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Research Article

Applications of Umbral Calculus Associated with p-Adic Invariant Integrals on Z_p

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Recently, Dere and Simsek (2012) have studied the applications of umbral algebra to some special functions. In this paper, we investigate some properties of umbral calculus associated with p-adic invariant integrals on $\mathbf{Z}_p$. From our properties, we can also derive some interesting identities of Bernoulli polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbf{Z}_p, \mathbf{Q}_p$, and $\mathbf{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbf{Q}_p$, respectively.

Let $\mathbf{N} \cup \{0\}$. Let $UD(\mathbf{Z}_p)$ be space of uniformly differentiable functions on $\mathbf{Z}_p$. For $f \in UD(\mathbf{Z}_p)$, the p-adic invariant integral on $\mathbf{Z}_p$ is defined by

$$\int_{\mathbf{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}$$

see [1, 2].

From (1.1), we have

$$\int_{\mathbf{Z}_p} f(x+n)d\mu(x) - \int_{\mathbf{Z}_p} f(x)d\mu(x) = \sum_{l=0}^{n} f'(l), \quad n \in \mathbf{N}, \tag{1.2}$$

where $f'(l) = (df(x)/dx)|_{x=l}$ (see [1–6]). Let $\mathbf{F}$ be the set of all formal power series in the variable t over $\mathbf{C}_p$ with

$$\mathbf{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbf{C}_p \right\}. \tag{1.3}$$

Let $\mathbb{P} = \mathbf{C}_p[x]$ and let $\mathbb{P}^*$ denote the vector space of all linear functional on $\mathbb{P}$.

The formal power series,

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathbf{F}, \tag{1.4}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t) \mid x^n \rangle = a_n, \quad \forall n \geq 0, \tag{1.5}$$

see [7, 8].

In particular, by (1.4) and (1.5), we get

$$\left\langle t^k \mid x^n \right\rangle = n! \delta_{n,k}, \tag{1.6}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [7]). Here, $\mathbf{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functional on $\mathbb{P}$, so an element $f(t)$ of $\mathbf{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathbf{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra.

The order $o(f(t))$ of power series $f(t)(\neq 0)$ is the smallest integer k for which a_k does not vanish. We define $o(f(t)) = \infty$ if $f(t) = 0$. From the definition of order, we note that $o(f(t)g(t)) = o(f(t)) + o(g(t))$ and $o(f(t) + g(t)) \geq \min\{o(f(t)), o(g(t))\}$.

The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $1/f(t)$, if and only if $o(f(t)) = 0$.

Such a series is called invertible series. A series $f(t)$ for which $o(f(t)) = 1$ is called a delta series (see [7, 8]). Let $f(t), g(t) \in \mathbf{F}$. Then, we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle = \langle g(t) \mid f(t)p(x) \rangle. \tag{1.7}$$

By (1.5) and (1.6), we get

$$\langle e^{yt} \mid x^n \rangle = y^n, \qquad \langle e^{yt} \mid p(x) \rangle = p(y), \tag{1.8}$$

see [7].

Notice that for all $f(t)$ in $\mathbf{F}$,

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k, \tag{1.9}$$

and for all polynomials $p(x)$,

$$p(x) = \sum_{k \geq 0} \frac{\langle t^k \mid p(x) \rangle}{k!} x^k, \tag{1.10}$$

see [7, 8].

Let $f_1(t), f_2(t), \ldots, f_m(t) \in \mathbf{F}$. Then, we have

$$\left\langle f_1(t) f_2(t) \cdots f_m(t) \mid x^n \right\rangle = \sum \binom{n}{i_1, \ldots, i_m} \left\langle f_1(t) \mid x^{i_1} \right\rangle \cdots \left\langle f_m(t) \mid x^{i_m} \right\rangle, \tag{1.11}$$

where the sum is over all nonnegative integers $i_1, i_2, \ldots, i_m$ such that $i_1 + \cdots + i_m = n$ (see [8]).

By (1.10), we get

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{n} \frac{\langle t^l \mid p(x) \rangle}{l!} l(l-1) \cdots (l-k+1) x^{l-k}. \tag{1.12}$$

Thus, from (1.12), we have

$$p^{(k)}(0) = \left\langle t^k \mid p(x) \right\rangle = \left\langle 1 \mid p^{(k)}(x) \right\rangle, \tag{1.13}$$

see [7].

By (1.13), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k(p(x))}{dx^k}. \tag{1.14}$$

Thus, by (1.14), we see that

$$e^{yt} p(x) = p(x+y). \tag{1.15}$$

Let us assume that $s_n(x)$ is a polynomial of degree n. Suppose that $f(t), g(t) \in \mathbf{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then, there exists a unique sequence $s_n(x)$ of polynomials satisfying $\langle g(t) f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k}$ for all $n, k \geq 0$.

