overline{p_m}| = |p_n p_m|$. However, the converse is not true; there are logharmonic functions $f = h\overline{g}$ defined in $\mathbb{C}$ which are not logharmonic polynomials and have the property $\lim_{z\to\infty} f(z) = \infty$. The function $f(z) = ze^{z}\overline{e^{-z}}$ is such an example. Note that there are harmonic polynomials $p_n(z) + \overline{p_m(z)}$ which do not satisfy $\lim_{z\to\infty} f(z) = \infty$. However, if it is assumed that $a(\infty)$ exists and $|a(\infty)| \neq 1$, then the following result [2] is deduced.

Theorem 4.1. *Let $f = h\overline{g}$ be a logharmonic function in $\mathbb{C}$ such that $\lim_{z\to\infty} f(z) = \infty$. If $\lim_{z\to\infty} a(z) = a(\infty)$ exists and if $|a(\infty)| \neq 1$, then f is a polynomial.*

Denote by $NZ(f-w, D)$ the cardinality of $Z(f-w, D)$, that is, the number of zeros of $f-w$ in D, multiplicity not counted. The polynomial $f(z) = |z|^2$ has the property that $NZ(f-1, \mathbb{C}) = \infty$. On the other hand, using Theorem 2.3, it follows that a univalent logharmonic mapping in $\mathbb{C}$ is necessarily a polynomial which is either of the form $f(z) = \text{const} \cdot (z-a)\overline{(z-a)}^{m}$ or of its conjugate, where $\text{const} \neq 0$, $a \in \mathbb{C}$, and $m = 0, 1, 2, \ldots$. There are functions of the form $f = h\overline{g}$ which are not polynomials but have the property that $NZ(f-w, \mathbb{C})$ is finite and uniformly bounded for all $w \in \mathbb{C}$. For example, the function $f(z) = ze^{z}\overline{e^{z}} - w$ has at most two zeros for all fixed $w \in \mathbb{C}$. The following result was shown in [2].

Theorem 4.2. *Let $f = h\overline{g}$ be a logharmonic function in $\mathbb{C}$ such that $NZ(f-w, G)$ is finite for at least two different values of w, $\lim_{z\to\infty} a(z) = a(\infty)$ exists with $|a(\infty)| \neq 1$, then f is a polynomial.*

An upper bound on the number of w-points of a logharmonic polynomial can be readily obtained by using Bezout's theorem [39].

Theorem 4.3 (see [39]). *Let $p(x, y)$ and $q(x, y)$ be polynomials in the real variables x and y with real coefficients. If $\deg(p) = n$ and $\deg(q) = m$, then either p and q have at most nm common zeros or they have infinitely many zeros.*

Wilmshurst [40] has shown that Bezout's theorem gives a sharp upper bound for the number of zeros of a harmonic polynomial and hence for polyanalytic polynomials (see, e.g., [41, 42]). However, this is not true for logharmonic polynomials.

Let $f = p_n\overline{p_m}$ be a logharmonic polynomial of degree $n+m$. Then $f(z) - w = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} z^k \overline{z}^j$. The functions $P(z) = \mathrm{Re}(f(z))$ and $Q(z) = \mathrm{Im}(f(z))$ are real-valued polynomials in x and y and are of degree $n+m$. Applying Bezout's theorem, we conclude with the following estimate.

Theorem 4.4. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$. Then either $f-w$ has infinitely many zeros or $f-w$ has at most $(n+m)^2$ zeros for all $w \in \mathbb{C}$.*

The bound is not the best possible. Indeed, a quadratic polynomial is of the form $f(z) = p_2(z)$, $\overline{f(z)} = p_2(z)$, or $f(z) = a(z+b)\overline{(z+c)}$. In all three cases, $f-w$ has either infinitely many zeros or it has at most two.

Observe that the logharmonic polynomial $f(z) = (z-1)/(\overline{z}+1)$ is 2-valent and omits the half-plane $\mathrm{Re}(w) < -1$ and that $|a| \not\equiv 1$. However, the situation changes if $|a(\infty)| \neq 1$ and we have the following result [2].

Theorem 4.5. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, and suppose that $n > m$. Fix $w \in \mathbb{C}$ such that $Z(f - w, \mathbb{C}) \cap (\partial D \cup S_E(D))$ is empty. Then the number of zeros $VZ(f - w, S_E(\mathbb{C}))$ of $f - w$, and hence also the valency $V(f, \mathbb{C})$ of f in $\mathbb{C}$, is at least $n - m$. The bound is best possible.*

The following result is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, and suppose that $n > m$. Then*

(i) $f(\mathbb{C}) = \mathbb{C}$,

(ii) *for almost all $w \in \mathbb{C}$, the function $f - w$ has at least $n - m$ disjoint zeros.*

The next result characterizes polynomials of finite valency [2].

Theorem 4.7. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, such that $p_n \not\equiv \mathrm{const} \cdot p_m$. Then the cardinality $NZ(f - w, \mathbb{C})$ of the zero set $Z(f - w, \mathbb{C})$ is finite (hence, by Bezout's theorem, uniformly bounded) for all $w \in \mathbb{C}$.*

Remark 4.8. If $p_n \equiv \mathrm{const} \cdot p_m$, then the image lies on a straight line.

5. Subclasses of Logharmonic Mappings

5.1. Spirallike Logharmonic Mappings

Let Ω be a simply connected domain if $\mathbb{C}$ contains the origin. We say that Ω is α-spirallike, $-\pi/2 < \alpha < \pi/2$, if $w \in \Omega$ implies that $w \exp(-te^{i\alpha}) \in \Omega$ for all $t \geq 0$. If $\alpha = 0$, the domain Ω is called starlike (with respect to the origin). We will use the following notations.

(a) S^{α}_{Lh} is the set of all univalent logharmonic mappings f in U satisfying $f(0) = 0$, $h(0) = g(0) = 1$, and $f(U)$ is an α-spirallike domain.

