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Research Article

Positive Solutions for Nonlinear First-Order *m*-Point Boundary Value Problem on Time Scales

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By means of fixed-point theorems, we investigate the existence of positive solutions for nonlinear first-order m-point boundary value problem $x^{\Delta}(t) + a(t)x(\sigma(t)) = f(t, x(\sigma(t)))$, $t \in [t_1, t_m] \subset \mathbb{T}$, $x(t_1) = \sum_{k=2}^{m-1} \alpha_k x(t_k) + \alpha_1 x(\sigma(t_m))$, where $\mathbb{T}$ is a time scale, $0 \leq t_1 < t_2 < \cdots < t_{m-1} < t_m$, $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \geq 0$ are given constants.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 (see [1]). The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, and social sciences; see the monographs of Aulbach and Hilger [2], Bohner and Peterson [3, 4], and Lakshmikantham et al. [5] and the references therein.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus. The closed interval in $\mathbb{T}$ is defined as

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}, \tag{1.1}$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

In this study, we consider the nonlinear first-order m-point boundary value problem

$$x^{\Delta}(t) + a(t)x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [t_1, t_m] \subset \mathbb{T},$$
$$x(t_1) = \sum_{k=2}^{m-1} \alpha_k x(t_k) + \alpha_1 x(\sigma(t_m)), \tag{1.2}$$

where $\mathbb{T}$ is a time scale, $0 \leq t_1 < t_2 < \cdots < t_{m-1} < t_m$, $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \geq 0$ are given constants. a is regressive and rd-continuous, and $f : [t_1, \sigma(t_m)] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

In [6], Cabada studied the following first-order periodic boundary value problem on time scales:

$$u^{\Delta}(t) = f(t, u(t)), \quad t \in [a, b] \subset \mathbb{T},$$
$$u(a) = u(\sigma(b)). \tag{1.3}$$

He developed the monotone method in the presence of lower and upper solutions to obtain the existence of extremal solutions. When $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_{m-1} = 0$, and $a(t) \equiv 0$, BVP (1.2) is reduced to (1.3).

In [7], Sun studied the first-order boundary value problem

$$x^{\Delta}(t) = f(x(\sigma(t))), \quad t \in [0, T] \subset \mathbb{T},$$
$$x(0) = \beta x(\sigma(T)), \tag{1.4}$$

where $0 < \beta < 1$. Some existence results for at least two positive solutions were established, by using Avery-Henderson fixed-point theorem. When $\alpha_2 = \cdots = \alpha_{m-1} = 0$ and $a(t) \equiv 0$, BVP (1.2) is reduced to (1.5).

In [8], Shu and Chunhua are concerned with the existence of three positive solutions for the following nonlinear first-order boundary value problem on time scale:

$$x^{\Delta}(t) = f(x(\sigma(t))), \quad t \in [0, T] \subset \mathbb{T},$$
$$x(0) = \eta x(\sigma(T)), \tag{1.5}$$

where $T > 0$ is fixed, $0, T \in \mathbb{T}$, and $f : [0, \infty) \to [0, \infty)$ is continuous. When $\alpha_2 = \cdots = \alpha_{m-1} = 0$ and $a(t) \equiv 0$, BVP (1.2) is reduced to (1.5).

Sun and Li [9] studied the following first-order periodic boundary value problem on time scales:

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = g(t, x(\sigma(t))), \quad t \in [0, T] \subset \mathbb{T},$$
$$x(0) = x(\sigma(T)). \tag{1.6}$$

Conditions for the existence of at least one solution were obtained by using novel inequalities and the Schaefer fixed-point theorem. When $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_{m-1} = 0$, BVP (1.2) is reduced to (1.6).

In [10], Tian and Ge studied the existence and uniqueness results for first-order three-point boundary value problem

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [0, T] \subset \mathbb{T},$$
$$x(0) - \alpha x(\xi) = \beta x(\sigma(T)), \tag{1.7}$$

by using several well-known fixed-point theorems. When $\alpha_3 = \cdots = \alpha_{m-1} = 0$, BVP (1.2) is reduced to (1.7).

Motivated by [6–10], we establish some new and more general results for the existence of positive solutions for the problem (1.2) by applying fixed-point theorems in cones.

We have arranged the paper as follows. In Section 2, we give some lemmas which are needed later. In Section 3, we apply the Krasnosel'skii fixed-point theorem, Avery-Henderson fixed-point theorem, and Leggett-Williams fixed-point theorem to prove the existence of at least one, two, and three positive solutions to BVP (1.2). In Section 4, as an application, the examples are included to illustrate our results.