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

The Sheffer sequence for $(g(t), t)$ is called the Appell sequence for $g(t)$, or $s_n(x)$ is Appell for $g(t)$, which is indicated by $s_n(x) \sim (g(t), t)$.

For $p(x) \in \mathbb{P}$, it is known that

$$\begin{gathered}\left\langle f(t) \mid x p(x)\right\rangle = \left\langle \partial_t f(t) \mid p(x)\right\rangle = \left\langle f'(t) \mid p(x)\right\rangle, \\ \left\langle e^{yt} - 1 \mid p(x)\right\rangle = p(y) - p(0),\end{gathered} \tag{1.16}$$

see [7, 8].

Let $s_n(x) \sim (g(t), f(t))$. Then, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid s_k(x)\rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathbf{F}, \tag{1.17}$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\left\langle g(t) f(t)^k \mid p(x)\right\rangle}{k!} s_k(x), \quad p(x) \in \mathbb{P}, \tag{1.18}$$

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad \text{for any } y \in \mathbf{C}_p, \tag{1.19}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, and

$$f(t) s_n(x) = n s_{n-1}(x), \tag{1.20}$$

see [7, 8].

We recall that the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.21}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [1–16]).

In the special case, $x = 0, B_n(0) = B_n$ are called the nth Bernoulli numbers. By (1.21), we easily get

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l. \tag{1.22}$$

Thus, by (1.22), we see that $B_n(x)$ is a monic polynomial of degree n. It is easy to show that

$$B_0 = 1, \qquad B_n(1) - B_n = \delta_{1,n}, \tag{1.23}$$

see [13–15].

From (1.2), we can derive the following equation:

$$\int_{\mathbf{Z}_p} f(x+1) d\mu(x) - \int_{\mathbf{Z}_p} f(x) d\mu(x) = f'(0). \tag{1.24}$$

Let us take $f(x) = e^{tx} \in UD(\mathbf{Z}_p)$. Then, from (1.21), (1.22), (1.23), and (1.24), we have

$$\int_{\mathbf{Z}_p} x^n d\mu(x) = B_n, \qquad \int_{\mathbf{Z}_p} (x+y)^n d\mu(y) = B_n(x), \tag{1.25}$$

where $n \geq 0$ (see [1, 2]). Recently, Dere and simsek have studied applications of umbral algebra to some special functions (see [7]). In this paper, we investigate some properties of umbral calculus associated with p-adic invariant integrals on $\mathbf{Z}_p$. From our properties, we can derive some interesting identities of Bernoulli polynomials.

2. Applications of Umbral Calculus Associated with p-Adic Invariant Integrals on Z_p

Let $s_n(x)$ be an Appell sequence for $g(t)$. By (1.19), we get

$$\frac{1}{g(t)} x^n = s_n(x), \quad \text{iff} \quad x^n = g(t) s_n(x). \tag{2.1}$$

Let us take $g(t) = ((e^t - 1)/t) \in \mathbf{F}$. Then, $g(t)$ is clearly invertible series. From (1.21) and (2.1), we have

$$\sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{2.2}$$

Thus, by (2.2), we get

$$\frac{1}{g(t)} x^n = B_n(x), \quad tB_n(x) = B_n'(x) = nB_{n-1}(x), \quad (n \geq 0). \tag{2.3}$$

From (1.21), (2.1), and (2.3), we note that $B_n(x)$ is an Appell sequence for $g(t) = (e^t - 1)/t$.

Let us take the derivative with respect to t on both sides of (2.2). Then, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{B_k(x)}{k!} k t^{k-1} &= \frac{x g(t) e^{xt} - e^{xt} g'(t)}{g(t)^2} \\ &= \sum_{k=0}^{\infty} \left\{ x \frac{x^k}{g(t)} - \frac{x^k}{g(t)} \frac{g'(t)}{g(t)} \right\} \frac{t^k}{k!}.\end{aligned} \tag{2.4}$$

Thus, by (2.4), we get

$$B_{k+1}(x) = x \frac{x^k}{g(t)} - \frac{x^k}{g(t)} \frac{g'(t)}{g(t)} = \left(x - \frac{g'(t)}{g(t)} \right) B_k(x), \tag{2.5}$$

where $k \geq 0$.

$$\int_{\mathbf{Z}_p} e^{(x+y+1)t} d\mu(y) - \int_{\mathbf{Z}_p} e^{(x+y)t} d\mu(y) = te^{xt}. \tag{2.6}$$

