

Research Article

The Tracial Class Property for Crossed Products by Finite Group Actions

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We define the concept of tracial \mathcal{C} -algebra of C^* -algebras, which generalize the concept of local \mathcal{C} -algebra of C^* -algebras given by H. Osaka and N. C. Phillips. Let \mathcal{C} be any class of separable unital C^* -algebras. Let A be an infinite dimensional simple unital tracial \mathcal{C} -algebra with the (SP)-property, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then $A \rtimes_{\alpha} G$ is a simple unital tracial \mathcal{C} -algebra.

1. Introduction

In this paper, our purpose is to prove that certain classes of separable unital C^* -algebras are closed under crossed products by finite group actions with the tracial Rokhlin property.

The term “tracial” has been widely used to describe the properties of C^* -algebras since Lin introduced the concept of tracial rank of C^* -algebras in [1]. The notion of tracial rank was motivated by the Elliott program of classification of nuclear C^* -algebras. C^* -algebras with tracial rank no more than k for some $k \in \mathbb{N}$ are C^* -algebras that can be locally approximated by C^* -subalgebras in $\mathcal{D}^{(k)}$ after cutting out a “small” approximately central projection p . The term “tracial” come from the fact that, in good cases, the projection p is “small” if $\tau(p) < \varepsilon$ for every tracial state τ on A . The C^* -algebras of tracial rank zero can be determined by K -theory and hence can be classified. For example, Lin proved that if a simple separable amenable unital C^* -algebra A has tracial rank zero and satisfies the Universal Coefficient Theorem, then A is a simple AH-algebra with slow dimension growth and with real rank zero [2, 3]. In [4], Fang discovered the classification of certain nonsimple C^* -algebras with tracial rank zero.

These successes suggest that one consider “tracial” versions of other C^* -algebra concepts. In [5], Yao and Hu introduced the concept of tracial real rank of C^* -algebras. In [6], Fan and Fang introduced the concept of tracial stable rank of C^* -algebras. In [7, 8], Elliott and Niu and Fang and Fan studied the general concept of tracial approximation of properties of C^* -algebras. The concept of the Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes [9]. Then Herman and Ocneanu [10] and Rørdam [11] and Kishimoto [12] introduced the Rokhlin property to a much more general context of C^* -algebras. In [13], Phillips introduced the concept of tracial Rokhlin property of finite group actions, which is more universal than the Rokhlin property. In [14], Osaka and Phillips introduced the concepts of local class property and approximate class property of unital C^* -algebras and proved that these two properties are closed under crossed products by finite actions with the Rokhlin property.

Inspired by these papers, we introduce the concept of tracial class property of C^* -algebras and prove that, for appropriate classes of C^* -algebras, the tracial class property is closed under crossed products by finite group actions with the tracial Rokhlin property. As consequences, we get analogs of results in [13–18] such as the following ones. Let A be a separable simple unital C^* -algebra, and let α be an action of a finite group G on A which has the tracial Rokhlin property. If A is an AF-algebra, then $A \rtimes_{\alpha} G$ has tracial rank zero. If A is an AT-algebra with the (SP)-property, then $A \rtimes_{\alpha} G$ has tracial rank no more than one. If A has stable rank one and real rank zero, then the induced crossed product $A \rtimes_{\alpha} G$ has these two properties.

2. Definitions and Preliminaries

We denote by $\mathcal{D}^{(0)}$ the class of finite dimensional C^* -algebras and by $\mathcal{D}^{(k)}$ the class of C^* -algebras with the form $p(C(X) \otimes F)p$, where $F \in \mathcal{D}^{(0)}$, X is a finite CW complex with dimension k , and $p \in C(X) \otimes F$ is a projection.

Let p, q be projections in A and $a \in A_+$. If p is Murray-von Neumann equivalent to q , then we write $[p] = [q]$. If p is Murray-von Neumann equivalent to a subprojection of aAa , then we write $[p] \leq [a]$.

Let A be a C^* -algebra, and let \mathcal{F} be a subset of A , $a, b, x \in A$, $\varepsilon > 0$. If $\|a - b\| < \varepsilon$; then we write $a \approx_{\varepsilon} b$. If there exists an element $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$, then we write $x \in_{\varepsilon} \mathcal{F}$.

