Research Article

On the Stability of Trigonometric Functional Equations in Distributions and Hyperfunctions

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We consider the Hyers-Ulam stability for a class of trigonometric functional equations in the spaces of generalized functions such as Schwartz distributions and Gelfand hyperfunctions.

1. Introduction

Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed the following question [1].

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d\left(f\left(xy\right), f\left(x\right) f\left(y\right)\right) \leq \epsilon. \tag{1}$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d\left(f\left(x\right), h\left(x\right)\right) \leq \delta_\epsilon \tag{2}$$

for all $x \in G_1$?

This problem was solved affirmatively by Hyers [2] under the assumption that G_2 is a Banach space. After the result of Hyers, Aoki [3] and Bourgin [4, 5] treated with this problem; however, there were no other results on this problem until 1978 when Rassias [6] treated again with the inequality of Aoki [3]. Generalizing Hyers' result, he proved that if a mapping $f : X \rightarrow Y$ between two Banach spaces satisfies

$$\left\| f\left(x + y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \Phi\left(x, y\right), \quad \text{for } x, y \in X \tag{3}$$

with $\Phi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ ($\epsilon \geq 0$, $0 \leq p < 1$), then there exists a unique additive function $A : X \rightarrow Y$ such

that $\|f(x) - A(x)\| \leq 2\epsilon|x|^p/(2 - 2^p)$ for all $x \in X$. In 1951 Bourgin [4, 5] stated that if Φ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j < \infty$ for each $x \in X$, then there exists a unique additive function $A : X \rightarrow Y$ such that $\|f(x) - A(x)\| \leq \sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j$ for all $x \in X$. Unfortunately, there was no use of these results until 1978 when Rassias [7] treated with the inequality of Aoki [3]. Following Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–10, 10–25]. In 1990 Székelyhidi [24] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. We refer the reader to [9, 10, 18, 19, 25] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [24] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional inequalities

$$
\begin{aligned}
\left| f\left(x - y\right) - f\left(x\right) g\left(y\right) + g\left(x\right) f\left(y\right)\right| &\leq \psi\left(y\right), \\
\left| g\left(x - y\right) - g\left(x\right) g\left(y\right) - f\left(x\right) f\left(y\right)\right| &\leq \psi\left(y\right),
\end{aligned}
\tag{4}
$$

where $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ and $\psi : \mathbb{R}^n \rightarrow [0, \infty)$ is a continuous function. As a distributional version of the inequalities (4), we

consider the inequalities for the generalized functions $u, v \in \mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$),

$$\left\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \right\| \leq \psi(y),$$

$$\left\| v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y \right\| \leq \psi(y), \quad (5)$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, and $\psi : \mathbb{R}^n \rightarrow [0, \infty)$ denotes a continuous infraexponential function of order 2 (resp., a function of polynomial growth). For the proof we employ the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^n, \ t > 0. \quad (6)$$

For the first step, convolving $E_t(x)E_s(y)$ in both sides of (5) we convert (5) to the Hyers-Ulam stability problems of *trigonometric-hyperbolic type* functional inequalities, respectively,

$$\left| U(x - y, t + s) - U(x, t)V(y, s) + V(x, t)U(y, s) \right|$$

$$\leq \Psi(y, s),$$

$$\left| V(x - y, t + s) - V(x, t)V(y, s) - U(x, t)U(y, s) \right|$$

$$\leq \Psi(y, s), \quad (7)$$

for all $x, y \in \mathbb{R}^n, t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, given by

$$U(x, t) = u * E_t(x) = \left\langle u_y, E_t(x - y) \right\rangle, \quad (8)$$

$$V(x, t) = v * E_t(x), \quad (9)$$

which are solutions of the heat equation, and

$$\Psi(y, s) = \int \psi(\eta) E_s(\eta - y) d\eta = (\psi * E_s)(y). \quad (10)$$

For the second step, using similar idea of Székelyhidi [24] we prove the Hyers-Ulam stabilities of inequalities (7). For the final step, taking initial values as $t \rightarrow 0^+$ for the results we arrive at our results.

