Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 861438, 6 pages
doi:10.1155/2012/861438

Research Article

Solitary-Solution Formulation for Differential-Difference Equations Using an Ancient Chinese Algorithm

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Received 6 December 2011; Accepted 10 January 2012

Academic Editor: Allan C. Peterson

This paper applies an ancient Chinese algorithm to differential-difference equations, and a solitary-solution formulation is obtained. The discrete mKdV lattice equation is used as an example to elucidate the solution procedure.

1. Introduction

Discrete nonlinear lattices have been the focus of considerable attention in various branches of science. Many differential equations on the nanoscales are invalid, but the problems arising can be well modeled by differential-difference equations [1–5]. As is well known, there are many physically interesting problems such as charge fluctuations in net work, ladder type, electric circuits, phenomena in crystals, and molecular chains, which all can be modelled by differential-difference equations [6–9].

Recently many analytical methods were proposed to solve differential-difference equations, such as the exp-function method [10–12], the variational iteration method [13–15], the homotopy perturbation method [16–19], and the parameterized perturbation method [17]. A complete review on various analytical methods is available in [18, 19]. In this paper we will apply an ancient Chinese algorithm to the discussed problem.

2. Amplitude-Frequency Formulation for Nonlinear Oscillators

The ancient Chinese algorithm was first applied to nonlinear oscillators in 2006 [18], where a simple frequency formulation is given. Consider a generalized nonlinear oscillator in the form

$$u'' + f(u) = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{2.1}$$

We use two trial functions

$$u_1(t) = A\cos\omega_1 t, \tag{2.2}$$

$$u_2 = A\cos\omega_2 t, \tag{2.3}$$

which are, respectively, the solutions of the following linear oscillator equations:

$$\begin{aligned} u'' + \omega_1^2 u = 0, \\ u'' + \omega_2^2 u = 0, \end{aligned} \tag{2.4}$$

where ω_1 and ω_2 are trial frequencies which can be chosen freely; for example, we can set $\omega_1 = 1$ and $\omega_2 = 2$ or $\omega_1 = 1$ and $\omega_2 = \omega$, where ω is assumed to be the frequency of the nonlinear oscillator.

Substituting (2.2) and (2.3) into, respectively, (2.1), we obtain the following residuals:

$$\begin{aligned} R_1(t) &= -A\omega_1^2\cos\omega_1 t + f(A\cos\omega_1 t), \\ R_2(t) &= -A\omega_2^2\cos\omega_2 t + f(A\cos\omega_2 t). \end{aligned} \tag{2.5}$$

According to the ancient Chinese algorithm, an amplitude-frequency formulation for nonlinear oscillators was proposed [18].

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)}. \tag{2.6}$$

Other modifications of the frequency formulation are shown in [20–23]. The frequency formulation is now widely used to solve various nonlinear oscillators [20–25], and the basic idea of the Chinese algorithm can also be used to solve other nonlinear problems [26–29].

3. Solitary-Solution Formulation for Differential-Difference Equations

Suppose the differential-difference equation we discuss in this paper is in the following nonlinear polynomial form:

$$\frac{du_n(t)}{dt} = f(u_{n-1}, u_n, u_{n+1}), \tag{3.1}$$

where $u_n = u(n,t)$ is a dependent variable, and t is a continuous variable, and $n, p_i \in Z$.

Using the basic idea of the ancient Chinese algorithm, we choose two trial functions in the following forms:

$$\begin{aligned} u_{n,1}(n,t) &= f(\xi_n + \omega_1 t), \\ u_{n,2}(n,t) &= g(\xi_n + \omega_2 t), \end{aligned} \tag{3.2}$$

where $\xi_n = nd + \xi_0$, ξ_0 is arbitrary, and f and g are known functions. If a periodic solution is searched for, f and g must be periodic functions; if a solitary solution is solved, f and g must be of solitary structures. In this paper a bell solitary solution of a differential-difference equation is considered, and trial functions are chosen as follows:

$$u_{n,1}(n,t) = \frac{A}{e^{\xi_n + \omega_1 t} + e^{-(\xi_n + \omega_1 t)} + B}, \quad \omega_1 = 1, \tag{3.3}$$

$$u_{n,2}(n,t) = \frac{A}{e^{\xi_n + \omega t} + e^{-(\xi_n + \omega t)} + B}, \quad \omega_2 = \omega. \tag{3.4}$$

For u_n, u_{n-1}, and u_{n+1} should be compatible; then, from (3.3), we have

$$\begin{aligned} u_{n-1,1}(n,t) &= \frac{A}{e^{\xi_n - d + t} + e^{-(\xi_n - d + t)} + B}, \\ u_{n+1,1}(n,t) &= \frac{A}{e^{\xi_n + d + t} + e^{-(\xi_n + d + t)} + B}, \end{aligned} \tag{3.5}$$

and from (3.4), we have

$$u_{n-1,2}(n,t) = \frac{A}{e^{\xi_n - d + \omega t} + e^{-(\xi_n - d + \omega t)} + B}, \tag{3.6}$$

$$u_{n+1,2}(n,t) = \frac{A}{e^{\xi_n + d + \omega t} + e^{-(\xi_n + d + \omega t)} + B}. \tag{3.7}$$

Solution Procedure

Step 1. Define residual function

$$\widetilde{R}(t) = \frac{du_n(t)}{dt} - f(u_{n-1}, u_n, u_{n+1}). \tag{3.8}$$

Substituting (3.3)–(3.7) into (3.1), we can obtain, respectively, the residual functions $\widetilde{R}_1$ and $\widetilde{R}_2$

$$\begin{aligned} \widetilde{R}_1(t) &= \frac{du_{n,1}(t)}{dt} - f(u_{n-1,1}, u_{n,1}, u_{n+1,1}), \\ \widetilde{R}_2(t) &= \frac{du_{n,2}(t)}{dt} - f(u_{n-1,2}, u_{n,2}, u_{n+1,2}). \end{aligned} \tag{3.9}$$

Step 2. Solitary-solution formulation is constructed as follows:

$$\omega^2 = \frac{\omega_1^2 \widetilde{R}_2(0) - \omega_2^2 \widetilde{R}_1(0)}{\widetilde{R}_2(0) - \widetilde{R}_1(0)}, \tag{3.10}$$

where $\omega_1 = 1$ and $\omega_2 = \omega$.

