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Research Article

Various Heteroclinic Solutions for the Coupled Schrödinger-Boussinesq Equation

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Various closed-form heteroclinic breather solutions including classical heteroclinic, heteroclinic breather and Akhmediev breathers solutions for coupled Schrödinger-Boussinesq equation are obtained using two-soliton and homoclinic test methods, respectively. Moreover, various heteroclinic structures of waves are investigated.

1. Introduction

The existence of the homoclinic and heteroclinic orbits is very important for investigating the spatiotemporal chaotic behavior of the nonlinear evolution equations (NEEs). In recent years, exact homoclinic and heterclinic solutions were proposed for some NEEs like nonlinear Schrödinger equation, Sine-Gordon equation, Davey-Stewartson equation, Zakharov equation, and Boussinesq equation [1–7].

The coupled Schrödinger-Boussinesq equation is considered as

$$\begin{gathered} iE_t + E_{xx} + \beta_1 E - NE = 0, \\ 3N_{tt} - N_{xxxx} + 3\left(N^2\right)_{xx} + \beta_2 N_{xx} - \left(|E|^2\right)_{xx} = 0, \end{gathered} \tag{1}$$

with the periodic boundary condition

$$E(x,t) = E(x+l,t), \qquad N(x,t) = N(x+l,t), \tag{2}$$

where l, β_1, β_2 are real constants, $E(x,t)$ is a complex function, and $N(x,t)$ is a real function. Equation (1) has also appeared in [8] as a special case of general systems governing the stationary propagation of coupled nonlinear upper-hybrid and magnetosonic waves in magnetized plasma. The complete integrability of (1) was studied by Chowdhury et al. [9], and N-soliton solution, homoclinic orbit solution, and rogue solution were obtained by Hu et al. [10], Dai et al. [11–13], and Mu and Qin [14].

2. Linear Stability Analysis

It is easy to see that $(e^{i\theta_0}, \beta_1)$ is a fixed point of (1), and θ_0 is an arbitrary constant. We consider a small perturbation of the form

$$E = e^{i\theta_0}(1+\epsilon), \qquad N = \beta_1(1+\phi), \tag{3}$$

where $|\epsilon(x,t)| \ll 1$, $|\phi(x,t)| \ll 1$. Substituting (3) into (1), we get the linearized equations

$$\begin{gathered} i\epsilon_t + \epsilon_{xx} - \beta_1\phi = 0, \\ 3\phi_{tt} - \phi_{xxxx} + \left(\beta_2 + 2\beta_1^2\right)\phi_{xx} - \epsilon_{xx} - \bar{\epsilon}_{xx} = 0. \end{gathered} \tag{4}$$

Assume that ϵ and ϕ have the following forms:

$$\begin{gathered} \epsilon = Ge^{i\mu_n x+\sigma_n t} + He^{-i\mu_n x+\sigma_n t}, \\ \phi = C\left(e^{i\mu_n x+\sigma_n t} + e^{-i\mu_n x+\sigma_n t}\right), \end{gathered} \tag{5}$$

where G, H are complex constants, and C is a real number; $\mu_n = 2\pi n/l$, and σ_n is the growth rate of the nth modes.

Substituting (5) into (4), we have

$$\begin{aligned} G\left(i\sigma_n-\mu_n^2\right)&=\beta_1 C,\\ H\left(i\sigma_n-\mu_n^2\right)&=\beta_1 C,\\ \left(3\sigma_n^2-\mu_n^4-\mu_n^2\left(\beta_2+2\beta_1^2\right)\right)C&=-\left(G+\overline{H}\right)\nu_n^2,\\ \left(3\sigma_n^2-\mu_n^4-\mu_n^2\left(\beta_2+2\beta_1^2\right)\right)C&=-\left(H+\overline{G}\right)\mu_n^2. \end{aligned} \tag{6}$$