(b) $S^{\alpha} = \{f \in S^{\alpha}_{Lh}$ and $f \in H(U)\}$.

(c) $S^{*}_{Lh} = S^{0}_{Lh}$ and $S^{*} = S^{0}$, for which $f(U)$ is starlike (with respect to the origin).

To each $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in S^{\alpha}_{Lh}$, we associate the analytic function $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}$, $\psi(0) = 0$. Abdulhadi and Hengartner [8] gave a representation theorem for mappings in the class S^{α}_{Lh}.

Theorem 5.1. (a) *If $f \in S^{\alpha}_{Lh}$, then $\psi \in S^{\alpha}$.*

(b) *For any given $\psi \in S^{\alpha}$ and $a \in B(U)$, there are h and g in $H(U)$ uniquely determined such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}$,

(iii) $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ *is a solution of* (1.1) *in S^{α}_{Lh}, where* $\beta = (\overline{a(0)}(1 + a(0)))/(1 - |a(0)|^2)$.

Remark 5.2. Theorem 5.1 has no equivalence for the class of all convex univalent logharmonic mappings. Indeed, $\varphi(z) = z$ is a convex mapping, $a(z) = z^4 \in B(U)$, but $f(z) = z/|1 - z^4|^{1/2}$ is not a convex mapping.

Remark 5.3. Theorem 5.1 asserts that the class S^{α}_{Lh}, α fixed in $(-\pi/2, \pi/2)$, is isomorphic to $S^{\alpha} \times B(U)$.

The following result is an immediate consequence of Theorem 5.1.

Corollary 5.4. *If $f \in S^{\alpha}_{Lh}$, then $f(rz)/r \in S^{\alpha}_{Lh}$ for all $r \in (0,1)$. In other words, level sets inherit the property of being α-spirallike.*

The next result is an integral representation for $f \in S^{\alpha}_{Lh}$ [8].

Theorem 5.5. *A function $f \in S^{\alpha}_{Lh}$ if and only if there are two probability measures μ and ν on the Borel sets of ∂U and an $a(0) \in U$ such that*

$$f(z) = z|z|^{2\beta} \cdot \exp\left\{ \int_{\partial U \times \partial U} K(z, \zeta, \xi; a(0)) d\mu(\zeta) d\nu(\xi) \right\}, \tag{5.1}$$

where

$$\begin{aligned}
\beta &= \frac{\overline{a(0)}(1 + a(0))}{1 - |a(0)|^2}, \\
K(z, \zeta, \xi; a(0)) &= -2\cos(\alpha) \cdot e^{i\alpha} \cdot \log(1 - \zeta z) + 2e^{i\alpha} \operatorname{Re}\left\{ e^{i\alpha} \log(1 - \zeta z) \right\} + T(z, \zeta, \xi; a(0)), \\
T(z, \zeta, \xi; a(0)) &= 2e^{i\alpha} \operatorname{Re} \frac{e^{i\alpha}(1 + a(0))\left(1 - \overline{a(0)} e^{-2i\alpha}\right)\zeta + e^{-i\alpha}\left(1 + \overline{a(0)}\right)\left(1 - a(0)e^{2i\alpha}\right)\xi}{(\zeta - \xi)|1 - a(0)e^{2i\alpha}|^2} \\
&\quad \times \log \frac{1 - \xi z}{1 - \zeta z},
\end{aligned} \tag{5.2}$$

if $|\zeta| = |\xi| = 1$, $\zeta \neq \xi$, and

$$T(z, \zeta, \zeta; a(0)) = 4\cos(\alpha) \cdot e^{i\alpha} \cdot \operatorname{Re} \frac{\zeta z}{(1 - \zeta z)} \frac{1 - |a(0)|^2}{|1 - a(0)e^{2i\alpha}|^2}. \tag{5.3}$$

Observe that S^{α}_{Lh} is not compact, but Theorem 5.5 can be used to solve extremal problems over the class of mappings in S^{α}_{Lh} with a given $a(0) = 0$.

We have seen in Corollary 5.4 that if f is a univalent logharmonic mapping in U, $f(0) = 0$, and if $f(U)$ is starlike, then $f(|z| < r)$ is starlike (with respect to the origin) for all $r \in (0,1)$. The next result proved in [8] shows that this property may fail whenever $f(0) \neq 0$.

Theorem 5.6. *For each $z_0 \in U \setminus \{0\}$, there are univalent logharmonic mappings f such that $f(z_0) = 0$, $f(U)$ is starlike (with respect to the origin), but no level set $f(|z| < r)$, $|z_0| = \rho < r < 1$, is starlike.*

5.2. Close-to-Starlike Logharmonic Mappings

5.2.1. Logharmonic Mappings with Positive Real Part

Let P_{Lh} be the set of all logharmonic mappings R in U which are of the form $R = H\overline{G}$, where H and G are in $H(U)$, $H(0) = G(0) = 1$, such that $\mathrm{Re}(R(z)) > 0$ for all $z \in U$. In particular, the set P of all analytic functions p in U with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ in U is a subset of P_{Lh}.

The next result [43] describes the connection between P_{Lh} and P.

Theorem 5.7. *A function $R = H\overline{G} \in P_{Lh}$ if and only if $p = H/G \in P$.*

As a consequence of Theorem 5.7, it follows that R admits the representation

$$R(z) = p(z) \exp 2\,\mathrm{Re} \int_0^z \frac{a(s)}{1 - a(s)} \frac{p'(s)}{p(s)} ds, \tag{5.4}$$

where $a \in B(U)$ and p is an analytic function with positive real part normalized by $p(0) = 1$.

The following result [43] is a distortion theorem for the class P_{Lh}.

