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Research Article

Asymptotic Behavior of a Class of Evolution Variational Inequalities

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We establish a new LaSalle's invariance principle and discuss the asymptotic behavior of a class of first-order evolution variational inequalities.

1. Introduction

Nonsmooth systems, roughly speaking, are those systems whose trajectories may not be differentiable everywhere. Usually nonsmooth dynamical systems are represented as differential inclusions, complementarity systems, evolution variational inequalities, and so on [1]. Since they play important roles in numerous fields, there appeared an increasing interest in the study of their dynamics in recent years.

In this paper, we consider a class of typical nonsmooth dynamical systems given by the following first-order evolution variational inequalities:

$$\left\langle \frac{dx(t)}{dt} + f(x(t)), v - x(t) \right\rangle + \varphi(v) - \varphi(x(t)) \geq 0, \quad \forall v \in \mathbb{R}^n. \tag{1.1}$$

It is known that many important mechanical systems arising from applications can be transformed into an variational inequality as above. In case φ is a proper convex and lower semicontinuous function from $\mathbb{R}^n$ to $\mathbb{R}^1 \cup \{+\infty\}$ and f is a continuous operator with $f + \omega I$ being monotone for some $\omega \geq 0$, Adly and Goeleven [2] made a systematic study on the asymptotic behavior of the system (1.1). The existence and uniqueness of solutions were established, and the asymptotic behavior was discussed.

In this present work, we are basically interested in the case where f is only continuous. On the other hand, to avoid some technical difficulties, we will always assume φ is a proper convex and lower semicontinuous function from $\mathbb{R}^n$ to $\mathbb{R}^1$. Note that, in our case, (1.1) may fail to have uniqueness. The main purpose is to establish a LaSalle's invariance principle and discuss asymptotic stability of the equilibria of the system.

LaSalle's invariance principle plays a key role in stability analysis and control. In the past decades, there appeared many important extensions. Results closely related to ours can be found in [2–6] and so forth.

This paper is organized as follows. In Section 2, we provide some basic definitions and auxiliary results. In Section 3, we develop a LaSalle's invariance theorem and discuss the strong stability and strong asymptotic stability of the system.

2. Preliminaries

This section is concerned with some preliminary works. For convenience, we will denote by $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{R}^n$ with the corresponding norm $\|\cdot\|$.

2.1. Subdifferential

Let V be a function from $\mathbb{R}^n$ to $\mathbb{R}^1$. For $x, v \in \mathbb{R}^n$, define

$$D_v V(x) = \lim_{h\to 0^+} \frac{V(x+hv) - V(x)}{h} \in [-\infty, +\infty]. \tag{2.1}$$

$D_v V(x)$ is said to be the *derivative of V at x in the direction v*.

If $D_v V(x)$ exists for all directions v, we say that V is *differentiable* at x.

Definition 2.1. The following closed convex subset (possibly empty)

$$\partial V(x) := \{p \in \mathbb{R}^n \mid \forall v \in \mathbb{R}^n,\ \langle p, v\rangle \le D_v V(x)\} \tag{2.2}$$

is called the subdifferential of V at x, and we say that the elements p of $\partial V(x)$ are the subgradients of V at x.

It is known that if V is differentiable at x in the classical sense, then

$$\partial V(x) = \{\nabla V(x)\}. \tag{2.3}$$

Proposition 2.2 (see [7]). *If V is a convex function from $\mathbb{R}^n$ to $\mathbb{R}^1$, then, for each fixed x, the mapping $v \to D_v V(x)$ is convex and positively homogeneous with the following inequalities hold:*

$$-\infty \le V(x) - V(x-v) \le D_v V(x) \le V(x+v) - V(x) \le +\infty. \tag{2.4}$$

Furthermore,

$$\partial V(x) = \{p \in \mathbb{R}^n \mid V(x) - V(y) \le \langle p, x-y\rangle, \forall y \in \mathbb{R}^n\}. \tag{2.5}$$

Proposition 2.3 (see [7]). *Let one assume that $V : \mathbb{R}^n \to \mathbb{R}^1$ is convex and lower semicontinuous. Then,*

(a) *for each x, $\partial V(x)$ is a nonempty and bounded set,*

(b) *the mapping $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \to D_v V(x)$ is upper semicontinuous (and thus $\partial V(\cdot)$ is upper semicontinuous as well),*

(c) *the following regularity property holds:*

$$D_v V(x) = \limsup_{\substack{h \to 0^+ \\ y \to x}} \frac{V(y + hv) - V(y)}{h}. \tag{2.6}$$

2.2. Some Basic Facts on the Evolution Variational Inequality

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex and lower semicontinuous function, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Consider the following evolution variational inequality.