Solving (6), we obtain that

$$\sigma_n^2=\frac{\mu_n^2\left(\beta_2+2\beta_1^2\right)-2\mu_n^4\pm\sqrt{\Delta}}{6}, \tag{7}$$

with

$$\begin{aligned} \Delta&=4\mu_n^8+\mu_n^4\left(\beta_2+2\beta_1^2\right)^2-4\mu_n^6\left(\beta_2+2\beta_1^2\right)\\ &\quad+12\mu_n^4\left(\mu_n^4+\mu_n^2\left(\beta_2+2\beta_1^2\right)-2\beta_1\right). \end{aligned} \tag{8}$$

Obviously, (7) implies that $\mu_n^2(\beta_2+2\beta_1^2)-2\mu_n^4>0$; then,

$$\mu_n^2<\frac{\beta_2+2\beta_1^2}{2}. \tag{9}$$

3. Various Heterclinic Breather Solutions

Set

$$E(x,t)=e^{-iat}u(x,t),\qquad N(x,t)=v_0+v(x,t). \tag{10}$$

Substituting (10) into (1), we get

$$\begin{aligned} &iu_t+u_{xx}+(a+\beta_1-v_0)u=uv,\\ &3v_{tt}-v_{xxxx}+(6v_0+\beta_2)v_{xx}+3\left(v^2\right)_x x=\left(|u|^2\right)_{xx}. \end{aligned} \tag{11}$$

We can choose a, v_0 such that $a+\beta_1-v_0=0$.

By using the following transformation

$$u=\frac{g(x,t)}{f(x,t)},\qquad v=-2(\ln f(x,t))_{xx}. \tag{12}$$

Equation (11) can be reduced into the following bilinear form:

$$\begin{aligned} &\left(iD_t+D_x^2\right)g\cdot f=0,\\ &\left(3D_t^2+(6v_0+\beta_2)D_x^2-D_x^4-\lambda\right)f\cdot f+gg^*=0, \end{aligned} \tag{13}$$

where $g(x,t)$ is an unknown complex function and $f(x,t)$ is a real function, g^* is conjugate function of $g(x,t)$, and λ is an integration constant. The Hirota bilinear operators $D_x^m D_t^n$ are defined by

$$\begin{aligned} &D_x^m D_t^n f(x,t)\cdot g(x,t)\\ &=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\left[f(x,t)\,g\left(x',t'\right)\right]_{x'=x,t'=t}. \end{aligned} \tag{14}$$

We use three test functions to investigate the variation of the heterclinic solution for the coupled Schrödinger-Boussinesq equation (1). (1) We seek the following forms of the heterclinic solution:

$$\begin{aligned} g&=1+b_1\cos(px)e^{\Omega t+\gamma}+b_2e^{2\Omega t+2\gamma},\\ f&=1+b_3\cos(px)e^{\Omega t+\gamma}+b_4e^{2\Omega t+2\gamma}, \end{aligned} \tag{15}$$

where b_1, b_2 are complex numbers and b_3, b_4 are real numbers. b_i $(i=1,2,3,4)$, p, Ω, γ will be determined later.

Choosing $v_0=\beta_1$, then $a=0$. Substituting (15) into the (13), we have the following relations among these constants:

$$\begin{aligned} &\lambda=1,\qquad b_1=\frac{i\Omega+p^2}{i\Omega-p^2}b_3,\\ &b_2=\left(\frac{i\Omega+p^2}{i\Omega-p^2}\right)^2 b_4,\qquad b_4=\frac{\Omega^2+p^4}{4\Omega^2}b_3^2,\\ &\left(3\Omega^2-p^4-(6\beta_1+\beta_2)p^2\right)\left(\Omega^2+p^4\right)=2p^4. \end{aligned} \tag{16}$$