Definition 2.1 (see [19, Definition 3.6.2], [5, Definition 1.4.], and [6, Definition 2.1]). Let A be a simple unital C^* -algebra and $k \in \mathbb{N}$. A is said to have tracial rank no more than k ; write $\text{TR}(A) \leq k$; (tracial real rank zero, write $\text{TRR}(A) = 0$; tracial stable rank one, write $\text{Tsr}(A) = 1$), if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$ and any nonzero positive element $b \in A$, there exist a nonzero projection $p \in A$ and a C^* -subalgebra $B \subseteq A$ with $1_B = p$ and $B \in \mathcal{D}^{(k)}$ ($\text{RR}(B) = 0$; $\text{tsr}(B) = 1$, resp.) such that

- (1) $\|pa - ap\| < \varepsilon$ for any $a \in \mathcal{F}$,
- (2) $pap \in_{\varepsilon} B$ for all $a \in \mathcal{F}$,
- (3) $[1 - p] \leq [b]$.

If, furthermore, $\text{TR}(A)k - 1$, then we say $\text{TR}(A) = k$.

Lemma 2.2 (see [5, Theorem 3.3], [6, Theorem 3.3]). *Let A be a simple unital C^* -algebra. If $\text{TRR}(A) = 0$, then $\text{RR}(A) = 0$. If $\text{Tsr}(A) = 1$ and has the (SP)-property, then $\text{tsr}(A) = 1$.*

Definition 2.3 (see [13, Definition 1.2]). Let A be an infinite dimensional finite simple separable unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the tracial Rokhlin property if, for every $\varepsilon > 0$, every finite set $\mathcal{F} \subseteq A$, every positive element $b \in A$, there are mutually orthogonal projections $\{e_g : g \in G\}$ such that

- (1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in \mathcal{F}$,
- (3) with $e = \sum_{g \in G} e_g$, $[1 - e] \leq [b]$.

Lemma 2.4 (see [13, Corollary 1.6]). *Let A be an infinite dimensional finite simple separable unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then $A \rtimes_{\alpha} G$ is simple.*

Lemma 2.5 (see [20, Theorem 4.2]). *Let A be a simple unital C^* -algebra with the (SP)-property, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a discrete group G on A . Suppose that the normal subgroup $N = \{g \in G \mid \alpha_g \text{ is inner on } A\}$ of G is finite; then any nonzero hereditary C^* -subalgebra of the crossed product $A \rtimes_{\alpha} G$ has a nonzero projection which is Murray-von Neumann equivalent to a projection in $A \rtimes_{\alpha} N$.*

If the action $\alpha : G \rightarrow \text{Aut}(A)$ has the tracial Rokhlin property, then each α_g is outer for all $g \in G \setminus \{1\}$. So $N = \{g \in G \mid \alpha_g \text{ is inner on } A\} = \{1\}$. Since $A \rtimes_{\alpha} N = A \rtimes_{\alpha} \{1\} \cong A$, by Lemma 2.5 we have the following lemma.

Lemma 2.6. *Let A be an infinite dimensional finite simple separable unital C^* -algebra with the (SP)-property, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property; then any nonzero hereditary C^* -subalgebra of the crossed product $A \rtimes_{\alpha} G$ has a nonzero projection which is Murray-von Neumann equivalent to a projection in A .*

Lemma 2.7 (see [19, Lemma 3.5.6]). *Let A be a simple C^* -algebra with the (SP)-property, and let $p, q \in A$ be two nonzero projections. Then there are nonzero projections $p_1 \leq p$, $q_1 \leq q$ such that $[p_1] = [q_1]$.*

Definition 2.8 (see [14, Definition 1.1]). Let \mathcal{C} be a class of separable unital C^* -algebras. We say that \mathcal{C} is finitely saturated if the following closure conditions hold.

- (1) If $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.
- (2) If $A_i \in \mathcal{C}$ for $i = 1, 2, \dots, n$, then $\bigoplus_{k=1}^n A_k \in \mathcal{C}$.
- (3) If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.
- (4) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Moreover, the finite saturation of a class \mathcal{C} is the smallest finitely saturated class which contains \mathcal{C} .

Definition 2.9 (see [14, Definition 1.2]). Let \mathcal{C} be a class of separable unital C^* -algebras. We say that \mathcal{C} is flexible if.