(VP) For any given $x_0 \in \mathbb{R}^n$, find a $x(\cdot) \in C([0, T); \mathbb{R}^n)$ with $dx/dt \in L^\infty_{\text{loc}}(0, T; \mathbb{R}^n)$, such that

$$\begin{gathered} \left\langle \frac{dx(t)}{dt} + f(x(t)), v - x(t) \right\rangle + \varphi(v) - \varphi(x(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq 0, \\ x(0) = x_0. \end{gathered} \tag{2.7}$$

Thanks to Proposition 2.2, one can easily rewrite (2.7) as the initial value problem of a differential inclusion:

$$\frac{dx(t)}{dt} + f(x(t)) \in -\partial\varphi(x(t)), \qquad x(0) = x_0. \tag{2.8}$$

By Definition 2.1 and Proposition 2.3, we see that, for each $x \in \mathbb{R}^n$, $\partial\varphi(x)$ is a nonempty compact and convex subset of $\mathbb{R}^n$; moreover, the multifunction $\partial\varphi(\cdot)$ is upper semicontinuous in x. This guarantees by the basic theory on differential inclusions (see, e.g., [7–9], etc.) the local existence of solutions for (2.7).

Let $x(\cdot)$ be a solution to differential inclusion (2.7) defined on $[0, +\infty)$. Then, the *ω-limit set* $\omega(x(\cdot))$ is defined as

$$\omega(x(\cdot)) := \{ y \in \mathbb{R}^n \mid \exists t_n \longrightarrow +\infty \text{ such that } x(t_n) \longrightarrow y \}. \tag{2.9}$$

We infer from [10] that the following basic facts on ω-limit sets hold.

Proposition 2.4. *If a solution $x(\cdot)$ of (2.7) is bounded on $[0, +\infty)$, then $\omega(x(\cdot))$ is a nonempty compact weakly invariant set, namely, for each $y \in \Omega$, there is a complete solution $x(\cdot)$ on $\mathbb{R}^1$ which is contained in $\omega(x(\cdot))$ with $x(0) = y$. Moreover,*

$$\lim_{t \to +\infty} d(x(t), \omega(x(\cdot))) = 0. \tag{2.10}$$

3. LaSalle's Invariance Principle

We are now ready to establish a LaSalle's invariance principle for (2.7). For convenience, we will denote by $E_\Omega(f,\varphi,V)$ the set

$$E_\Omega(f,\varphi,V) := \{x \in \Omega \mid \langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = 0\}. \tag{3.1}$$

In case $\Omega = \mathbb{R}^n$, we simply write $E_\Omega(f,\varphi,V)$ as $E(f,\varphi,V)$.

3.1. Invariance Principle

In this subsection, we provide a LaSalle's invariance principle for the system (2.7) involving a mapping f that is only assumed continuous. The approach followed by Adly and Goeleven [2] has been proved with f being continuous and $f + \omega I$ monotone.

The main result is contained in the following theorem. The weak invariance of ω-limit set plays an important role in the proof of the theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be closed. Assume that there exists $V \in C^1(\Omega;\mathbb{R})$ such that*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \ge 0, \quad x \in \Omega. \tag{3.2}$$

Let $\mathcal{M}_\Omega$ be the largest weakly invariant set of $E_\Omega(f,\varphi,V)$.