2. Generalized Functions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions (see [26–29] for more details of these spaces). We use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n!, |x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 1 (see [29]). One denotes by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \quad (11)$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\| \cdot \|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions, and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.

Definition 2 (see [26]). One denotes by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n} \frac{\left| x^\alpha \partial^\beta \varphi(x) \right|}{h^{|\alpha|} k^{|\beta|} \alpha!^{1/2} \beta!^{1/2}} < \infty \quad (12)$$

for some $h, k > 0$. One says that $\varphi_j \rightarrow 0$ as $j \rightarrow \infty$ if $\|\varphi_j\|_{h,k} \rightarrow 0$ as $j \rightarrow \infty$ for some h, k, and one denotes by $\mathcal{G}'$ the dual space of $\mathcal{G}$ and calls its elements Gelfand hyperfunctions.

It is well known that the following topological inclusions hold:

$$\mathcal{G} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{G}'. \quad (13)$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$\left| \varphi(x + iy) \right| \leq C \exp\left(-a|x|^2 + b|y|^2\right), \quad x, y \in \mathbb{R}^n \quad (14)$$

for some a, b, and $C > 0$ (see [26]).

By virtue of Theorem 6.12 of [27, p. 134] we have the following.

Definition 3. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$, with $n_1 \geq n_2$, and let $\lambda : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ be a smooth function such that, for each $x \in \mathbb{R}^{n_1}$, the Jacobian matrix $\nabla\lambda(x)$ of λ at x has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{G}'(\mathbb{R}^{n_2}) \rightarrow \mathcal{G}'(\mathbb{R}^{n_1})$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. One calls $\lambda^* u$ the pullback of u by λ which is often denoted by $u \circ \lambda$.

In particular, let $\lambda : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be defined by $\lambda(x, y) = x - y, x, y \in \mathbb{R}^n$. Then in view of the proof of Theorem 6.12 of [27, p. 134] we have

$$\left\langle u \circ \lambda, \varphi(x, y) \right\rangle = \left\langle u, \int \varphi(x - y, y) dy \right\rangle. \quad (15)$$

Definition 4. Let $u_x \in \mathcal{G}'(\mathbb{R}^{n_1}), u_y \in \mathcal{G}'(\mathbb{R}^{n_2})$. Then the tensor product $u_x \otimes u_y$ of u_x and u_y, defined by

$$\left\langle u_x \otimes u_y, \varphi(x, y) \right\rangle = \left\langle u_x, \left\langle u_y, \varphi(x, y) \right\rangle \right\rangle \quad (16)$$

for $\varphi(x, y) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

For more details of pullback and tensor product of distributions we refer the reader to Chapter V-VI of [27].

3. Main Theorems

Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Then f is said to be an *infraexponential function of order 2 (resp.,*

a *function of polynomial growth*) if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ (resp., there exist positive constants C, N, and d) such that

$$|f(x)| \leq C_\epsilon e^{\epsilon |x|^2} \quad \left[\text{resp.} \leq C|x|^N + d\right] \tag{17}$$

for all $x \in \mathbb{R}^n$. It is easy to see that every infraexponential function f of order 2 (resp., every function of polynomial growth) defines an element of $\mathscr{G}'(\mathbb{R}^n)$ (resp., $\mathscr{S}'(\mathbb{R}^n)$) via the correspondence

$$\langle f, \varphi \rangle = \int f(x) \varphi(x)\, dx \tag{18}$$

for $\varphi \in \mathscr{G}(\mathbb{R}^n)$ (resp. $\mathscr{S}(\mathbb{R}^n)$).