- (1) for every $A \in \mathcal{C}$, every $n \in \mathbb{N}$, and every nonzero projection $p \in M_n(A)$, the corner $pM_n(A)p$ is semiprojective and finitely generated;
- (2) for every $A \in \mathcal{C}$ and every ideal $I \subseteq A$, there is an increasing sequence $I_1 \subseteq I_2 \subseteq \dots$ of ideals of A such that $\bigcup_{n=1}^{\infty} I_n = I$ and such that for every n the C^* -algebra A/I_n is in the finite saturation of \mathcal{C} .

Example 2.10. (1) Let $\mathcal{C} = \{\bigoplus_{i=1}^n M_{k(i)} \mid n, k(i) \in \mathbb{N}\}$; that is, \mathcal{C} contains all finite dimensional algebras. \mathcal{C} is finitely saturated and flexible.

(2) Let $\mathcal{C} = \{\bigoplus_{i=1}^n C(X_i, M_{k(i)}) \mid n, k(i) \in \mathbb{N}; \text{ each } X_i \text{ is a closed subset of the circle}\}$. We can show that \mathcal{C} is finitely saturated and flexible.

(3) Let $\mathcal{C} = \{f \in \bigoplus_{i=1}^n C([0, 1], M_{k(i)}) \mid n, k(i) \in \mathbb{N}, f(0) \text{ is scalar}\}$. We can also show that \mathcal{C} is finitely saturated and flexible.

(4) For some $d \in \mathbb{N}$, let \mathcal{C}_d contain all the C^* -algebras $\bigoplus_{i=1}^n p_i C(X_i, M_{k(i)}) p_i$, where $n, k(i) \in \mathbb{N}$, each p_i is a nonzero projection in $C(X_i, M_{k(i)})$, and each X_i is a compact metric space with covering dimension at most d . The class \mathcal{C}_d is not flexible for $d \neq 0$ (see [14] Example 2.9).

Definition 2.11 (see [16, Definition 1.4]). Let \mathcal{C} be a class of separable unital C^* -algebras. A unital approximate \mathcal{C} -algebra is a C^* -algebra which is isomorphic to an inductive limit $\lim_{n \rightarrow \infty} (A_n, \phi_n)$, where each A_n is in the finite saturation of \mathcal{C} and each homomorphism $\phi_n : A_n \rightarrow A_{n+1}$ is unital.

Definition 2.12 (see [14, Definition 1.5]). Let \mathcal{C} be a class of separable unital C^* -algebras. Let A be a separable unital C^* -algebra. We say that A is a unital local \mathcal{C} -algebra if, for every $\varepsilon > 0$ and every finite subset $\mathcal{F} \subset A$, there is a C^* -algebra B in the finite saturation of \mathcal{C} and a $*$ -homomorphism $\phi : B \rightarrow A$ such that $a \in_{\varepsilon} \phi(B)$ for all $a \in \mathcal{F}$.

By [14] Proposition 1.6, if \mathcal{C} is a finitely saturated flexible class of separable unital C^* -algebras, then every unital local \mathcal{C} -algebra is a unital approximate \mathcal{C} -algebra. The converse is clear.

Let \mathcal{C} be a class as (1) of Example 2.10. Then a unital AF-algebra is a unital approximate \mathcal{C} -algebra and is a unital local \mathcal{C} -algebra.

Let \mathcal{C} be a class as (2) of Example 2.10. Then a unital AT-algebra is a unital approximate \mathcal{C} -algebra and is a unital local \mathcal{C} -algebra.

Definition 2.13. Let A be a simple unital C^* -algebra, and let \mathcal{C} be a class of separable unital C^* -algebra. We say that A is a tracial \mathcal{C} -algebra if, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any nonzero positive element $b \in A$, there exist a nonzero projection $p \in A$, a C^* -algebra B in the finite saturation of \mathcal{C} , and a $*$ -homomorphism $\phi : B \rightarrow A$ with $1_{\phi(B)} = p$, such that

- (1) $\|pa - ap\| < \varepsilon$ for any $a \in \mathcal{F}$,
- (2) $pap \in_{\varepsilon} \phi(B)$ for all $a \in \mathcal{F}$,
- (3) $[1 - p] \leq [b]$ in A .