2. Preliminaries

Let $\mathcal{B}$ denote the Banach space $C[t_1, \sigma(t_m)]$ with the norm $\|x\| = \sup_{t\in[t_1,\sigma(t_m)]}|x(t)|$. For $h \in \mathcal{B}$, we consider the following linear boundary value problem:

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = h(t), \quad t \in [t_1, t_m] \subset \mathbb{T},$$
$$x(t_1) = \sum_{k=2}^{m-1}\alpha_k x(t_k) + \alpha_1 x(\sigma(t_m)). \tag{2.1}$$

Lemma 2.1. *For $h \in \mathcal{B}$, BVP* (2.1) *has the unique solution*

$$x(t) = \frac{1}{e_a(t,t_1)}\left\{\Gamma\left[\frac{\alpha_1\int_{t_1}^{\sigma(t_m)} e_a(s,t_1)h(s)\Delta s}{e_a(\sigma(t_m),t_1)} + \sum_{k=2}^{m-1}\frac{\alpha_k\int_{t_1}^{t_k} e_a(s,t_1)h(s)\Delta s}{e_a(t_k,t_1)}\right] + \int_{t_1}^{t} e_a(s,t_1)h(s)\Delta s\right\}, \quad t \in [t_1, \sigma(t_m)], \tag{2.2}$$

where $\Gamma = [1 - \sum_{k=2}^{m-1}(\alpha_k/e_a(t_k,t_1)) - (\alpha_1/e_a(\sigma(t_m),t_1))]^{-1}$.

Proof. From $x^{\Delta}(t) + a(t)x(\sigma(t)) = h(t)$, we have

$$x(t) = \frac{1}{e_a(t,t_1)}\left[x(t_1) + \int_{t_1}^{t} e_a(s,t_1)h(s)\Delta s\right]. \tag{2.3}$$

By using the boundary condition, we get

$$\left[1-\frac{\alpha_1}{e_a(\sigma(t_m),t_1)}-\sum_{k=2}^{m-1}\frac{\alpha_k}{e_a(t_k,t_1)}\right]x(t_1)=\frac{\alpha_1\int_{t_1}^{\sigma(t_m)}e_a(s,t_1)h(s)\Delta s}{e_a(\sigma(t_m),t_1)}+\sum_{k=2}^{m-1}\frac{\alpha_k\int_{t_1}^{t_k}e_a(s,t_1)h(s)\Delta s}{e_a(t_k,t_1)}. \tag{2.4}$$

Thus, x satisfies (2.2). □

Let $G(t,s)$ be Green's function for the boundary value problem

$$\begin{gathered}x^{\Delta}(t)+a(t)x(\sigma(t))=h(t),\quad t\in[t_1,t_m]\subset\mathbb{T},\\ x(t_1)=\sum_{k=2}^{m-1}\alpha_k x(t_k)+\alpha_1 x(\sigma(t_m)).\end{gathered} \tag{2.5}$$

By Lemma 2.1, we obtain

$$G(t,s)=\begin{cases}G_1(t,s), & t_1\le s\le\sigma(s)\le t_2,\\ G_2(t,s), & t_2\le s\le\sigma(s)\le t_3,\\ \vdots & \\ G_{m-2}(t,s), & t_{m-2}\le s\le\sigma(s)\le t_{m-1},\\ G_{m-1}(t,s), & t_{m-1}\le s\le t_m,\end{cases} \tag{2.6}$$

where

$$G_j(t,s)=\begin{cases}\dfrac{e_a(s,t_1)}{e_a(t,t_1)}\left\{\Gamma\left[\dfrac{\alpha_1}{e_a(\sigma(t_m),t_1)}+\displaystyle\sum_{k=j+1}^{m-1}\dfrac{\alpha_k}{e_a(t_k,t_1)}\right]+1\right\}, & \sigma(s)\le t,\\ \dfrac{\Gamma e_a(s,t_1)}{e_a(t,t_1)}\left[\dfrac{\alpha_1}{e_a(\sigma(t_m),t_1)}+\displaystyle\sum_{k=j+1}^{m-1}\dfrac{\alpha_k}{e_a(t_k,t_1)}\right], & t\le s,\end{cases} \tag{2.7}$$

for all $j=1,2,\dots,m-1$.