Theorem 5.8. *Let $R(z) = H(z)\overline{G(z)} \in P_{Lh}$, and suppose that $a(0) = 0$. Then for $z \in U$*

(i) $\exp(-2|z|/(1 - |z|)) \le |R(z)| \le \exp(2|z|/(1 - |z|))$,

(ii) $|R_z(z)| \le (2/(1 - |z|)(1 - z|^2)) \exp(2|z|/(1 - |z|))$,

(iii) $|R_{\overline{z}}(z)| \le (2|z|/(1 - |z|)(1 - |z|^2)) \exp(2|z|/(1 - |z|))$.

Equality occurs for the right inequalities if R is a function of the form $R_0(\zeta z)$, $|\zeta| = 1$, where

$$R_0(z) = \frac{1+z}{1-z}\left|\frac{1+z}{1-z}\right| \exp\left(\mathrm{Re}\,\frac{2z}{1-z}\right), \tag{5.5}$$

and equality occurs for the left inequalities if R is of the form

$$\frac{1}{R_0(\zeta z)}, \qquad |\zeta| = 1. \tag{5.6}$$

5.2.2. Close-to-Starlike Logharmonic Mappings

Let $F(z) = z|z|^{2\beta}h\overline{g}$ be a logharmonic mapping. The function F is close to starlike if F is a product between a starlike logharmonic mapping $f(z) = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ which is a solution of (1.1) with respect to $a \in B(U)$ and a logharmonic mapping with positive real part $R \in P_{Lh}$ with the same second dilatation function a.

The geometric interpretation for a close-to-starlike logharmonic mappings is that the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than π.

Denote by CST_{Lh} the set of all close-to-starlike logharmonic mappings. It contains in particular the set CST of all analytic close-to-starlike functions which was introduced by Reade [44] in 1955. Also, the set S^*_{Lh} of all starlike univalent logharmonic mappings is a subset of CST_{Lh} (take $R(z) \equiv 1$ in the product). Furthermore, if $F(z) = z|z|^{2\beta}h\overline{g}$ is a logharmonic mapping with respect to $a \in B(U)$ satisfying $h(0) = g(0) = 1$ and $\operatorname{Re} F(z)/z|z|^{2\beta} > 0$, then F is a close-to-starlike logharmonic mapping, where

$$f(z) = z|z|^{2\beta}\left|\exp\left(\int_0^z \frac{a(s)/s}{1-a(s)}ds\right)\right|^2. \tag{5.7}$$

On the other hand, a mapping $F \in \mathrm{CST}_{Lh}$ need not necessarily be univalent. For example, take $F(z) = z(1+z)$, where $z \in S^*$ and $1+z \in P$.

Our next result is a representation theorem for the class CST_{Lh} proved in [43].

Theorem 5.9. (a) *Let $F = z|z|^{2\beta}h\overline{g}$ be in CST_{Lh}. Then $\varphi = zh/g \in CST$.*

(b) *Given any $\varphi \in CST$ and $a \in B(U)$, there are h and g in $H(U)$ uniquely determined such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\varphi(z) = zh/g$,

(iii) $F = z|z|^{2\beta}h\overline{g}$ *is in CST_{Lh} which is a solution of* (1.1) *with respect to the given a.*

Corollary 5.10. *$F \in CST_{Lh}$ if and only if $F(rz)/r \in CST_{Lh}$ for all $r \in (0,1)$.*

In the next result the radius of univalence and the radius of starlikeness are determined for those mappings in the set CST_{Lh} [43].

Theorem 5.11. *Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then F maps the disc $|z| < R$, $R \le 2-\sqrt{3}$, onto a starlike domain. The upper bound is best possible for all $a \in B(U)$.*

Combining Theorems 5.8 and 5.11 with $\alpha = 0$, we obtain the following distortion theorem for the class CST_{Lh}.

Theorem 5.12. *Let $F = zh\overline{g} \in CST_{Lh}$. Then, for every $z \in U$,*

(a) $|z|\exp(-2|z|/(1-|z|) - 4|z|/(1+|z|)) \le |F(z)| \le |z|\exp(6|z|/(1-|z|))$,

(b) $|F_z(z)| \le ((|z|^2+4|z|+1)/(1-|z|)^2(1+|z|))\exp(6|z|/(1-|z|))$,

(c) $|F_{\overline{z}}(z)| \le (|z|(|z|^2+4|z|+1)/(1-|z|)^2(1+|z|))\exp(6|z|/(1-|z|))$.

Equality holds for the right inequalities if F is a function of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)}\frac{1+\zeta z}{1-\zeta z}\left|\frac{1-\zeta z}{1+\zeta z}\right|\exp\left(\operatorname{Re}\left[\frac{4\eta z}{1-\eta z}+\frac{2\zeta z}{1-\zeta z}\right]\right), \tag{5.8}$$

where $|\eta| = |\zeta| = 1$, and for the left inequalities if F is a function of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)}\frac{1+\zeta z}{1-\zeta z}\left|\frac{1-\zeta z}{1+\zeta z}\right|\exp\left(\operatorname{Re}\left[\frac{4\eta z}{1-\eta z}-\frac{2\zeta z}{1-\zeta z}\right]\right), \tag{5.9}$$

where $|\eta| = |\zeta| = 1$.

5.3. Typically Real Logharmonic Mappings

A logharmonic mapping f is said to be typically real if and only if f is real whenever z is real and if f is normalized by $f(0) = 0$ and $h(0)\overline{g(0)} = 1$, or equivalently by $f(0) = 0$ and $h(0) = g(0) = 1$. Denote by T_{Lh} the class of all orientation-preserving typically real logharmonic mappings. Since f is orientation preserving and univalent on the interval $(-1, 1)$, it follows that f is of the form (2.10). Furthermore, if $f \in T_{Lh}$, then β (and hence, also $a(0)$) has to be real and yields the relation

$$\operatorname{Im} z \operatorname{Im} f(z) > 0, \quad \forall\, z \in U \setminus \mathbb{R}. \tag{5.10}$$

The class T_{Lh} is a compact convex set with respect to the topology of locally uniform convergence, and it contains, in particular, the set T of all analytic typically real functions.