Therefore, we have the heterclinic solution for (1) as:

$$\begin{aligned} E(x,t)&=\frac{e^{\Omega t+\gamma}+b_1\cos(px)+b_2e^{\Omega t+\gamma}}{\sqrt{b_4}\left(2\cosh\left(\Omega t+\gamma+\ln\sqrt{b_4}\right)+b_3\cos(px)\right)},\\ N(x,t)&\\ &=\beta_1+\frac{2b_3p^2\left(2\sqrt{b_4}\cos(px)\cosh\left(\Omega t+\gamma+\ln\sqrt{b_4}\right)+b_3\right)}{b_4\left(2\cosh\left(\Omega t+\gamma+\ln\sqrt{b_4}\right)+b_3\cos(px)\right)^2}. \end{aligned} \tag{17}$$

It is easy to see that $(E,N)\to(1,\beta_1)$ as $t\to-\infty$ and $(E,N)\to(((i\Omega+p^2)/(i\Omega-p^2))^2,\beta_1)$ as $t\to+\infty$. After giving some constants in (17), we find that the shape of the heterclinic orbit for Schrödinger-Boussinesq equation likes the hook, and the orbits are heterclinic to two different fixed points (see Figure 1 with $\beta_1=1$, $\beta_2=-2$, $p=1$, and $\gamma=1$).

(2) We take ansatz of extended homoclinic test approach for (13) as follows:

$$\begin{aligned} f(x,t)&=e^{-p_1(x-\alpha t)-\eta_0}+b_3\cos\left(p(x+\alpha t)+\eta_1\right)\\ &\quad+b_4e^{p_1(x-\alpha t)+\eta_0},\\ g(x,t)&=e^{-i\theta}\left(e^{-p_1(x-\alpha t)-\eta_0}+b_1\cos\left(p(x+\alpha t)+\eta_1\right)\right.\\ &\quad\left.+b_2e^{p_1(x-\alpha t)+\eta_0}\right), \end{aligned} \tag{18}$$

where the parameters $p, p_1, \alpha, \eta_0, \eta_1, b_s$ $(s=1,2,3,4)$ will be determined later, b_1 and b_2 are complex numbers, and b_3 and b_4 are real numbers. Substituting (18) into (13) and choosing $v_0=\beta_1$, we get the following relations among the parameters:

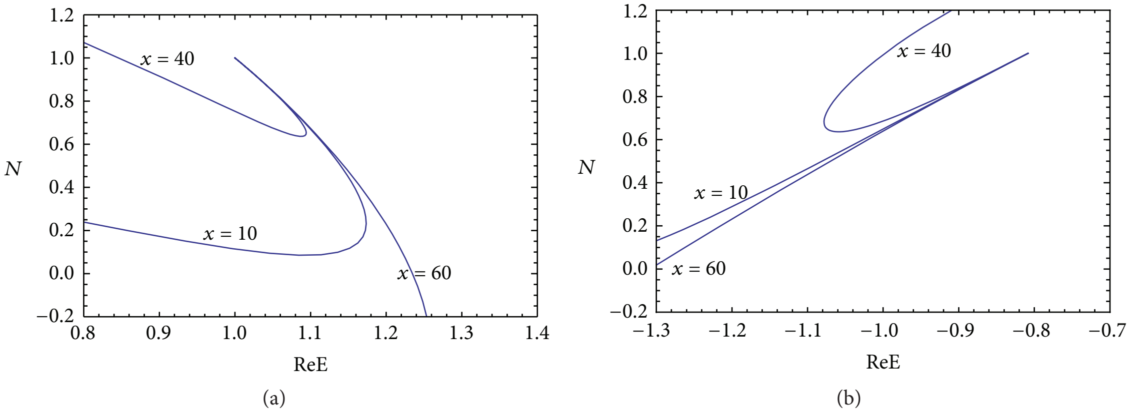

FIGURE 1: Hook heteroclinic orbits for Schrödinger-Boussinesq equation as $t \to -\infty$ (a) and $t \to +\infty$ (b).