Thus, by (2.6), we get

$$\int_{\mathbf{Z}_p} (x+y+1)^n d\mu(y) - \int_{\mathbf{Z}_p} (x+y)^n \mu(y) = nx^{n-1}, \quad (n \geq 0). \tag{2.7}$$

From (1.25) and (2.7), we have

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad (n \geq 0). \tag{2.8}$$

By (2.5), we see that

$$g(t)B_{k+1}(x) = g(t)xB_k(x) - g'(t)B_k(x), \tag{2.9}$$

Thus, by (2.9), we have

$$\left(e^t - 1\right)B_{k+1}(x) = \left(e^t - 1\right)xB_k(x) - \left(e^t - g(t)\right)B_k(x), \quad (k \geq 0), \tag{2.10}$$

and we can derive the following equation.

From (2.3) and (2.10),

$$B_{k+1}(x+1) - B_{k+1}(x) = (x+1)B_k(x+1) - xB_k(x) - B_k(x+1) + x^k, \quad (k \geq 0). \tag{2.11}$$

By (2.8) and (2.11), we see that

$$B_{k+1}(x+1) = B_{k+1}(x) + (k+1)x^k. \tag{2.12}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. *For $k \in \mathbf{Z}_+$, one has*

$$B_{k+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)B_k, \tag{2.13}$$

where $g'(t) = dg(t)/dt$.

Corollary 2.2. *For ≥ 0, one has*

$$B_{k+1}(x+1) = B_{k+1}(x) + (k+1)x^k. \tag{2.14}$$

Let us consider the linear functional $f(t)$ that satisfies

$$\langle f(t) \mid p(x) \rangle = \int_{\mathbf{Z}_p} p(u) d\mu(u), \tag{2.15}$$

for all polynomials $p(x)$. It can be determined from (1.9) that

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k = \sum_{k=0}^{\infty} \int_{\mathbf{Z}_p} u^k d\mu(u) \frac{t^k}{k!} \\ &= \int_{\mathbf{Z}_p} e^{ut} d\mu(u). \end{aligned} \tag{2.16}$$

By (1.24) and (2.16), we get

$$f(t) = \int_{\mathbf{Z}_p} e^{ut} d\mu(u) = \frac{t}{e^t - 1}. \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.3. *For $p(x) \in \mathbf{P}$, one has*

$$\left\langle \int_{\mathbf{Z}_p} e^{ut} d\mu(u) \mid p(x) \right\rangle = \int_{\mathbf{Z}_p} p(u) d\mu(u). \tag{2.18}$$

That is

$$\left\langle \frac{t}{e^t - 1} \mid p(x) \right\rangle = \int_{\mathbf{Z}_p} p(u) d\mu(u). \tag{2.19}$$

In particular, one has

$$B_n = \left\langle \int_{\mathbf{Z}_p} e^{ut} d\mu(u) \mid x^n \right\rangle. \tag{2.20}$$

From (1.24), one has

$$\begin{aligned} \sum_{n=0} \int_{\mathbf{Z}_p} (x+y)^n d\mu(y) \frac{t^n}{n!} &= \int_{\mathbf{Z}_p} e^{(x+y)t} d\mu(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbf{Z}_p} e^{yt} d\mu(y) x^n \frac{t^n}{n!}. \end{aligned} \tag{2.21}$$

By (1.25) and (2.21), we get

$$B_n(x) = \int_{\mathbf{Z}_p} (x+y)^n d\mu(y) = \int_{\mathbf{Z}_p} e^{yt} d\mu(y) x^n, \tag{2.22}$$

where $n \geq 0$.

Therefore, by (2.22), we obtain the following theorem.

Theorem 2.4. *For $p(x) \in \mathbb{P}$, we have*

$$\begin{aligned}\int_{\mathbf{Z}_p} p(x+y) d\mu(y) &= \int_{\mathbf{Z}_p} e^{yt} d\mu(y) p(x) \\ &= \frac{t}{e^t - 1} p(x).\end{aligned} \tag{2.23}$$

In particular, one obtains

$$\begin{aligned}B_n(x) &= \int_{\mathbf{Z}_p} (x+y)^n d\mu(y) = \int_{\mathbf{Z}_p} e^{yt} d\mu(y) x^n \\ &= \frac{t}{e^t - 1} x^n.\end{aligned} \tag{2.24}$$