5.3.1. Basic Properties of Mappings from T_{Lh}

The following representation theorem for typically real logharmonic mappings was proved in [45].

Theorem 5.13. (a) *If $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is in T_{Lh}, then $\phi = zh/g \in T$.*

(b) *Given $\phi \in T$ and $a \in B(U)$ such that $\beta \in \mathbb{R}$ and $a(0) \in \mathbb{R}$, there are uniquely determined mappings h and g in $H(U)$ such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\phi(z) = zh/g$,

(iii) $F = z|z|^{2\beta}h\overline{g}$ *is in T_{Lh} which is a solution of* (1.1) *with respect to the given a.*

As a consequence of Theorem 5.13, it follows that

$$f(z) = zh(z)\overline{g(z)} = \phi(z)\left|g(z)\right|^2, \tag{5.11}$$

where

$$\begin{gathered} g(z) = \exp \int_0^z \frac{a(s)\phi'(s)}{(1 - a(s))\phi(s)} ds, \\ zh(z) = \phi(z)g(z). \end{gathered} \tag{5.12}$$

Denote by T_{Lh}^0 the subclass of T_{Lh} ($\beta = 0$) consisting of all mappings F from T_{Lh} for which $\phi = zh/g = z/(1 - z^2)$. Then F is of the form

$$F(z) = \frac{z}{1 - z^2} \exp 2 \operatorname{Re} \int_0^z \frac{a(s)(1 + s^2)}{s(1 - a(s))(1 - s^2)} ds. \tag{5.13}$$

The next theorem links the class T_{Lh} with the class P_{Lh}.

Theorem 5.14 (see [45]). *Let $f(z) = zh(z)\overline{g(z)} \in T_{Lh}$ with respect to $a \in B(U)$, and $a(0) = 0$. Then there exist an $R \in P_{Lh}$ and an $F \in T^0_{Lh}$, such that both functions are logharmonic with respect to the same a and*

$$f(z) = F(z)R(z). \tag{5.14}$$

The next result is a distortion theorem for the class T^0_{Lh}.

Theorem 5.15 (see [45]). *Let $F(z) = zh(z)\overline{g(z)} \in T^0_{Lh}$. Then, for $z \in U$,*

(a) $|F(z)| \leq |z| \exp(2|z|/(1-|z|))$,

(b) $|F_z(z)| \leq ((1+|z|^2)/(1-|z|^2)(1-|z|)) \exp(2|z|/(1-|z|))$,

(c) $|F_{\overline{z}}(z)| \leq (|z|(1+|z|^2)/(1-|z|^2)(1-|z|)) \exp(2|z|/(1-|z|))$.

Equality holds if and only if F is of the form $\overline{\eta}F_0(\eta z)$, $|\eta| = 1$, where

$$F_0(z) = \frac{z}{1-z^2}\left|1-z^2\right| \exp\left(\operatorname{Re}\left(\frac{2z}{1-z}\right)\right). \tag{5.15}$$

Combining Theorems 5.8, 5.14, and 5.15, the following distortion theorem is obtained for the class T_{Lh}.

Theorem 5.16. *Let $f(z) = zh(z)\overline{g(z)} \in T_{Lh}$. Then, for $z \in U$,*

(a) $|f(z)| \leq |z| \exp(4|z|/(1-|z|))$,

(b) $|f_z(z)| \leq ((1+|z|)/(1-|z|^2)) \exp(4|z|/(1-|z|))$,

(c) $|f_{\overline{z}}(z)| \leq (|z|(1+|z|)/(1-|z|^2)) \exp(4|z|/(1-|z|))$.

Equality holds if f is of the form $\overline{\eta}f_0(\eta z)$, $|\eta| = 1$, where

$$f_0(z) = \frac{z(1-\overline{z})}{1-z} \exp\left(\operatorname{Re}\left(\frac{4z}{1-z}\right)\right). \tag{5.16}$$

Remark 5.17. The function f_0 given in (5.16) plays the role of the Koebe mapping in the set of logharmonic mappings (see, e.g., [1, 6]).

The next result gives the radius of univalence and the radius of starlikeness for the mappings in the set T_{Lh} [45].

Theorem 5.18 (see [45]). *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in T_{Lh}$. Then f maps the disc $\{z : |z| < R_0\}$, where $R_0 = (1+\sqrt{5}-\sqrt{2+2\sqrt{5}})/2$, onto a starlike domain. The upper bound is the best possible for all $a \in B(U)$.*

5.3.2. Univalent Mappings in T_{Lh}

Now we consider univalent mappings in T_{Lh}. For analytic typically real functions, it is known that if $t(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in the unit disc U, then t belongs to T if and only if the image $t(U)$ is a domain symmetric with respect to the real axis.

One might consider a similar problem in T_{Lh}. Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent logharmonic mapping in the unit disc U, and $h(0) = g(0) = 1$, $\beta > -1/2$. Observe that β (and hence $a(0)$) is real. Is it true that f belongs to T_{Lh} if and only if the image of $f(U)$ is a symmetric domain with respect to the real axis?

The answer is negative in both directions. Indeed, the function

$$f(z) = z\left(1 + \frac{iz}{8}\right)\left(1 - \frac{i\overline{z}}{8}\right) \tag{5.17}$$

is a normalized univalent logharmonic typically real mapping, but $f(U)$ is not symmetric with respect to the real axis. On the other hand, the function $f(z) = z(1 + i\overline{z})/(1 - iz)$ is a univalent logharmonic mapping from U onto U, and $f(U)$ is symmetric with respect to the real axis, but f is not typically real (for more details, see [45]).

Additional conditions on a and on the image domain $\Omega = f(U)$ are needed in order to get an affirmative answer to the question posed above.