Lemma 2.2. *Green's function $G(t,s)$ in* (2.6) *has the following properties:*

(i) $G(t,s)\ge 0$ *for* $(t,s)\in[t_1,\sigma(t_m)]\times[t_1,t_m]$.

(ii) $m\le G(t,s)\le M$, *where* $m=\Gamma\alpha_1/(e_a(\sigma(t_m),t_1))^2$ *and* $M=\Gamma\sum_{k=1}^{m-1}\alpha_k+e_a(t_{m-1},t_1)$.

(iii) $G(t,s)\ge(m/M)\sup_{(t,s)\in[t_1,\sigma(t_m)]\times[t_1,t_m]}G(t,s)$ *for* $(t,s)\in[t_1,\sigma(t_m)]\times[t_1,t_m]$.

Let $\mathcal{B}$ denote the Banach space $C[t_1,\sigma(t_m)]$ with the norm $\|x\|=\max_{t\in[t_1,\sigma(t_m)]}|x(t)|$. Define the cone $P\subset\mathcal{B}$ by

$$P=\left\{x\in\mathcal{B}:x(t)\ge 0,x(t)\ge\frac{m}{M}\|x\|\ \text{ on }[t_1,\sigma(t_m)]\right\}. \tag{2.8}$$

Equation (1.2) is equivalent to the nonlinear integral equation

$$x(t) = \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s. \tag{2.9}$$

We can define the operator $A : P \to \mathcal{B}$ by

$$Ax(t) = \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s. \tag{2.10}$$

Therefore solving (2.9) in P is equivalent to finding fixed-points of the operator A.

From Lemma 2.2, $Ax(t) \geq 0$ for $t \in [t_1, \sigma(t_m)]$. In addition, by using Lemma 2.2 we get

$$\begin{aligned} Ax(t) &= \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\geq \frac{m}{M} \sup_{(t,s)\in[t_1,\sigma(t_m)]\times[t_1,t_m]} G(t,s) \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &\geq \frac{m}{M} \sup_{t\in[t_1,\sigma(t_m)]} \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &= \frac{m}{M}\|Ax\|. \end{aligned} \tag{2.11}$$

So, we have $A : P \to P$.

3. Main Results

To prove the existence of at least one positive solution for the BVP (1.2), we will need the following (Krasnosel'skii) fixed-point theorem.

Theorem 3.1 (Krasnosel'skii fixed-point theorem [11]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \longrightarrow K \tag{3.1}$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *hold. Then A has a fixed-point in* $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 3.2. *Let there exist numbers r, R satisfying $0 < r < R < \infty$ such that for $t \in [t_1, \sigma(t_m)]$*

$$f(t,x) < \frac{x}{M\sigma(t_m)} \quad \text{for } x \in [0,r], \qquad f(t,x) \geq \frac{Mx}{m^2\sigma(t_m)} \quad \text{for } x \in [R,\infty). \tag{3.2}$$

Then BVP (1.2) *has at least one positive solution x satisfying $r \leq x(t) \leq RM/m$, $t \in [t_1, \sigma(t_m)]$.*

Proof. It is easy to check by the Arzela-Ascoli theorem that the operator $A : P \to P$ is completely continuous. Let us now define two bounded open sets as follows:

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < r\}, \qquad \Omega_2 = \left\{x \in \mathcal{B} : \|x\| < \frac{RM}{m}\right\}. \tag{3.3}$$

Then $\overline{\Omega_1} \subset \Omega_2$. For $x \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned} Ax(t) &= \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\leq M \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &\leq M \frac{\int_{t_1}^{\sigma(t_m)} x(\sigma(s))\Delta s}{M\sigma(t_m)} \leq r = \|x\|. \end{aligned} \tag{3.4}$$

Hence $\|Ax\| \leq \|x\|$ for $x \in P \cap \partial\Omega_1$.

If $x \in P \cap \partial\Omega_2$, then $\|x\| = RM/m$ and $x(t) \geq (m/M)\|x\| = R$ for $t \in [t_1, \sigma(t_m)]$. We have

$$\begin{aligned} Ax(t) &= \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\geq m \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &\geq m \frac{M \int_{t_1}^{\sigma(t_m)} x(\sigma(s))\Delta s}{m^2\sigma(t_m)} \\ &\geq \frac{RM}{m} = \|x\|. \end{aligned} \tag{3.5}$$

Thus $\|Ax\| \geq \|x\|$ for $x \in P \cap \partial\Omega_2$. By the first part of Theorem 3.1, A has a fixed-point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. Therefore, the BVP (1.2) has at least one positive solution satisfying $r \leq x(t) \leq RM/m$, $t \in [t_1, \sigma(t_m)]$. □

Now, we will apply the following (Avery-Henderson) fixed-point theorem to prove the existence of at least two positive solutions to BVP (1.2).