$$p^2 = 3p_1^2, \qquad \lambda = 1,$$
$$p_1^2 = \frac{3}{4}\alpha^2 - \frac{1}{4}\beta_2 - \frac{3}{2}\beta_1, \qquad \alpha^2 = \frac{(\beta_2 + 6\beta_1)^2 - 2}{4(\beta_2 + 6\beta_1)},$$
$$b_1 = \frac{b_3(i\alpha - 2p_1)}{i\alpha + 2p_1}, \qquad b_2 = \frac{b_4(i\alpha - 2p_1)^2}{(i\alpha + 2p_1)^2}, \tag{19}$$
$$b_3 = \pm\frac{2p_1\sqrt{(3\alpha^2 - 4p_1^2)b_4}}{p\sqrt{\alpha^2 + 4p_1^2}}.$$

From (19), we get the restrictive conditions with

$$-\sqrt{2} < \beta_2 + 6\beta_1 < 0, \qquad b_4 < 0. \tag{20}$$

Denote that $(i\alpha - 2p_1)/(i\alpha + 2p_1) = e^{i\theta_0}$. Then, substituting (10) into (1) and employing (19), we obtain the solution of the coupled Schrödinger-Boussinesq equation as follows:

$$E(x,t) = e^{i(\theta_0 - \theta)}\frac{2\sqrt{-b_4}\sinh\left(p_1(x-\alpha t) + \eta_0 + \ln\left(\sqrt{-b_4}\right) + i\theta_0\right) - b_3\cos\left(p(x+\alpha t) + \eta_1\right)}{2\sqrt{-b_4}\sinh\left(p_1(x-\alpha t) + \eta_0 + \ln\left(\sqrt{-b_4}\right)\right) - b_3\cos\left(p(x+\alpha t) + \eta_1\right)},$$
$$N(x,t) = \beta_1 - \frac{8\sqrt{-b_4}b_3p_1^2\sinh\left(p_1(x-\alpha t) + \eta_0 + \ln\left(\sqrt{-b_4}\right)\right)\cos\left(p(x+\alpha t) + \eta_1\right)}{\left(2\sqrt{-b_4}\sinh\left(p_1(x-\alpha t) + \eta_0 + \ln\left(\sqrt{-b_4}\right)\right) - b_3\cos\left(p(x+\alpha t) + \eta_1\right)\right)^2} \tag{21}$$
$$- \frac{2\left(-4\sqrt{-b_4}pp_1b_3\cosh\left(p_1(x-\alpha t) + \eta_0 + \ln\sqrt{-b_4}\right)\sin\left(p(x+\alpha t) + \eta_1\right) + \left(4b_4 - 3b_3^2\right)p_1^2\right)}{\left(2\sqrt{-b_4}\sinh\left(p_1(x-\alpha t) + \eta_0 + \ln\left(\sqrt{-b_4}\right)\right) - b_3\cos\left(p(x+\alpha t) + \eta_1\right)\right)^2},$$

where η_0, η_1 are arbitrary numbers.

Solution in (21) is a heteroclinic breather wave solution. It is easy to see that $(E, N) \to (e^{-i(\theta+2\theta_0)}, \beta_1)$ as $t \to -\infty$ and $(E, N) \to (e^{-i\theta}, \beta_1)$ as $t \to +\infty$. Given some constants in (21), this kind of the heterclinic orbit likes a spiral, and it is heterclinic to the points $(e^{-i(\theta+2\theta_0)}, \beta_1)$ and $(e^{-i\theta}, \beta_1)$ (see Figure 2 with $\beta_1 = -1.5$, $\beta_2 = 8$, and $b_4 = -4$).

Note that $(e^{-i(\theta+2\theta_0)}, \beta_1)$ and $(e^{-i\theta}, \beta_1)$ are two different fixed points of (21), which is a heteroclinic solution (see Figure 3). This wave also contains the periodic wave, and its amplitude periodically oscillates with the evolution of time, which shows that this wave has breather effect. The previous results combined with (21) show that interaction between a solitary wave and a periodic wave with the same velocity α and opposite propagation direction can form a heteroclinic breather flow. This is a new phenomenon of physics in the stationary propagation of coupled nonlinear upper-hybrid and magnetosonic waves in magnetized plasma.