Theorem 5.19 (see [45]). *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent (orientation-preserving) logharmonic mapping in the unit disc U and normalized by $f(0) = 0$, $h(0) = \overline{g(0)} = 1$. Suppose that the second dilatation function a has real coefficients, that is, $a(z) \equiv \overline{a(\overline{z})}$. (Observe that the condition $a(0)$ real or equivalently β real is automatically satisfied.)*

(a) *If f is typically real, then $f(U)$ is symmetric with respect to the real axis.*

(b) *If $|a| \leq k < 1$ in U and $f(U)$ is a strictly starlike Jordan domain symmetric with respect to the real axis, then f is typically real.*

5.4. Starlike Logharmonic Mappings of Order α

Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent logharmonic mapping. We say that f is starlike logharmonic mapping of order α if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = \operatorname{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \quad 0 \leq \alpha < 1, \tag{5.18}$$

for all $z \in U$. Denote by $\mathrm{ST}_{Lh}(\alpha)$ the set of all starlike logharmonic mappings of order α. If $\alpha = 0$, we get the class of starlike logharmonic mappings. Also, let $\mathrm{ST}(\alpha) = \{f \in \mathrm{ST}_{Lh}(\alpha)$ and $f \in H(U)\}$.

In this section, we obtain two representation theorems [46] for functions in $\mathrm{ST}_{Lh}(\alpha)$. In the first, we establish the connection between the classes $\mathrm{ST}_{Lh}(\alpha)$ and $\mathrm{ST}(\alpha)$. The second is an integral representation theorem.

Theorem 5.20. *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping in U, $0 \notin hg(U)$. Then $f \in ST_{Lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in ST(\alpha)$.*

Theorem 5.21. *A function $f = zh\overline{g} \in ST_{Lh}(\alpha)$ with $a(0) = 0$ if and only if there are two probability measures μ and ν such that*

$$f(z) = z\exp\left(\int_{\partial U\times\partial U} K(z,\zeta,\xi)d\mu(\zeta)d\nu(\xi)\right), \tag{5.19}$$

where

$$K(z,\zeta,\xi) = (1-\alpha)\log\left(\frac{1+\overline{\zeta z}}{1-\zeta z}\right) + T(z,\zeta,\xi),$$

$$T(z,\zeta,\xi) = \begin{cases} -2(1-\alpha)\operatorname{Im}\left(\dfrac{\zeta+\xi}{\zeta-\xi}\right)\arg\left(\dfrac{1-\xi z}{1-\zeta z}\right) - 2\alpha\log|1-\xi z|, & \text{if } |\zeta| = |\xi| = 1,\ \zeta\neq\xi, \\ (1-\alpha)\operatorname{Re}\left(\dfrac{4\zeta z}{1-\zeta z}\right) - 2\alpha\log|1-\zeta z|, & \text{if } |\zeta| = |\xi| = 1,\ \zeta = \xi. \end{cases} \tag{5.20}$$

Remark 5.22. Theorem 5.21 can be used to solve extremal problems for the class $\mathrm{ST}_{Lh}(\alpha)$ with $a(0) = 0$. For example, see Theorem 5.23.

The following is a distortion theorem for the class $\mathrm{ST}_{Lh}(\alpha)$ with $a(0) = 0$.

Theorem 5.23 (see [46]). *Let $f = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$ with $a(0) = 0$. Then, for $z \in U$,*

$$\frac{|z|}{(1+|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le \frac{|z|}{(1-|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{4|z|}{1-|z|}\right). \tag{5.21}$$

Equalities occur if and only if $f(z) = \overline{\zeta} f_0(\zeta z)$, $|\zeta| = 1$, where

$$f_0(z) = z\left(\frac{1-\overline{z}}{1-z}\right)\frac{1}{(1-\overline{z})^{2\alpha}}\exp\left((1-\alpha)\operatorname{Re}\frac{4z}{1-z}\right). \tag{5.22}$$

The next result gives sharp coefficient estimates for functions h and g in the starlike logharmonic mapping $f = zh(z)\overline{g(z)}$.

Theorem 5.24 (see [6]). *Let $f = zh(z)\overline{g(z)} \in ST_{Lh}(0)$ with $a(0) = 0$, and put*

$$h(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^k\right), \qquad g(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right). \tag{5.23}$$

Then

$$|a_n| \le 2 + \frac{1}{n}, \qquad |b_n| \le 2 - \frac{1}{n} \tag{5.24}$$

for all $n \geq 1$. *Equality holds for the mapping*

$$f(z) = z\frac{1-\overline{ze^{i\alpha}}}{1-ze^{i\alpha}} \exp\left(\frac{4ze^{i\alpha}}{1-ze^{i\alpha}}\right), \qquad \alpha \in (0, 2\pi). \tag{5.25}$$

5.5. Functions with Logharmonic Laplacian

We consider the class of all continuous complex-valued functions $F = u + iv$ in a domain $D \subseteq C$ such that the Laplacian of F is logharmonic. Note that $\log(\Delta F)$ is harmonic in D if it satisfies the Laplace equation $\Delta(\log(\Delta F)) = 0$, where

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}. \tag{5.26}$$

In any simply connected domain D, we can write

$$F = r^2 L + H, \qquad z = re^{i\theta}, \tag{5.27}$$

where L is logharmonic and H is harmonic in D. It is known that L and H can be expressed as

$$\begin{aligned} L &= h_1 \overline{g_1}, \\ H &= h_2 + \overline{g_2}, \end{aligned} \tag{5.28}$$

where h_1, g_1, h_2, and g_2 are analytic in D. Denote by $L_{Lh}(U)$ the set of all functions of the form (5.27) which are defined in the unit disc U.

Note that the composition $L \circ \phi$ of a logharmonic function L with an analytic function ϕ is logharmonic and, also, the composition $H \circ \phi$ of a harmonic function H with analytic function ϕ is harmonic, while this is not true for the function F. Also, if $F_1(z) = r^2 L_1(z)$ and $F_2(z) = r^2 L_2(z)$ are in $L_{Lh}(U)$, where L_1 and L_2 are logharmonic with respect to the same a, then $F_1^{\alpha} F_2^{\beta}$, $\alpha + \beta = 1$, is also in $L_{Lh}(U)$.