Using the similar proof of Lemma 3.6.5 of [19] about the tracial rank of unital hereditary C^* -subalgebras of a simple unital C^* -algebra, we get the following one.

Lemma 2.14. *Let \mathcal{C} be any finitely saturated class of separable unital C^* -algebras. Let p be a projection in a simple unital C^* -algebra A with the (SP)-property. If A is a tracial \mathcal{C} -algebra, so also is pAp .*

For $n \in \mathbb{N}$, $\delta > 0$, a unital C^* -algebra A , if $w_{i,j}$, for $1 \leq i, j \leq n$, are elements of A , such that $\|w_{i,j}\| \leq 1$ for $1 \leq i, j \leq n$, such that $\|w_{i,j}^* - w_{j,i}\| < \delta$ for $1 \leq i, j \leq n$, such that $\|w_{i_1,j_1} w_{i_2,j_2} - \delta_{j_1,i_2} w_{i_1,j_2}\| < \delta$ for $1 \leq i_1, i_2, j_1, j_2 \leq n$, and such that $w_{i,i}$ are mutually orthogonal projections, we say that $w_{i,j}$ ($1 \leq i, j \leq n$) form a δ -approximate system of $n \times n$ matrix units in A .

By perturbation of projections (see Theorem 2.5.9 of [19]), we have Lemma 2.15.

Lemma 2.15. *For any $n \in \mathbb{N}$, any $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon) > 0$ such that, whenever $(f_{i,j})_{1 \leq i, j \leq n}$ is a system of matrix units for M_n , whenever B is a unital C^* -algebra, and whenever $w_{i,j}$, for $1 \leq i, j \leq n$, are elements of B which form a δ -approximate system of $n \times n$ matrix units, then there exists a $*$ -homomorphism $\phi : M_n \rightarrow B$ such that $\phi(f_{i,i}) = w_{i,i}$ for $1 \leq i \leq n$ and $\|\phi(f_{i,j}) - w_{i,j}\| < \varepsilon$ for $1 \leq i, j \leq n$.*

3. Main Results

Theorem 3.1. *Let \mathcal{C} be any class of separable unital C^* -algebras. Let A be an infinite dimensional finite simple unital tracial \mathcal{C} -algebra with the (SP)-property, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then $A \rtimes_\alpha G$ is a simple unital tracial C^* -algebra.*

Proof. By Lemma 2.4, $A \rtimes_\alpha G$ is a simple unital C^* -algebra. By Definition 2.13, it suffices to show the following.

For any $\varepsilon > 0$, any finite subset $\mathcal{F} = \mathcal{F}_0 \cup \{u_g \mid g \in G\} \subset A \rtimes_\alpha G$, where \mathcal{F}_0 is a finite subset of the unit ball of A and $u_g \in A \rtimes_\alpha G$ is the canonical unitary implementing the automorphism α_g , and any nonzero positive element $b \in A \rtimes_\alpha G$, there exist a nonzero projection $p \in A \rtimes_\alpha G$, a C^* -algebra B in the finite saturation of \mathcal{C} , and a $*$ -homomorphism $\phi : B \rightarrow A \rtimes_\alpha G$ with $1_{\phi(B)} = p$, such that

- (1) $\|pa - ap\| < \varepsilon$ for any $a \in \mathcal{F}$,
- (2) $pap \in_\varepsilon \phi(B)$ for all $a \in \mathcal{F}$,
- (3) $[1 - p] \leq [b]$ in $A \rtimes_\alpha G$.

By Lemma 2.6, there exists a nonzero projection $q \in A$ such that $[q] \leq [b]$ in $A \rtimes_\alpha G$.

Since A is an infinite dimensional simple unital C^* -algebra with the (SP)-property, by [19, Lemma 3.5.7], there exist orthogonal nonzero projections $q_1, q_2 \in A$ such that $q_1 + q_2 \leq q$.

Set $n = \text{card}(G)$ and set $\varepsilon_0 = \varepsilon/48n$. Choose $\delta > 0$ according to Lemma 2.15 for n given above and ε_0 in place of ε . Moreover we may require $\delta < \min\{\varepsilon/72n, \varepsilon/(24n(n-1))\}$.