Then, for each $x_0 \in \Omega$ and each bounded solution $x(\cdot)$ of (2.7) in Ω, we have

$$\lim_{t\to+\infty} d(x(t), \mathcal{M}_\Omega) = 0. \tag{3.3}$$

Proof. For each $p \in -\partial\varphi(x)$, by Proposition 2.2, we find that

$$\langle p, x - y\rangle + \varphi(x) - \varphi(y) \le 0, \quad \forall y \in \mathbb{R}^n. \tag{3.4}$$

Taking $y = x - \nabla V(x)$ in (3.4), one gets

$$\langle \nabla V(x), p\rangle \le \varphi(x - \nabla V(x)) - \varphi(x). \tag{3.5}$$

Equation (3.2) then implies

$$\varphi(x - \nabla V(x)) - \varphi(x) \le \langle f(x), \nabla V(x)\rangle. \tag{3.6}$$

Now, by (3.5) and (3.6), we deduce that

$$\max_{p\in-\partial\varphi(x)} \langle \nabla V(x), p\rangle \le \langle \nabla V(x), f(x)\rangle, \quad x \in \Omega. \tag{3.7}$$

Set

$$E := \{x \in \Omega \mid \langle \nabla V(x), p - f(x) \rangle = 0 \text{ for some } p \in -\partial\varphi(x)\}. \tag{3.8}$$

Denote by $\mathcal{M}$ the largest weakly invariant set of E. In the following, we will check that, for each $x_0 \in \Omega$ and each bounded solution $x(\cdot)$ of (2.7) in Ω, we have

$$\lim_{t \to +\infty} d(x(t), \mathcal{M}) = 0. \tag{3.9}$$

By Proposition 2.4, we know that $\omega(x(\cdot))$ is a nonempty compact weakly invariant set, and

$$\lim_{t \to +\infty} d(x(t), \omega(x(\cdot))) = 0. \tag{3.10}$$

To prove (3.9), it suffices to check that

$$\omega(x(\cdot)) \subset \mathcal{M}. \tag{3.11}$$

Note that Ω is closed, we have $\omega(x(\cdot)) \subset \Omega$. Since for every $y \in \omega(x(\cdot))$, by the definition of ω-limit set, there is $t_n \to +\infty$ such that $x(t_n) \to y$. Here, $x(\cdot)$ is a bounded solution of (2.7) in Ω, and Ω is closed, hence $y \in \Omega$.

In what follows we first show that

$$V(y) \equiv \text{const.}, \qquad y \in \omega(x(\cdot)). \tag{3.12}$$

Indeed, by (2.8) and (3.7), we see that

$$\begin{aligned} \frac{dV(x(t))}{dt} &= \langle \nabla V(x(t)), \dot{x}(t) \rangle \\ &\le \max_{p \in -\partial\varphi(x(t))} \langle \nabla V(x(t)), p - f(x(t)) \rangle \\ &\le 0, \quad \text{a.e. } t \ge 0. \end{aligned} \tag{3.13}$$

It then follows from the proof of Lemma 2 in [2] that V is nonincreasing on $[0, +\infty)$. Moreover, V is bounded from below on $[0, +\infty)$ since $x([0, +\infty)) \subset \Omega$ and V is continuous on the closed set Ω. This provides with an existence of the limit of V. Hence,

$$\lim_{t \to +\infty} V(x(t)) := \lambda \tag{3.14}$$

exists.

For each $y \in \omega(x(\cdot))$, by definition of ω-limit set, there exists $t_n \to +\infty$ such that

$$x(t_n) \longrightarrow y, \quad n \longrightarrow +\infty. \tag{3.15}$$

Further, by continuity, we deduce that

$$V(y) = V\left(\lim_{n\to+\infty} x(t_n)\right) = \lim_{n\to+\infty} V(x(t_n)) = \lambda. \tag{3.16}$$

This verifies the validity of (3.12).

Now, we check that $\omega(x(\cdot)) \subset E$. Let $y_0 \in \omega(x(\cdot))$. As $\omega(x(\cdot))$ is weakly invariant, there exists a complete solution $y(\cdot)$ starting from y_0 with $y(t) \in \omega(x(\cdot))$ for all $t \geq 0$. By what we have just proved, it holds that

$$V(y(t)) \equiv \lambda, \quad \forall t \geq 0. \tag{3.17}$$

Take a sequence $t_n \to 0$ such that $y(t)$ is differentiable at each t_n with

$$\dot{y}(t_n) + f(y(t_n)) \in -\partial\varphi(y(t_n)). \tag{3.18}$$

Then,

$$\frac{dV(y(t_n))}{dt} = \langle \nabla V(y(t_n)), \dot{y}(t_n)\rangle = 0. \tag{3.19}$$