Denote the Jacobian of W by J_W. Then

$$J_W = |W_z|^2 - |W_{\overline{z}}|^2. \tag{5.29}$$

Also let

$$\begin{aligned} \lambda_W &= |W_z| - |W_{\overline{z}}|, \\ \Lambda_W &= |W_z| + |W_{\overline{z}}|. \end{aligned} \tag{5.30}$$

Then $J_W = \lambda_W \Lambda_W$.

5.5.1. The Univalence of Functions with Logharmonic Laplacian

First a lower bound for the area of the range of $F(z) = r^2 L(z)$ is established, where L is a starlike univalent logharmonic mapping.

Theorem 5.25 (see [47]). *Let $F(z) = r^2 L(z)$, where $L = h\overline{g}$ is starlike univalent logharmonic in U, with $g(0) = 1$ and $h'(0) = 1$. Let $A(r, F)$ denote the area of $F(U_r)$, where $U_r = \{z : |z| < r\}$, for $r < 1$. Then*

$$A(r,F) \geq 2\pi\left[-2r + r^2 - \frac{2r^3}{3} + \frac{r^4}{2} - \frac{r^5}{5} + \frac{r^6}{6} - \frac{r^8}{8} + 2\ln(1+r)\right]. \tag{5.31}$$

Equality holds if and only if $L_0(z) = r^2 z(1+\overline{z}/2)/(1+z/2)$ or one of its rotations.

Definition 5.26. Let L be logharmonic function in U. A complex-valued function of the form $F(z) = r^2 L(z)$ is starlike in U if it is orientation preserving, $F(0) = 0$, $F(z) \neq 0$ when $z \neq 0$ and the curve $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, $\partial \arg F(re^{it})/\partial t = \operatorname{Re}((zF_z - \overline{z}F_{\overline{z}})/F) > 0$.

Remark 5.27. Note that starlike functions are univalent in U.

The following theorem links starlike functions in $L_{Lh}(U)$ with the class of starlike analytic functions.

Theorem 5.28 (see [48]). *Let $F(z) = r^2 L(z)$, where $L(z) = h(z)\overline{g(z)}$, be a logharmonic function in U with respect to a, where $a \in B(U)$ with $a(0) = 0$. Then F is starlike univalent in U if and only if $\psi(z) = h(z)/g(z)$ is starlike univalent function in U.*

Corollary 5.29. *The function $r^2 L(z)$ is starlike for all conformal starlike functions L.*

A characterization of the logharmonic Laplacian solutions of the Dirichlet problem in the unit disc U is given in [48].

Theorem 5.30. *Let F^* be an orientation-preserving homeomorphism from ∂U onto ∂U, that is, $F^*(e^{it}) = e^{i\lambda(t)}$, where λ is continuous and strictly monotonically increasing on $[0, 2\pi]$. Furthermore, suppose that $\lambda(2\pi) = \lambda(0) + 2\pi$. Then $F(z) = \overline{z}|z|^2 h(z)/\overline{h(z)}$ is a univalent solution of the Dirichlet problem in U.*

For the general case $F(z) = r^2 L(z) + H(z)$, a sufficient condition is obtained that makes F locally univalent.

Theorem 5.31 (see [48]). *Let $F(z) = r^2 h_1(z)\overline{g_1(z)} + h_2(z) + \overline{g_2(z)}$ be in the class $L_{Lh}(U)$. Suppose that $\psi(z) = h_1(z)/g_1(z)$ is starlike univalent in U, and $|g_2'(z)| < |h_2'(z)|$ for $z \in U$. If*

$$\operatorname{Re}\left[g_2' \overline{\left(r^2 h_1 \overline{g_1}\right)_{\overline{z}}}\right] < \operatorname{Re}\left[h_2' \overline{\left(r^2 h_1 \overline{g_1}\right)_z}\right], \tag{5.32}$$

then $J_F(z) > 0$ for $z \neq 0$, and F is locally univalent.

5.5.2. Landau's Theorem for Functions with Logharmonic Laplacian

Lewy's famous theorem [49] states that a harmonic function W is locally univalent in D (univalent in some neighborhood of each point in D) if and only if its Jacobian does not vanish in D.

The classical Landau Theorem states that if f is analytic in the unit disc U with $f(0) = 0$, $f'(0) = 1$, and $|f(z)| < M$ for $z \in U$, then f is univalent in the disc $U_{\rho_0} = \{z : |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}, \tag{5.33}$$

and $f(U_{\rho_0})$ contains a disc U_{R_0} with $R_0 = M\rho_0^2$. This result is sharp, with the extremal function $f(z) = Mz(1 - Mz)/(M - z)$ (see [19]).

Chen et al. [50] obtained a version of Landau's Theorem for bounded harmonic mappings of the unit disc. Unfortunately their result is not sharp. Better estimates were given in [51] and later in [52].

Specifically, it was shown in [52] that if f is harmonic in the unit disc U with $f(0) = 0$, $J_f(0) = 1$, and $|f(z)| < M$ for $z \in U$, then f is univalent in the disc $U_{\rho_1} = \{z : |z| < \rho_1\}$ with

$$\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}, \tag{5.34}$$

and $f(U_{\rho_1})$ contains a disc U_{R_1} with $R_1 = \pi/4M - 2M(\rho_1^2/(1 - \rho_1))$. This result is the best known, but not sharp.

The following Schwarz lemma for harmonic mappings is due to Grigoryan [52].