Let $u, v \in \mathscr{G}'(\mathbb{R}^n)$ (resp., $\mathscr{S}'(\mathbb{R}^n)$). We prove the stability of the following functional inequalities:

$$\left\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \right\| \leq \psi(y), \tag{19}$$

$$\left\| v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y \right\| \leq \psi(y), \tag{20}$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, $\psi : \mathbb{R}^n \to [0, \infty)$ denotes a continuous infraexponential functional of order 2 (resp. a continuous function of polynomial growth) with $\psi(0) = 0$, and $\| \cdot \| \leq \psi$ means that $|\langle \cdot, \varphi \rangle| \leq \|\psi\varphi\|_{L^1}$ for all $\varphi \in \mathscr{G}(\mathbb{R}^n)$ (resp., $\mathscr{S}(\mathbb{R}^n)$).

In view of (14) it is easy to see that the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad t > 0, \tag{21}$$

belongs to the Gelfand space $\mathscr{G}(\mathbb{R}^n)$ for each $t > 0$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x - y)\rangle$ is well defined for all $u \in \mathscr{G}'(\mathbb{R}^n)$. It is well known that $U(x, t) = (u * E_t)(x)$ is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x, t) : x \in \mathbb{R}^n, t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions that is, for every $\varphi \in \mathscr{G}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int (u * E_t)(x) \varphi(x)\, dx. \tag{22}$$

We call $(u * E_t)(x)$ the Gauss transform of u.

A function A from a semigroup $\langle S, +\rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *an additive function* provided that $A(x + y) = A(x) + A(y)$, and $m : S \to \mathbb{C}$ is said to be *an exponential function* provided that $m(x + y) = m(x)m(y)$.

For the proof of stabilities of (19) and (20) we need the following.

Lemma 5 (see [15]). *Let S be a semigroup and $\mathbb{C}$ the field of complex numbers. Assume that $f, g : S \to \mathbb{C}$ satisfy the inequality; for each $y \in S$ there exists a positive constant M_y such that*

$$\left| f(x + y) - f(x) g(y) \right| \leq M_y \tag{23}$$

for all $x \in S$. Then either f is a bounded function or g is an exponential function.

Proof. Suppose that g is not exponential. Then there are $y, z \in S$ such that $g(y + z) \neq g(y)g(z)$. Now we have

$$
\begin{aligned}
f(x + y + z) &- f(x + y) g(z) \\
&= (f(x + y + z) - f(x) g(y + z)) \\
&\quad - g(z)(f(x + y) - f(x) g(y)) \\
&\quad + f(x)(g(y + z) - g(y) g(z)),
\end{aligned}
\tag{24}
$$

and hence

$$
\begin{aligned}
f(x) = {}& (g(y + z) - g(y) g(z))^{-1} \\
&\times ((f(x + y + z) - f(x + y) g(z)) \\
&\quad - (f(x + y + z) - f(x) g(y + z)) \\
&\quad + g(z)(f(x + y) - f(x) g(y))).
\end{aligned}
\tag{25}
$$

In view of (23) the right hand side of (25) is bounded as a function of x. Consequently, f is bounded. $\square$

Lemma 6 (see [30, p. 122]). *Let $f(x, t)$ be a solution of the heat equation. Then $f(x, t)$ satisfies*

$$\left| f(x, t) \right| \leq M, \quad x \in \mathbb{R}^n, \ t \in (0, 1) \tag{26}$$

for some $M > 0$, if and only if

$$f(x, t) = (f_0 * E_t)(x) = \int f_0(y) E_t(x - y)\, dy \tag{27}$$

for some bounded measurable function f_0 defined in $\mathbb{R}^n$. In particular, $f(x, t) \to f_0(x)$ in $\mathscr{G}'(\mathbb{R}^n)$ as $t \to 0^+$.

We discuss the solutions of the corresponding trigonometric functional equations

$$u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y = 0, \tag{28}$$

$$v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0, \tag{29}$$

in the space $\mathscr{G}'$ of Gelfand hyperfunctions. As a consequence of the results [8, 31, 32] we have the following.