Apply Definition 2.3 with \mathcal{F}_0 given above, with δ in place of ε , with q_1 in place of b . There exist mutually orthogonal projections $e_g \in A$ for $g \in G$ such that

- (1') $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$,
- (2') $\|e_g a - a e_g\| < \delta$ for all $g \in G$ and all $a \in \mathcal{F}_0$,
- (3') $[1 - e] \leq [q_1]$ in A , where $e = \sum_{g \in G} e_g$.

By Lemma 2.7, there are nonzero projections $v_1, v_2 \in A$ such that $v_1 \leq e_1$, $v_2 \leq q_2$ and $[v_1] = [v_2]$.

Define $w_{g,h} = u_{gh^{-1}} e_h$ for $g, h \in G$. By the proof of Theorem 2.2 of [14], we can estimate that $w_{g,h}$ ($g, h \in G$) form a δ -approximate system of $n \times n$ matrix units in $A \rtimes_\alpha G$. Moreover, $\sum_{g \in G} w_{g,g} = \sum_{g \in G} e_g = e$.

Let $(f_{g,h})_{g,h \in G}$ be a system of matrix units for M_n . By Lemma 2.15, there exists a $*$ -homomorphism $\phi_0 : M_n \rightarrow A \times_\alpha G$ such that

$$\|\phi_0(f_{g,h}) - w_{g,h}\| < \varepsilon_0 \quad (3.1)$$

for all $g, h \in G$, and $\phi_0(f_{g,g}) = e_g$ for all $g \in G$.

Set $E = M_n \otimes e_1 A e_1$. Define an injective unital $*$ -homomorphism $\phi_1 : E \rightarrow e(A \times_\alpha G)e$ by

$$\phi_1(f_{g,h} \otimes a) = \phi_0(f_{g,1}) a \phi_0(f_{1,h}) \quad (3.2)$$

for all $g, h \in G$ and $a \in e_1 A e_1$. Then

$$\phi_1(1_{M_n} \otimes e_1) = \sum_{g \in G} e_g = e, \quad \phi_1(f_{1,1} \otimes a) = a \quad (3.3)$$

for all $a \in e_1 A e_1$ and

$$\phi_1(f_{g,h} \otimes e_1) = \phi_0(f_{g,1}) e_1 \phi_0(f_{1,h}) = \phi_0(f_{g,1}) \phi_0(f_{1,1}) \phi_0(f_{1,h}) = \phi_0(f_{g,h}) = e_g \phi_0(f_{g,h}) e_h. \quad (3.4)$$

By (2'), for all $a \in \mathcal{F}_0$, we have

$$\|ae - ea\| \leq \sum_{g \in G} \|ae_g - e_g a\| < n\delta. \quad (3.5)$$

By (1'), for all $g \in G$, we get

$$\|u_g e - e u_g\| \leq \|u_g e u_{g^{-1}} - e\| = \left\| \sum_{h \in G} \alpha_g(e_h) - \sum_{h \in G} e_{gh} \right\| \leq n\delta. \quad (3.6)$$

For all $g \in G$, we have

$$\begin{aligned} \left\| e u_g e - \sum_{h \in G} \phi_1(f_{gh,h} \otimes e_1) \right\| &\leq \|e u_g e - u_g e\| + \left\| u_g e - \sum_{h \in G} \phi_1(f_{gh,h} \otimes e_1) \right\| \\ &< n\delta + \left\| u_g e - \sum_{h \in G} \phi_1(f_{gh,h} \otimes e_1) \right\| \\ &= n\delta + \left\| \sum_{h \in G} u_g e_h - \sum_{h \in G} \phi_1(f_{gh,h} \otimes e_1) \right\| \\ &= n\delta + \left\| \sum_{h \in G} w_{gh,h} - \sum_{h \in G} \phi_0(f_{gh,h}) \right\| < n\delta + n\varepsilon_0 < \frac{5\varepsilon}{144}. \end{aligned} \quad (3.7)$$

That is, for all $g \in G$, we have

$$eu_g e \in_{5\varepsilon/144} \phi_1(E). \quad (3.8)$$

Set $b = \sum_{g \in G} f_{g,g} \otimes e_1 \alpha_{g^{-1}}(a) e_1$; then $b \in E$. Using $\|e_g a e_h - a e_g e_h\| < \delta$, we get

$$\left\| eae - \sum_{g \in G} e_g a e_g \right\| \leq \sum_{g \neq h} \|e_g a e_h\| < n(n-1)\delta. \quad (3.9)$$