(3) Use the following forms of the heterclinic solution [14]:

$$g = b_1\cosh(\alpha t) + b_2\cos(px) + b_3\sinh(\alpha t),$$
$$f = b_4\cosh(\alpha t) + b_5\cos(px), \tag{22}$$

where b_1, b_2, b_3 are complex numbers and b_4, b_5 are real numbers. b_i $(i = 1, 2, 3, 4, 5)$, p, α will be determined later.

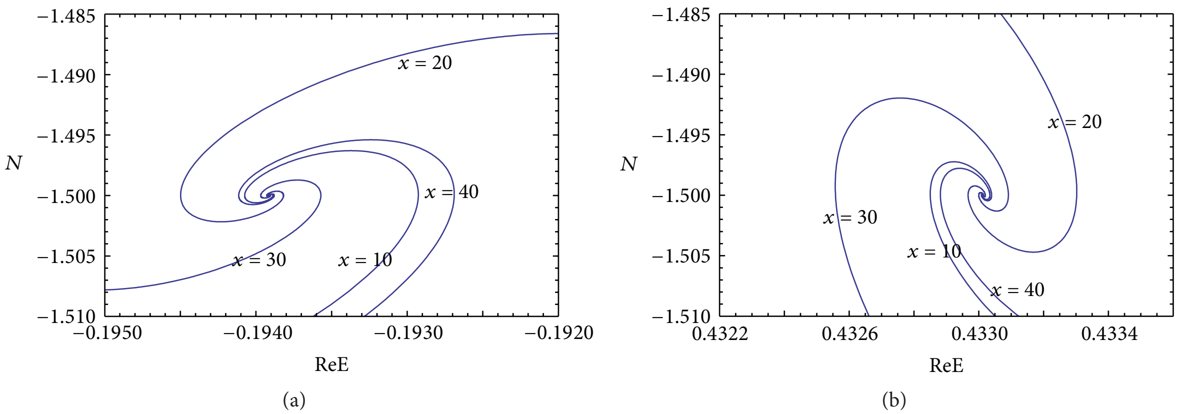

FIGURE 2: Spiral heteroclinic orbits for Schrödinger-Boussinesq equation as $t \to -\infty$ (a) and $t \to +\infty$ (b).

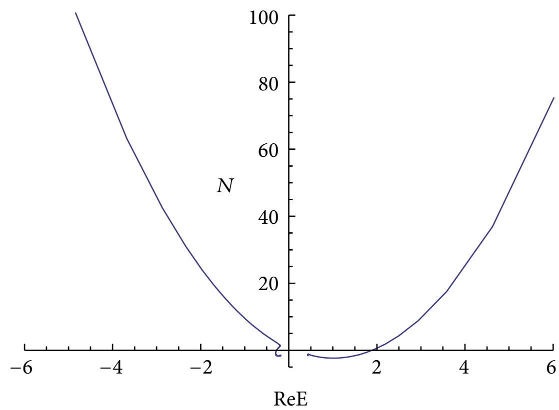

FIGURE 3: One heteroclinic orbit for Schrödinger-Boussinesq equation as $x = 0$.

We also choose $v_0 = \beta_1$ and substitute (22) into (13). We have the following relations among these constants:

$$\begin{gathered} i b_3 b_4 \alpha = b_2 b_5 p^2, \\ b_5 \left(b_1 + b_3\right)\left(i\alpha - p^2\right) = b_2 b_4 \left(i\alpha + p^2\right), \\ b_2 b_4 \left(i\alpha - p^2\right) = b_5 \left(b_1 - b_3\right)\left(i\alpha + p^2\right), \\ -b_4^2 + 12\alpha^2 b_4^2 - 2b_5^2 \cos^2\left(px\right) - 16 b_5^2 p^4 - 4 b_5^2 p^2 \left(6\beta_1 + \beta_2\right) \\ + b_1 b_1^* - b_3 b_3^* + 2 b_2 b_2^* \cos^2\left(px\right) = 0. \end{gathered} \tag{23}$$