Theorem 3.3 (see [12]). *Let P be a cone in a real Banach space E. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}. \tag{3.6}$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants r and M,

$$\phi(u) \le \theta(u) \le \eta(u), \quad \|u\| \le M\phi(u), \tag{3.7}$$

for all $u \in \overline{P(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \le \lambda\theta(u), \quad \forall\, 0 \le \lambda \le 1,\ u \in \partial P(\theta, q). \tag{3.8}$$

If $A : \overline{P(\phi, r)} \to P$ is a completely continuous operator satisfying

(i) *$\phi(Au) > r$ for all $u \in \partial P(\phi, r)$,*

(ii) *$\theta(Au) < q$ for all $u \in \partial P(\theta, q)$,*

(iii) *$P(\eta, p) \ne \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta, p)$,*

then A has at least two fixed-points u_1 and u_2 such that

$$p < \eta(u_1) \quad \text{with } \theta(u_1) < q, \qquad q < \theta(u_2) \quad \text{with } \phi(u_2) < r. \tag{3.9}$$

Theorem 3.4. *Suppose there exist numbers p, q, and r satisfying $0 < p < q < r$ such that the function f satisfies the following conditions:*

(i) *$f(t, x) > r/m$ for $t \in [t_{m-1}, \sigma(t_m)]$ and $x \in [r, rM/m(\sigma(t_m) - t_{m-1})]$;*

(ii) *$f(t, x) < q/M\sigma(t_m)$ for $t \in [t_1, \sigma(t_m)]$ and $x \in [0, qM/m]$;*

(iii) *$f(t, x) > p/m(\sigma(t_m) - t_{m-1})$ for $t \in [t_{m-1}, \sigma(t_m)]$ and $x \in [pm/M, p]$.*

Then the BVP (1.2) *has at least two positive solutions x_1 and x_2 such that*

$$\begin{gathered} p < \sup_{t \in [t_1, \sigma(t_m)]} x_1(t) \quad \text{with} \quad \sup_{t \in [t_{m-1}, \sigma(t_m)]} x_1(t) < q, \\ q < \sup_{t \in [t_{m-1}, \sigma(t_m)]} x_2(t) \quad \text{with} \quad \inf_{t \in [t_{m-1}, \sigma(t_m)]} x_2(t) < r. \end{gathered} \tag{3.10}$$

Proof . Let the nonnegative increasing continuous functionals ϕ, θ, and η be defined on the cone P by

$$\phi(x) = \inf_{t \in [t_{m-1}, \sigma(t_m)]} x(t), \qquad \theta(x) = \sup_{t \in [t_{m-1}, \sigma(t_m)]} x(t), \qquad \eta(x) = \sup_{t \in [t_1, \sigma(t_m)]} x(t). \tag{3.11}$$

For each $x \in P$, we have $\phi(x) \le \theta(x) \le \eta(x)$ and

$$\begin{aligned} x(t) &= \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\le \frac{M}{m} m \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &\le \frac{M}{m} \inf_{(t,s)\in[t_1,\sigma(t_m)]\times[t_1,t_m]} G(t,s) \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &\le \frac{M}{m} \inf_{t\in[t_1,\sigma(t_m)]} \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &= \frac{M}{m}\phi(x). \end{aligned} \tag{3.12}$$

Then $\|x\| \le (M/m)\phi(x)$. In addition, $\theta(0) = 0$ and for all $x \in P$, $\lambda \in [0,1]$ we obtain $\theta(\lambda x) = \lambda\theta(x)$.

Now we will verify the remaining conditions of Theorem 3.3.

Claim 1. If $x \in \partial P(\phi, r)$, then $\phi(Ax) > r$. Since $x \in \partial P(\phi, r)$, we have $r = \inf_{t\in[t_{m-1},\sigma(t_m)]} x(t) \le \|x\| \le rM/m$ for $t \in [t_{m-1}, \sigma(t_m)]$. Then, we get

$$\begin{aligned} \phi(Ax) &= \int_{t_1}^{\sigma(t_m)} \min_{t\in[t_{m-1},\sigma(t_m)]} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\ge m \int_{t_{m-1}}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &> r \end{aligned} \tag{3.13}$$

by hypothesis (i).