We also have

$$\begin{aligned} \|\phi_0(f_{g,1})e_1 - u_g e_1\| &\leq \|\phi_0(f_{g,1}) - u_g e_1\| = \|\phi_0(f_{g,1}) - w_{g,1}\| < \varepsilon_0, \\ \|\phi_0(f_{1,g}) - e_1 u_{g^{-1}}\| &\leq \|\phi_0(f_{1,g}) - u_{g^{-1}} e_g\| + \|u_{g^{-1}} e_g - e_1 u_{g^{-1}}\| < \varepsilon_0 + \delta, \\ \|e_1 \alpha_{g^{-1}}(a) e_1 - \alpha_{g^{-1}}(e_g a e_g)\| &< 2\delta. \end{aligned} \quad (3.10)$$

Then, for all $a \in \mathcal{F}_0$, we have

$$\begin{aligned} \|eae - \phi_1(b)\| &= \left\| eae - \phi_1 \left(\sum_{g \in G} f_{g,g} \otimes e_1 \alpha_{g^{-1}}(a) e_1 \right) \right\| \\ &= \left\| eae - \sum_{g \in G} \phi_0(f_{g,1}) e_1 \alpha_{g^{-1}}(a) e_1 \phi_0(f_{1,g}) \right\| \\ &< \left\| eae - \sum_{g \in G} u_g e_1 \alpha_{g^{-1}}(a) e_1 u_{g^{-1}} \right\| + 2n\varepsilon_0 + n\delta \\ &< \left\| eae - \sum_{g \in G} u_g \alpha_{g^{-1}}(e_g a e_g) u_{g^{-1}} \right\| + 3n\delta + 2n\varepsilon_0 \\ &= \left\| eae - \sum_{g \in G} e_g a e_g \right\| + 3n\delta + 2n\varepsilon_0 \\ &< n(n-1)\delta + 3n\delta + 2n\varepsilon_0 < \frac{\varepsilon}{24} + \frac{\varepsilon}{24} + \frac{\varepsilon}{24} = \frac{\varepsilon}{8}. \end{aligned} \quad (3.11)$$

That is, for all $a \in \mathcal{F}_0$,

$$eae \in_{\varepsilon/8} \phi_1(E). \quad (3.12)$$

By (3.8) and (3.12), we can write

$$eae \in_{\varepsilon/8} \phi_1(E) \quad (3.13)$$

for all $a \in \mathcal{F}$.

Write

$$\tilde{\mathcal{F}} = \left\{ b \mid b \in E, \|\phi_1(b) - eae\| < \frac{\varepsilon}{8} \text{ for } a \in \mathcal{F} \right\}. \quad (3.14)$$

By Lemma 2.14, E is a simple unital tracial \mathcal{C} -algebra. Apply Definition 2.13 with $\tilde{\mathcal{F}}$ given above, with $\varepsilon/8$ in place of ε and $f_{1,1} \otimes v_1$ in place of a . There exist a nonzero projection $p_0 \in E$, a C^* -algebra B in the finite saturation of \mathcal{C} , and a $*$ -homomorphism $\psi_0 : B \rightarrow E$ with $1_{\psi_0(B)} = p_0$, such that

$$(1'') \quad \|p_0b - bp_0\| < \varepsilon/8 \text{ for any } b \in \tilde{\mathcal{F}},$$

$$(2'') \quad p_0bp_0 \in_{\varepsilon/8} \psi_0(B) \text{ for all } b \in \tilde{\mathcal{F}},$$

$$(3'') \quad [1_E - p_0] \leq [f_{1,1} \otimes v_1] \text{ in } E.$$

Set $p = \phi_1(p_0)$ and $\phi = \phi_1 \circ \psi_0 : B \rightarrow e(A \times_\alpha G)e$.