Step 3. Combining the coefficients of e^{ξ_n} in (3.10), and setting them to be zero, we can solve the algebraic equations to find the values of ω, A, and B. Finally an explicit solution is obtained.

4. Application in Discrete mKdV Lattice

The famous mKdV lattice equation reads [10]

$$\frac{du_n}{dt} = \left(\alpha - u_n^2\right)(u_{n-1} - u_{n+1}), \tag{4.1}$$

where $\alpha \neq 0$.

Substituting (3.3)–(3.7) into (4.1), we obtain, respectively, the residual functions $\widetilde{R}_1$ and $\widetilde{R}_2$, and using the solitary-solution formulation, (3.10), we have

$$\begin{aligned}
&e^{3\xi_n}\left(\omega^3+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}-\omega-\alpha e^{-d}+\alpha e^{d}\right) - e^{-3\xi_n}\left(\omega^3+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}-\omega-\alpha e^{-d}+\alpha e^{d}\right)\\
&\quad + e^{2\xi_n}B\left(\omega^3 e^{-d}+\omega^3 e^{d}-2\omega^2\alpha e^{d}+2\omega^2\alpha e^{-d}-\omega e^{d}-\omega e^{-d}+2\alpha e^{d}-2\alpha e^{-d}\right)\\
&\quad - e^{-2\xi_n}B\left(\omega^3 e^{-d}+\omega^3 e^{d}-2\omega^2\alpha e^{d}+2\omega^2\alpha e^{-d}-\omega e^{d}-\omega e^{-d}+2\alpha e^{d}-2\alpha e^{-d}\right)\\
&\quad + e^{\xi_n}(\omega^3B^2-\omega^3+\omega^3e^{-2d}+\omega^3e^{2d}+\omega^2A^2e^{d}-\omega^2A^2e^{-d}+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}+\omega^2\alpha B^2e^{-d}\\
&\qquad\quad -\omega^2\alpha B^2e^{d}-\omega B^2+\omega-\omega e^{2d}-\omega e^{-2d}+A^2e^{-d}-A^2e^{d}+\alpha B^2e^{d}-\alpha B^2e^{-d}+\alpha e^{d}-\alpha e^{-d})\\
&\quad - e^{-\xi_n}(\omega^3B^2-\omega^3+\omega^3e^{-2d}+\omega^3e^{2d}+\omega^2A^2e^{d}-\omega^2A^2e^{-d}+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}+\omega^2\alpha B^2e^{-d}\\
&\qquad\quad -\omega^2\alpha B^2e^{d}-\omega B^2+\omega-\omega e^{2d}-\omega e^{-2d}+A^2e^{-d}-A^2e^{d}+\alpha B^2e^{d}-\alpha B^2e^{-d}+\alpha e^{d}-\alpha e^{-d})\\
&\quad = 0.
\end{aligned} \tag{4.2}$$

Setting the coefficients of $e^{i\xi_n}$ to be zero, we have

$$\begin{aligned}
&\omega^3+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}-\omega-\alpha e^{-d}+\alpha e^{d}=0,\\
&B\left(\omega^3 e^{-d}+\omega^3 e^{d}-2\omega^2\alpha e^{d}+2\omega^2\alpha e^{-d}-\omega e^{d}-\omega e^{-d}+2\alpha e^{d}-2\alpha e^{-d}\right)=0\\
&\omega^3B^2-\omega^3+\omega^3e^{-2d}+\omega^3e^{2d}+\omega^2A^2e^{d}-\omega^2A^2e^{-d}+\omega^2\alpha e^{-d}-\omega^2\alpha e^{d}+\omega^2\alpha B^2e^{-d},\\
&\quad -\omega^2\alpha B^2e^{d}-\omega B^2+\omega-\omega e^{2d}-\omega e^{-2d}+A^2e^{-d}-A^2e^{d}+\alpha B^2e^{d}-\alpha B^2e^{-d}+\alpha e^{d}-\alpha e^{-d}=0.
\end{aligned} \tag{4.3}$$

Solving (4.3) simultaneously, we have

$$\omega = 2\alpha \sinh(d),$$
$$A = 2\sqrt{-\alpha} \sinh(d), \tag{4.4}$$
$$B = 0.$$

We, therefore, obtain the following needed solitary solution:

$$u_n = \frac{2\sqrt{-\alpha}\sinh(d)}{e^{\xi_n + 2\alpha\sinh(d)t} + e^{-(\xi_n + 2\alpha\sinh(d)t)}} = \sqrt{-\alpha}\sinh(d)\operatorname{sech}\{2\alpha\sinh(d)t + nd + \xi_0\}. \tag{4.5}$$

5. Conclusions

Though there are many analytical methods, such as the exp-function method, the variational iteration method, and the homotopy perturbation method, for differential-difference equations, this paper suggests an effective and simple approach to such problems using the basic idea of the ancient mathematics, and the simple formulation can be used routinely by followers to various differential-difference equations.

Acknowledgment

The work is supported by a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD).

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