Claim 2. If $x \in \partial P(\theta, q)$, then $\theta(Ax) < q$. Since $x \in \partial P(\theta, q)$, $0 \le x(t) \le \|x\| \le (M/m)\phi(x) \le (M/m)\theta(x) = qM/m$ for $t \in [t_1, \sigma(t_m)]$. Thus, by hypothesis (ii) we have

$$\begin{aligned} \theta(Ax) &= \int_{t_1}^{\sigma(t_m)} \max_{t\in[t_{m-1},\sigma(t_m)]} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\le M \int_{t_1}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &< q. \end{aligned} \tag{3.14}$$

Claim 3. $P(\eta, p) \neq \emptyset$ and $\eta(Ax) > p$ for all $x \in \partial P(\eta, p)$. Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $x \in \partial P(\eta, p)$, we get $(m/M)p \leq \phi(x) \leq x(t) \leq \|x\| = p$ for $t \in [t_{m-1}, \sigma(t_m)]$. Hence, we obtain

$$\begin{aligned} \eta(Ax) &\geq \int_{t_1}^{\sigma(t_m)} G(t,s) f(s, x(\sigma(s)))\Delta s \\ &\geq m \int_{t_{m-1}}^{\sigma(t_m)} f(s, x(\sigma(s)))\Delta s \\ &> p \end{aligned} \tag{3.15}$$

by hypothesis (iii). This completes the proof. □

To prove the existence of at least three positive solutions for the BVP (1.2), we will apply the following (Leggett-Williams) fixed-point theorem.

Theorem 3.5 (see [13]). *Let P be a cone in the real Banach space E. Set*

$$\begin{aligned} P_r &:= \{x \in P : \|x\| < r\}, \\ P(\psi, a, b) &:= \{x \in P : a \leq \psi(x), \|x\| \leq b\}. \end{aligned} \tag{3.16}$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ is a completely continuous operator and ψ is a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \leq r$ such that the following condition hold:

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ *and* $\psi(Au) > q$ *for all* $u \in P(\psi, q, l)$;

(ii) $\|Au\| < p$ *for* $\|u\| \leq p$;

(iii) $\psi(Au) > q$ *for* $u \in P(\psi, q, r)$ *with* $\|Au\| > l$,

then A has at least three fixed-points u_1, u_2, and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p, \quad \psi(u_2) > q, \quad p < \|u_3\| \quad \text{with } \psi(u_3) < q. \tag{3.17}$$

Theorem 3.6. *Suppose that there exist numbers p, q, and r satisfying $0 < p < q < qM/m \leq r$ such that for $t \in [t_1, \sigma(t_m)]$ the function f satisfies the following conditions:*

(i) $f(t, x) \leq r/M\sigma(t_m)$, $x \in [0, r]$,

(ii) $f(t, x) > q/m\sigma(t_m)$, $x \in [q, qM/m]$,

(iii) $f(t, x) < p/M\sigma(t_m)$, $x \in [0, p]$.

Then (1.2) *has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\begin{aligned} &\sup_{t \in [t_1, \sigma(t_m)]} x_1(t) < p, \qquad q < \inf_{t \in [t_1, \sigma(t_m)]} x_2(t), \\ &p < \sup_{t \in [t_1, \sigma(t_m)]} x_3(t) \quad \text{with} \quad \inf_{t \in [t_1, \sigma(t_m)]} x_3(t) < q. \end{aligned} \tag{3.18}$$

Proof. Define the nonnegative continuous concave functional $\psi : P \to [0,\infty)$ to be $\psi(x) := \inf_{t\in[t_1,\sigma(t_m)]} x(t)$ and the cone P as in (2.8). For all $x \in P$, we have $\psi(x) \le \|x\|$. If $x \in \overline{P_r}$, then $0 \le x \le r$ and $f(t,x) \le r/M\sigma(t_m)$ from the hypothesis (i). Then we get

$$\begin{aligned}\|Ax\| &= \sup_{t\in[t_1,\sigma(t_m)]} \int_{t_1}^{\sigma(t_m)} G(t,s) f(s,x(\sigma(s)))\Delta s \\ &\le M \int_{t_1}^{\sigma(t_m)} f(s,x(\sigma(s)))\Delta s \\ &\le r\end{aligned} \tag{3.19}$$

by Lemma 2.2. This proves that $A : \overline{P_r} \to \overline{P_r}$. Similarly, by the hypothesis (iii), the condition (ii) of Theorem 3.5 is satisfied.