This implies that

$$\langle \nabla V(y(t_n)), p - f(y(t_n))\rangle = 0, \quad \text{for some } p \in -\partial\varphi(y(t_n)). \tag{3.20}$$

Thus, one deduces that $y(t_n) \in E$. By continuity of $y(\cdot)$, we know that $y_0 \in E$. This proves what we desired, and (3.11) follows directly from the weak invariance of $\omega(x(\cdot))$.

Finally, we verify that $E \subset E_\Omega(f, \varphi, V)$, which implies $\mathcal{M} \subset \mathcal{M}_\Omega$ and completes the proof of the theorem. Let $x \in E$. Then, by (3.8), we have

$$\langle \nabla V(x), p\rangle = \langle \nabla V(x), f(x)\rangle, \quad \text{for some } p \in -\partial\varphi(x). \tag{3.21}$$

Invoking (3.5) and (3.6), we find that

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = 0. \tag{3.22}$$

Thus, $x \in E_\Omega(f, \varphi, V)$. □

As a particular case of Theorem 3.1, we have the following.

Theorem 3.2. *Assume that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R}^1)$ such that*

(1) $\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq 0$, $x \in \mathbb{R}^n$,

(2) $V(x) \to +\infty$ *as* $\|x\| \to +\infty$, $x \in \mathbb{R}^n$.

Denote by $\mathcal{M}$ the largest weakly invariant set of $E(f, \varphi, V)$.

Then, for each $x_0 \in \mathbb{R}^n$, every solution $x(\cdot)$ of (2.7) *is bounded and*

$$\lim_{t \to +\infty} d(x(t), \mathcal{M}) = 0. \tag{3.23}$$

Proof. Let $x_0 \in \mathbb{R}^n$ be given. We set

$$\Omega := \{x \in \mathbb{R}^n \mid V(x) \leq V(x_0)\}. \tag{3.24}$$

Then, by assumption (2), we see that Ω is a bounded closed subset of $\mathbb{R}^n$. We infer from the proof of Theorem 3.1 that V is decreasing along any solution of (2.7). Thus, Ω is actually positively invariant. Therefore, by Theorem 3.1, one concludes that

$$\lim_{t \to \infty} d(x(t), \mathcal{M}_\Omega) = 0 \tag{3.25}$$

for each solution $x(\cdot)$, where $\mathcal{M}_\Omega$ is the largest weakly invariant subset of $E(f, \varphi, V) \cap \Omega$. Clearly, $\mathcal{M}_\Omega \subset \mathcal{M}$, and the conclusion follows. □

3.2. Asymptotic Stability of Equilibria

As simple applications of the LaSalle's invariance principle established above, we make some further discussions on the asymptotic behavior of the system (1.1). For this purpose, we denote by $\mathcal{E}(f, \varphi)$ the set of stationary solutions to (1.1), that is,

$$\mathcal{E}(f, \varphi) := \{z \in \mathbb{R}^n \mid \langle f(z), v - z \rangle + \varphi(v) - \varphi(z) \geq 0, \forall v \in \mathbb{R}^n\}. \tag{3.26}$$

In what follows, we will always assume that

$$f(0) \in -\partial\varphi(0), \tag{3.27}$$

so that $0 \in \mathcal{E}(f, \varphi)$ is the trivial stationary solution of (1.1).

Let us first prove the strong stability of the trivial stationary solution 0. For $r > 0$, we denote by $\overline{B}_r$ the closed ball of radius r,

$$\overline{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}. \tag{3.28}$$

Theorem 3.3. *Suppose that there exists $\sigma > 0$ and $V \in C^1(\overline{B}_\sigma; \mathbb{R})$ such that*

(1) $V(x) \geq a(\|x\|)$ *for all $x \in \overline{B}_\sigma$, where $a \in C([0, \sigma])$ satisfies $a(t) > 0$ $(\forall t \in (0, \sigma))$,*

(2) $V(0) = 0$,

(3) $\langle f(x), \nabla V(x) \rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq 0$, $x \in \overline{B}_\sigma$.