Solving (23), we get

$$\begin{gathered} b_1 = \frac{\left(p^4 - \alpha^2\right) b_2}{\alpha\sqrt{2\left(\alpha^2 + p^4\right)}}, \qquad b_3 = \pm i \frac{\sqrt{2} p^2 b_2}{\sqrt{\alpha^2 + p^4}}, \\ b_4^2 = \frac{\left(\alpha^2 + p^4\right) b_5^2}{2\alpha^2}. \end{gathered} \tag{24}$$

Therefore, we have the heterclinic solution for (1) as

$$\begin{gathered} E\left(x,t\right) = \frac{b_1 \cosh\left(\alpha t\right) + b_2 \cos\left(px\right) + b_3 \sinh\left(\alpha t\right)}{b_4 \cosh\left(\alpha t\right) + b_5 \cos\left(px\right)}, \\ N\left(x,t\right) = \beta_1 + 2\frac{b_5 p^2 \left(b_4 \cos\left(px\right)\cosh\left(\alpha t\right) + b_5\right)}{\left(b_4 \cosh\left(\alpha t\right) + b_5 \cos\left(px\right)\right)^2}. \end{gathered} \tag{25}$$

Giving some special parameters in (25), we see that the shape of the heterclinic orbits likes the arc (see Figure 4 with $\beta_1 = 1$, $\alpha = \sqrt{3}$, and $p = \sqrt{2}$). The fixed points are $(E, N) \to ((b_1 - b_3)/b_4, \beta_1)$ as $t \to -\infty$ and $(E, N) \to ((b_1 + b_3)/b_4, \beta_1)$ as $t \to +\infty$.

4. Conclusion

In this work, by using three special test functions in two-soliton method and homoclinic test method, we obtain three families of heteroclinic breather wave solution heteroclinic to two different fixed points, respectively. Moreover, we investigate different structures of these wave solutions. These results show that the Schrödinger-Boussinesq equation has the variety of heteroclinic structure. As the further work, we

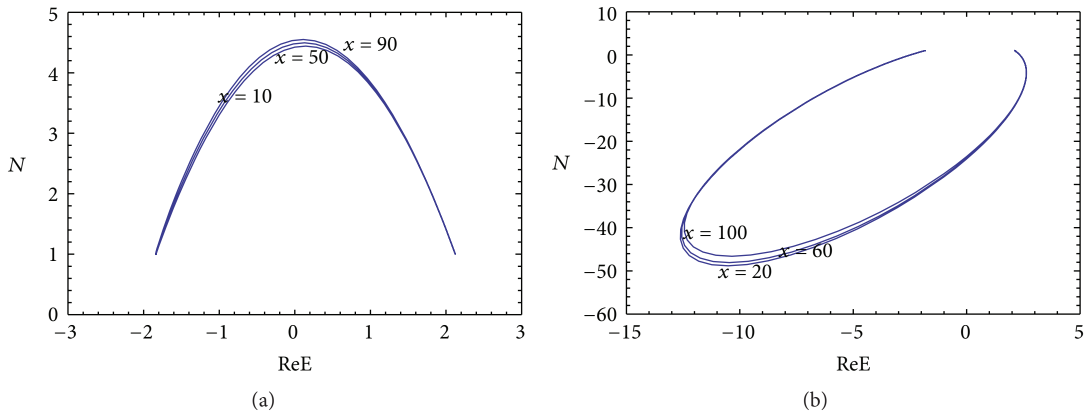

Figure 4: Arc Heteroclinic orbit for Schrödinger-Boussinesq equation as $t \rightarrow \pm\infty$ at $x = 10 * (2k + 1)$ (a) and $x = 10 * (4k + 2)$ (b), where $k = 0, 1, 2, \ldots$.

will consider whether there exist the spatiotemporal chaos for the coupled Schrödinger-Boussinesq equation or not.

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