Since $qM/m \in P(\psi,q,qM/m)$ and $\psi(qM/m) > q$, $\{y \in P(\psi,q,qM/m) : \psi(x) > q\} \neq \emptyset$. For all $x \in P(\psi,q,qM/m)$, we have $q \le \inf_{t\in[t_1,\sigma(t_m)]} x(t) \le \|x\| \le qM/m$ for $t \in [t_1,\sigma(t_m)]$. Using the hypothesis (ii) and Lemma 2.2, we find

$$\begin{aligned}\psi(Ax) &= \int_{t_1}^{\sigma(t_m)} \inf_{t\in[t_1,\sigma(t_m)]} G(t,s) f(s,x(\sigma(s)))\Delta s \\ &\ge m \int_{t_1}^{\sigma(t_m)} f(s,x(\sigma(s)))\Delta s \\ &> q.\end{aligned} \tag{3.20}$$

Hence, the condition (i) of Theorem 3.5 holds.

For the condition (iii) of Theorem 3.5, we suppose that $x \in P(\psi,q,r)$ with $\|Ax\| > qM/m$. Then, from Lemma 2.2 we obtain

$$\psi(Ax) = \inf_{t\in[t_1,\sigma(t_m)]} Ax(t) \ge \frac{m}{M}\|Ax\| > q. \tag{3.21}$$

This completes the proof. □

4. Examples

Example 4.1. Let $\mathbb{T} = \mathbb{Z}$. We consider the first-order four-point BVP as follows:

$$\begin{gathered} x^{\Delta}(t) + x(\sigma(t)) = \frac{x+5}{x^4+1}, \quad t \in [0,5] \subset \mathbb{T}, \\ x(0) = x(1) + x(2) + x(6). \end{gathered} \tag{4.1}$$

Taking $a(t) \equiv 1$, $t_1 = 0$, $t_2 = 1$, $t_3 = 2$, $t_4 = 5$, and $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we have $\Gamma = 64/15$, $m = 1/960$, and $M = 384/5$. If we take $p = 0.001$, $q = 0.01$, and $r = 0.02$; then all the assumptions in Theorem 3.4 are satisfied. Finally, BVP (4.1) has at least two positive solutions x_1 and x_2 such that

$$0.001 < \sup_{t\in[0,6]} x_1(t) \quad \text{with} \ \sup_{t\in[2,6]} x_1(t) < 0.01,$$
$$0.01 < \sup_{t\in[2,6]} x_2(t) \quad \text{with} \ \inf_{t\in[2,6]} x_2(t) < 0.02. \tag{4.2}$$

Example 4.2. Let $\mathbb{T} = \mathbb{N}_0^2$. We consider the first-order four-point BVP as follows:

$$x^{\Delta}(t) + x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [0,9] \subset \mathbb{T},$$
$$x(0) = x(1) + x(4) + x(16), \tag{4.3}$$

where $a(t) \equiv 1,\ t_1 = 0,\ t_2 = 1,\ t_3 = 4,\ t_4 = 9,\ \alpha_1 = \alpha_2 = \alpha_3 = 1$, and

$$f(t,x) = \begin{cases} \dfrac{x}{400}, & (t,x) \in [0,16] \times [0,1], \\ \left(6870 - \dfrac{1}{400}\right)x + \dfrac{2}{400} - 6870, & (t,x) \in [0,16] \times [1,2], \\ \dfrac{15x}{882434} + 6870 - \dfrac{15}{441217}, & (t,x) \in [0,16] \times [2,\infty). \end{cases} \tag{4.4}$$

Hence, we obtain $\Gamma = 384/143,\ m = 1/54912$, and $M = 2296/143$. If we take $p = 1,\ q = 2$, and $r = 1764870$; then all the assumptions in Theorem 3.6 are satisfied. Finally, BVP (4.3) has at least three positive solutions $x_1,\ x_2$, and x_3 such that

$$\sup_{t\in[0,16]} x_1(t) < 1, \quad 2 < \inf_{t\in[0,16]} x_2(t),$$
$$1 < \sup_{t\in[0,16]} x_3(t) \quad \text{with} \ \inf_{t\in[0,16]} x_3(t) < 2. \tag{4.5}$$

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