The higher order Bernoulli polynomials $B_n^{(r)}(x)$ are defined by

$$\begin{aligned}\int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{(x_1+x_2+\cdots+x_r+x)t} d\mu(x_1) \cdots d\mu(x_r) &= \left(\frac{t}{e^t - 1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.\end{aligned} \tag{2.25}$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the nth Bernoulli numbers of order r ($\in \mathbf{N}$). From (2.25), we note that

$$\begin{aligned}&\int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} (x_1 + \cdots + x_r)^n d\mu(x_1) \cdots d\mu(x_r) \\ &= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1, \ldots, i_r} \int_{\mathbf{Z}_p} x_1^{i_1} d\mu(x_1) \int_{\mathbf{Z}_p} x_2^{i_2} d\mu(x_2) \cdots \int_{\mathbf{Z}_p} x_r^{i_r} d\mu(x_r) \\ &= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r} = B_n^{(r)}.\end{aligned} \tag{2.26}$$

By (2.25) and (2.26), we get

$$B_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} x^l. \tag{2.27}$$

From (2.26) and (2.27), we note that $B_n^{(r)}(x)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}$. For $r \in \mathbf{N}$, let us assume that

$$g^{(r)}(t) = \left(\int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) \right)^{-1} = \left(\frac{e^t - 1}{t} \right)^r. \tag{2.28}$$

By (2.28), we easily see that $g^{(r)}(t)$ is an invertible series. From (2.25) and (2.28), we have

$$\begin{aligned} \frac{e^{xt}}{g^{(r)}(t)} &= \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu(x_1) \cdots d\mu(x_r) \\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \\ tB_n^{(r)}(x) &= nB_{n-1}^{(r)}(x). \end{aligned} \tag{2.29}$$

From (2.29), we note that $B_n^{(r)}$ is an Appell sequence for $g^{(r)}(t)$. Therefore, by (2.29), we obtain the following theorem.

Theorem 2.5. *For $p(x) \in \mathbb{P}$ and $r \in \mathbf{N}$, one has*

$$\int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} p(x_1 + \cdots + x_r + x) d\mu(x_1) \cdots d\mu(x_r) = \left(\frac{t}{e^t - 1} \right)^r p(x). \tag{2.30}$$

In particular, the Bernoulli polynomials of order r are given by

$$B_n^{(r)}(x) = \left(\frac{t}{e^t - 1} \right)^r x^n = \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) x^n. \tag{2.31}$$

That is

$$B_n^{(r)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^r, t \right). \tag{2.32}$$

Let us consider the linear functional $f^{(r)}(t)$ that satisfies

$$\left\langle f^{(r)}(t) \mid p(x) \right\rangle = \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} p(x_1 + \cdots + x_r) d\mu(x_1) \cdots d\mu(x_r), \tag{2.33}$$

for all polynomials $p(x)$. It can be determined from (1.9) that

$$\begin{aligned} f^{(r)}(t) &= \sum_{k=0}^{\infty} \frac{\langle f^{(r)}(t) \mid x^k \rangle}{k!} t^k \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^k d\mu(x_1) \cdots d\mu(x_r) \frac{t^k}{k!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) \\ &= \left(\frac{t}{e^t - 1} \right)^r. \end{aligned} \tag{2.34}$$

Therefore, by (2.34), we obtain the following theorem.

Theorem 2.6. *For $p(x) \in \mathbb{P}$, one has*

$$\begin{aligned} &\left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) \mid p(x) \right\rangle \\ &\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} p(x_1 + \cdots + x_r) d\mu(x_1) \cdots d\mu(x_r). \end{aligned} \tag{2.35}$$

That is

$$\left\langle \left(\frac{t}{e^t - 1} \right)^r \mid p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} p(x_1 + \cdots + x_r) d\mu(x_1) \cdots d\mu(x_r). \tag{2.36}$$

In particular, one gets

$$B_n^{(r)} = \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) \mid x^n \right\rangle. \tag{2.37}$$

Remark 2.7. From (1.11), we note that

$$\begin{aligned} &\left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu(x_1) \cdots d\mu(x_r) \mid x^n \right\rangle \\ &\quad = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1, \ldots, i_r} \left\langle \int_{\mathbb{Z}_p} e^{x_1 t} d\mu(x_1) \mid x^{i_1} \right\rangle \cdots \left\langle \int_{\mathbb{Z}_p} e^{x_r t} d\mu(x_r) \mid x^{i_r} \right\rangle. \end{aligned} \tag{2.38}$$

By Theorems 2.3 and 2.6 and (2.38), we get

$$B_n^{(r)} = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r}. \tag{2.39}$$

Let $s_n(x)$ be the Sheffer sequence for $(g(t), f(t))$.

Then the Sheffer identity is given by

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) s_{n-k}(x), \tag{2.40}$$

see [7, 8], where $p_k(y) = g(t)s_k(y)$. From Theorem 2.5 and (2.40), we have

$$B_n^{(r)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(r)}(x) x^k. \tag{2.41}$$

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