Then, the stationary solution 0 is strongly stable, that is, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $x(t)$ of (1.1) *with $\|x(0)\| \leq \delta$, one has*

$$\|x(t)\| \leq \varepsilon, \quad \forall t \geq 0. \tag{3.29}$$

Proof. For any $\delta < \sigma$, let m be the minimum value of V on the boundary of $\overline{B}_\delta$. Then, by assumptions (1) and (2), we find that $m > 0$. Let

$$U = \left\{ x \in \overline{B}_\delta \mid V(x) \le m \right\}. \tag{3.30}$$

It is clear that U is a neighborhood of 0. Since V is decreasing along each solution in $\overline{B}_\delta$, one trivially checks that U is strongly positive invariant. This implies the desired result. □

Propositions 3.4–3.6 below can be proved by the same arguments as the ones in [2]. We omit the details.

Proposition 3.4. *Suppose that* Ω *is a subset of* $\mathbb{R}^n$*, and there exists* $V \in C^1(\Omega;\mathbb{R})$ *such that*

$$\langle f(x), \nabla V(x) \rangle + \varphi(x) - \varphi(x - \nabla V(x)) \ge 0, \quad x \in \Omega. \tag{3.31}$$

Then,

$$(\mathcal{E}(f,\varphi) \cap \Omega) \subset E_\Omega(f,\varphi,V). \tag{3.32}$$

Proposition 3.5. *Suppose that there exist a* $\sigma > 0$ *and* $V \in C^1(\overline{B}_\sigma;\mathbb{R})$ *such that*

(1) $\langle f(x), \nabla V(x) \rangle + \varphi(x) - \varphi(x - \nabla V(x)) \ge 0$, $x \in \overline{B}_\sigma$,

(2) $E(f,\varphi,V) \cap \overline{B}_\sigma = \{0\}$.

Then, the stationary solution 0 is isolated in $\mathcal{E}(f,\varphi)$.

Proposition 3.6. *Suppose that there exists* $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that*

$$\langle f(x), \nabla V(x) \rangle + \varphi(x) - \varphi(x - \nabla V(x)) \ge 0, \quad x \in \mathbb{R}^n. \tag{3.33}$$

Then, $\mathcal{E}(f,\varphi) \subset E(f,\varphi,V)$.

Now, we can easily prove the following result.

Theorem 3.7. *Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\overline{B}_\sigma;\mathbb{R})$ *such that (1)–(3) in Theorem 3.3 hold; moreover,*

$$E(f,\varphi,V) \cap \overline{B}_\sigma = \{0\}. \tag{3.34}$$

Then, the trivial stationary solution 0 is strongly asymptotically stable.

Proof. The strong stability is readily implied in Theorem 3.3. Define

$$\Omega = \left\{ x \in \overline{B}_\sigma \mid V(x) \le m \right\}. \tag{3.35}$$

Then, as in the proof of Theorem 3.3, we know that Ω is a strongly positively invariant neighborhood of 0. Applying Theorem 3.1, we deduce that, for $x_0 \in \Omega$,

$$\lim_{t \to +\infty} d(x(t), \mathcal{M}) = 0. \tag{3.36}$$

On the other hand, by (3.34), we see that $\mathcal{M} = \{0\}$. Therefore, the trivial stationary solution 0 is strongly asymptotically stable. □

Theorem 3.8. *Suppose that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1) $V(x) \geq a(\|x\|)$ *for all* $x \in \mathbb{R}^n$*, where* $a : \mathbb{R}^+ \to \mathbb{R}$ *is a continuous strictly increasing function with* $a(0) = 0$*,*

(2) $V(0) = 0$*,*

(3) $\langle f(x), \nabla V(x) \rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq 0$, $x \in \mathbb{R}^n$*,*

(4) $E(f, \varphi, V) = \{0\}$.

Then, the trivial stationary solution to (2.7) *is globally strongly asymptotically stable.*

Proof. The strong asymptotic stability can be directly deduced from Theorem 3.8. Repeating the same argument as in Theorem 3.2, we can show that each solution of the system approaches $\mathcal{M} = \{0\}$. □

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