For every $a \in \mathcal{F}$, there exists $b \in \tilde{\mathcal{F}}$ such that $\phi_1(b) \approx_{\varepsilon/8} eae$. Then

$$pa = pea \approx_{n\delta} peae \approx_{\varepsilon/8} p\phi_1(b) = \phi_1(p_0b) \approx_{\varepsilon/8} \phi_1(bp_0) \approx_{n\delta+\varepsilon/4} ap. \quad (3.15)$$

That is,

$$\|pa - ap\| < 2n\delta + \frac{\varepsilon}{2} < \varepsilon. \quad (3.16)$$

Let $c \in B$ such that $p_0bp_0 \approx_{\varepsilon/8} \psi_0(c)$. Then

$$pap = peap \approx_{\varepsilon/8} p\phi_1(b)p = \phi_1(p_0bp_0) \approx_{\varepsilon/8} \phi_1(\psi_0(c)) = \phi(c). \quad (3.17)$$

Hence,

$$pap \in_\varepsilon \phi(B). \quad (3.18)$$

By (3''), in $A \times_\alpha G$, $[e - p] = [\phi_1(1_E - p_0)] \leq [\phi_1(f_{1,1} \otimes v_1)] = [v_1] = [v_2]$. Therefore,

$$[1 - p] = [1 - e] + [e - p] \leq [q_1] + [v_2] \leq [q_1] + [q_2] \leq [q] \leq [b]. \quad (3.19)$$

From (3.16), (3.18), and (3.19), $A \times_\alpha G$ is a simple unital tracial \mathcal{C} -algebra. \square

Corollary 3.2. *Let A be an infinite dimensional separable simple unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. If A is an AF-algebra, then the induced crossed product $A \times_\alpha G$ has tracial rank zero. If A is an $A\mathbb{T}$ -algebra with the (SP)-property, then the induced crossed product $A \times_\alpha G$ has tracial rank no more than one.*

Proof. If A is an AF-algebra, then A is a unital local \mathcal{C} -algebra, where \mathcal{C} is a class of C^* -algebras satisfying condition (1) of Example 2.10. By Theorem 3.1, we know that $A \times_\alpha G$ is a simple unital tracial \mathcal{C} -algebra. By the definition of tracial rank zero, $\text{TR}(A \times_\alpha G) = 0$.

If A is an $A\mathbb{T}$ -algebra, then A is a unital local \mathcal{C} -algebra, where \mathcal{C} is a class of C^* -algebra satisfying condition (2) of Example 2.10. By Theorem 3.1, we know that $A \times_\alpha G$ is a simple unital tracial \mathcal{C} -algebra. Since the covering dimension of closed subsets of the circle is no more than one, by the definition of tracial rank, $\text{TR}(A \times_\alpha G) \leq 1$. \square

It should be noted that the AF-part was proved by Phillips in [13] Theorem 2.6.

Corollary 3.3. *Let A be an infinite dimensional finite separable simple unital C^* -algebra with the (SP)-property, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. If A has stable rank one, then the induced crossed product $A \times_\alpha G$ has stable rank one. If A has real rank zero, then the induced crossed product $A \times_\alpha G$ has real rank zero.*

Proof. Let \mathcal{C} be the class of all separable unital C^* -algebras with stable rank one. By Theorems 3.1.2, 3.18, and 3.19 in [19], we have that \mathcal{C} is finitely saturated and satisfies condition (2) of Definition 2.9. By Theorem 3.1, the crossed product $A \times_\alpha G$ is a simple unital tracial \mathcal{C} -algebra, that is, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A \times_\alpha G$, and any nonzero positive element $b \in A \times_\alpha G$, there exist a nonzero projection $p \in A \times_\alpha G$, a C^* -algebra B in \mathcal{C} , and a $*$ -homomorphism $\phi : B \rightarrow A \times_\alpha G$ with $1_{\phi(B)} = p$, such that

- (1) $\|pa - ap\| < \varepsilon$ for any $a \in \mathcal{F}$,
- (2) $pap \in_\varepsilon \phi(B)$ for all $a \in \mathcal{F}$,
- (3) $[1 - p] \leq [b]$ in $A \times_\alpha G$.

Hence, $\text{Tsr}(A \times_\alpha G) = 1$. By Lemma 2.2, $\text{tsr}(A \times_\alpha G) = 1$.

Let \mathcal{C} be the class of all separable unital C^* -algebras with real rank zero. We can use the same argument to show that the crossed product $A \times_\alpha G$ is a simple unital tracial \mathcal{C} -algebra. Hence $\text{TRR}(A \times_\alpha G) = 0$. By Lemma 2.2, $\text{RR}(A \times_\alpha G) = 0$. \square

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