Lemma 5.32 (Schwarz lemma). *Let f be a harmonic mapping of the unit disc U with $f(0) = 0$ and $f(U) \subset U$. Then*

$$\begin{gathered} \left|f(z)\right| \leq \frac{4}{\pi}\arctan|z| \leq \frac{4}{\pi}|z|, \\ \Lambda_f(0) \leq \frac{4}{\pi}. \end{gathered} \tag{5.35}$$

Recently Mao et al. [53] established the Schwarz lemma for logharmonic mappings, through which two versions of Landau's theorem for these mappings were obtained.

The next theorem gives Landau's theorem for functions with logharmonic Laplacian of the form $F = r^2L(z)$.

Theorem 5.33 (see [47]). *Let L be logharmonic in U such that $L(0) = 0$, $J_L(0) = 1$, and $|L(z)| < M$ for $z \in U$. Then there is a constant $0 < \rho_2 < 1$ such that $F = r^2L$ is univalent in the disc $|z| < \rho_2$, ρ_2 is the solution of the equation $1 = 2\rho_2 M/(1-\rho_2^2) - 2M\rho_2/(1-\rho_2^2)^2$, and $f(U_{\rho_2})$ contains a disc U_{R_2} with $R_2 = \rho_2^2 - 2M\rho_2^4/(1 - \rho_2^2)$. This result is not sharp.*

Finally we give a Landau theorem for functions of logharmonic Laplacian of the form $F = r^2L + K$.

Theorem 5.34 (see [47]). *Let $F = r^2L + K$, $z = re^{i\theta}$ be in $L_{Lh}(U)$, where L is logharmonic and K is harmonic in the unit disc U, such that $L(0) = K(0) = 0$, $J_F(0) = 1$, and $|L|$ and $|K|$ are both bounded by M. Then there is a constant $0 < \rho_3 < 1$ such that F is univalent in $|z| < \rho_3$. Specifically ρ_3 satisfies*

$$\frac{\pi}{4M} - 2\rho_3 M - 2M\left(\frac{\rho_3^3}{\left(1-\rho_3^2\right)^2} + \frac{1}{\left(1-\rho_3\right)^2} - 1\right) = 0, \tag{5.36}$$

and $F(U_{\rho_3})$ contains a disc U_{R_3}, where

$$R_3 = \frac{\pi}{4M}\rho_3 - \rho_3^2 M\frac{1}{1-\rho_3^2} - 2M\frac{\rho_3^2}{1-\rho_3}. \tag{5.37}$$

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References

[1] Z. Abdulhadi and D. Bshouty, "Univalent functions in $H \cdot \overline{H}(D)$," *Transactions of the American Mathematical Society*, vol. 305, no. 2, pp. 841–849, 1988.
[2] Z. Abdulhadi and W. Hengartner, "Polynomials in $H\overline{H}$," *Complex Variables. Theory and Application*, vol. 46, no. 2, pp. 89–107, 2001.
[3] Z. Abdulhadi and W. Hengartner, "Canonical point mappings in $H\overline{H}$," *Bulletin of the Australian Mathematical Society*, vol. 56, no. 2, pp. 239–242, 1997.
[4] Z. Abdulhadi and W. Hengartner, "One pointed univalent logharmonic mappings," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 2, pp. 333–351, 1996.
[5] Z. Abdulhadi, W. Hengartner, and J. Szynal, "Univalent logharmonic ring mappings," *Proceedings of the American Mathematical Society*, vol. 119, no. 3, pp. 735–745, 1993.
[6] Z. Abdulhadi and W. Hengartner, "Univalent logharmonic extensions onto the unit disk or onto an annulus," in *Current Topics in Analytic Function Theory*, pp. 1–12, World Scientific Publishing, River Edge, NJ, USA, 1992.
[7] Z. Abdulhadi and W. Hengartner, "Univalent harmonic mappings on the left half-plane with periodic dilatations," in *Univalent Functions, Fractional Calculus, and Their Applications (Kōriyama, 1988)*, Ellis Horwood series in mathematics and its applications, pp. 13–28, Ellis Horwood, Chichester, UK, 1989.
[8] Z. Abdulhadi and W. Hengartner, "Spirallike logharmonic mappings," *Complex Variables. Theory and Application*, vol. 9, no. 2-3, pp. 121–130, 1987.
[9] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, New York, NY, USA, 2nd edition, 1973.
[10] P. Duren and W. Hengartner, "A decomposition theorem for planar harmonic mappings," *Proceedings of the American Mathematical Society*, vol. 124, no. 4, pp. 1191–1195, 1996.
[11] Z. Abdulhadi and Y. Abumuhanna, "A factorization theorem for logharmonic mappings," *International Journal of Mathematics and Mathematical Sciences*, no. 14, pp. 853–856, 2003.
[12] W. Hengartner and G. Schober, "Curvature estimates for some minimal surfaces," in *Complex Analysis*, pp. 87–100, Birkhäuser, Basel, Switzerland, 1988.
[13] J. C. C. Nitsche, *Vorlesungen über Minimalflächen*, Springer, Berlin, Germany, 1975.
[14] R. Osserman, *A Survey of Minimal Surfaces*, Dover, New York, NY, USA, 2nd edition, 1986.
[15] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae*, vol. 9, pp. 3–25, 1984.

[16] W. Hengartner and G. Schober, "Univalent harmonic functions," *Transactions of the American Mathematical Society*, vol. 299, no. 1, pp. 1–31, 1987.
[17] W. Hengartner and G. Schober, "Harmonic mappings with given dilatation," *Journal of the London Mathematical Society*, vol. 33, no. 3, pp. 473–483, 1986.
[18] W. Hengartner and G. Schober, "Univalent harmonic mappings onto parallel slit domains," *The Michigan Mathematical Journal*, vol. 32, no. 2, pp. 131–134, 1985.
[19] H. Kneser, "Losung der Aufgabe 41," *Jahresber. Deutsch Math. Verein*, vol. 35, pp. 123–124, 1926.
[20] T. Rado, "Aufgabe 41," *Jahresber. Deutsch Math. Verein*, vol. 35, p. 49, 1926.
[21] G. Choquet, "Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques," *Bulletin des Sciences Mathématiques*, vol. 69, pp. 156–165, 1945.
[22] Z. Abdulhadi, D. Bshouty, and W. Hengartner, "Minimal surfaces whose Gauss map covers periodically the pointed upper half-sphere exactly once," *Computational Methods and Function Theory*, vol. 2, no. 1, pp. 155–174, 2002.
[23] G. Opfer, "New extremal properties for constructing conformal mappings," *Numerische Mathematik*, vol. 32, no. 4, pp. 423–429, 1979.
[24] G. Opfer, "Conformal mappings onto prescribed regions via optimization techniques," *Numerische Mathematik*, vol. 35, no. 2, pp. 189–200, 1980.
[25] G. Opfer, "Solving complex approximation problems by semi-infinite–finite optimization techniques: a study on convergence," *Numerische Mathematik*, vol. 39, no. 3, pp. 411–420, 1982.
[26] D. Bshouty, N. Hengartner, and W. Hengartner, "A constructive method for starlike harmonic mappings," *Numerische Mathematik*, vol. 54, no. 2, pp. 167–178, 1988.
[27] W. Hengartner and L. Nadeau, "Univalent harmonic exterior mappings as solutions of an optimization problem," *Matematički Vesnik*, vol. 40, no. 3-4, pp. 233–240, 1988.
[28] W. Hengartner and J. Rostand, "A constructive method for univalent logharmonic mappings," in *Classical and Modern Potential Theory and Applications (Chateau de Bonas, 1993)*, vol. 430 of *NATO Science Series C: Mathematical and Physical Sciences*, pp. 273–291, Kluwer Academic, Dodrecht, The Netherlands, 1994.
[29] D. Bshouty and W. Hengartner, "Univalent solutions of the Dirichlet problem for ring domains," *Complex Variables. Theory and Application*, vol. 21, no. 3-4, pp. 159–169, 1993.
[30] Sh. Chen, S. Ponnusamy, and X. Wang, "Coefficient estimates and Landau-Bloch's constant for planar harmonic mappings," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 34, no. 2, pp. 255–265, 2011.
[31] P. Duren, *Harmonic Mappings in the Plane*, vol. 156 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.
[32] W. Hengartner and G. Schober, "Univalent harmonic exterior and ring mappings," *Journal of Mathematical Analysis and Applications*, vol. 156, no. 1, pp. 154–171, 1991.
[33] J. Eells and L. Lemaire, "A report on harmonic maps," *The Bulletin of the London Mathematical Society*, vol. 10, no. 1, pp. 1–68, 1978.
[34] J. Eells, Jr. and J. H. Sampson, "Harmonic mappings of Riemannian manifolds," *American Journal of Mathematics*, vol. 86, pp. 109–160, 1964.
[35] J. Jost, *Harmonic Maps between Surfaces*, vol. 1062 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1984.
[36] J. Jost, *Two-Dimensional Geometric Variational Problems*, Pure and Applied Mathematics, John Wiley & Sons, Chichester, UK, 1991.
[37] R. M. Schoen, "The role of harmonic mappings in rigidity and deformation problems," in *Complex Geometry (Osaka, 1990)*, vol. 143 of *Lecture Notes in Pure and Applied Mathematics*, pp. 179–200, Dekker, New York, NY, USA, 1993.
[38] J. C. C. Nitsche, *Lectures on Minimal Surfaces*, vol. 1, Cambridge University Press, Cambridge, 1989.
[39] G. T. Whyburn, *Topological Analysis*, Princeton Mathematical Series, No. 23, Princeton University Press, Princeton, NJ, USA, 2nd edition, 1964.
[40] A. S. Wilmshurst, "The valence of harmonic polynomials," *Proceedings of the American Mathematical Society*, vol. 126, no. 7, pp. 2077–2081, 1998.
[41] M. B. Balk, *Polyanalytic Functions*, vol. 63 of *Mathematical Research*, Akademie-Verlag, Berlin, Germany, 1991.
[42] M. B. Balk, "The fundamental theorem of algebra for polyanalytic polynomials," *Lietuvos Matematikos Rinkinys*, vol. 8, pp. 401–404, 1968.

[43] Z. Abdulhadi, "Close-to-starlike logharmonic mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 3, pp. 563–574, 1996.

[44] M. O. Reade, "On close-to-close univalent functions," *The Michigan Mathematical Journal*, vol. 3, pp. 59–62, 1955.

[45] Z. Abdulhadi, "Typically real logharmonic mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 31, no. 1, pp. 1–9, 2002.

[46] Z. Abdulhadi and Y. Abu Muhanna, "Starlike log-harmonic mappings of order α," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 4, Article 123, p. 6, 2006.

[47] Z. Abdulhadi, Y. Abu Muhanna, and R. M. Ali, "Landau's theorem for functions with logharmonic Laplacian," *Applied Mathematics and Computation*, vol. 218, pp. 6798–6802, 2012.

[48] Z. Abdulhadi, "On the univalence of functions with logharmonic Laplacian," *Applied Mathematics and Computation*, vol. 215, no. 5, pp. 1900–1907, 2009.

[49] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–692, 1936.

[50] H. Chen, P. M. Gauthier, and W. Hengartner, "Bloch constants for planar harmonic mappings," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3231–3240, 2000.

[51] M. Dorff and M. Nowak, "Landau's theorem for planar harmonic mappings," *Computational Methods and Function Theory*, vol. 4, no. 1, pp. 151–158, 2004.

[52] A. Grigoryan, "Landau and Bloch theorems for harmonic mappings," *Complex Variables and Elliptic Equations*, vol. 51, no. 1, pp. 81–87, 2006.

[53] Zh. Mao, S. Ponnusamy, and X. Wang, "Schwarzian derivative and Landau's theorem for logharmonic mappings," *Complex Variables and Elliptic Equations*.