Lemma 7. *The solutions $u, v \in \mathscr{G}'(\mathbb{R}^n)$ of (28) and (29) are equal, respectively, to the continuous solutions $f, g : \mathbb{R}^n \to \mathbb{C}$ of corresponding classical functional equations*

$$f(x - y) - f(x) g(y) + g(x) f(y) = 0, \tag{30}$$

$$g(x - y) - g(x) g(y) - f(x) f(y) = 0. \tag{31}$$

The continuous solutions (f, g) of the functional equation (30) are given by one of the following:

(i) $f = 0$ *and g is arbitrary,*

(ii) $f(x) = c_1 \cdot x$, $g(x) = 1 + c_2 \cdot x$ *for some $c_1, c_2 \in \mathbb{C}^n$,*

(iii) $f(x) = \lambda_1 \sin(c \cdot x)$ *and $g(x) = \cos(c \cdot x) + \lambda_2 \sin(c \cdot x)$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, $c \in \mathbb{C}^n$.*

Also, the continuous solutions (f, g) of the functional equation (31) are given by one of the following:

(i) $g(x) = \lambda$ and $f(x) = \pm\sqrt{\lambda - \lambda^2}$ *for some* $\lambda \in \mathbb{C}$,

(ii) $g(x) = \cos(c \cdot x)$ and $f(x) = \sin(c \cdot x)$ *for some* $c \in \mathbb{C}^n$.

For the proof of the stability of (19) we need the followings.

Lemma 8. *Let G be an Abelian group and let $U, V : G \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a nonnegative function $\Psi : G \times (0, \infty) \to \mathbb{R}$ such that*

$$\begin{aligned}
&|U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s)| \\
&\leq \Psi(y, s)
\end{aligned} \tag{32}$$

for all $x, y \in G$, $t, s > 0$. Then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M, \tag{33}$$

or else

$$U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) = 0 \tag{34}$$

for all $x, y \in G$, $t, s > 0$.

Proof. Suppose that inequality (33) holds only when $\lambda_1 = \lambda_2 = 0$. Let

$$\begin{aligned}
K(x, y, t, s) = {}& U(x + y, t + s) - U(x, t) V(-y, s) \\
&+ V(x, t) U(-y, s),
\end{aligned} \tag{35}$$

and choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily calculated that

$$\begin{aligned}
V(x, t) = {}& \lambda_0 U(x, t) + \lambda_1 U(x + y_1, t + s_1) \\
&- \lambda_1 K(x, y_1, t, s_1),
\end{aligned} \tag{36}$$

where $\lambda_0 = V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = -1/U(-y_1, s_1)$. By (35) we have

$$\begin{aligned}
U(x + (y + z), t + (s + r)) = {}& U(x, t) V(-y - z, s + r) \\
&- V(x, t) U(-y - z, s + r) \\
&+ K(x, y + z, t, s + r).
\end{aligned} \tag{37}$$

Also by (35) and (36) we have

$$\begin{aligned}
&U((x + y) + z, (t + s) + r) \\
&= U(x + y, t + s) V(-z, r) - V(x + y, t + s) U(-z, r) \\
&\quad + K(x + y, z, t + s, r) \\
&= (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\
&\qquad + K(x, y, t, s)) V(-z, r) \\
&\quad - (\lambda_0 U(x + y, t + s) + \lambda_1 U(x + y + y_1, t + s + s_1) \\
&\qquad - \lambda_1 K(x + y, y_1, t + s, s_1)) U(-z, r) \\
&\quad + K(x + y, z, t + s, r) \\
&= (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\
&\qquad + K(x, y, t, s)) V(-z, r) \\
&\quad - \lambda_0 (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\
&\qquad + K(x, y, t, s)) U(-z, r) \\
&\quad - \lambda_1 (U(x, t) V(-y - y_1, s + s_1) \\
&\qquad - V(x, t) U(-y - y_1, s + s_1) \\
&\qquad + K(x, y + y_1, t, s + s_1)) U(-z, r) \\
&\quad + \lambda_1 K(x + y, y_1, t + s, s_1) U(-z, r) \\
&\quad + K(x + y, z, t + s, r).
\end{aligned} \tag{38}$$

From (37) and (38) we have

$$\begin{aligned}
&(V(-y, s) V(-z, r) - \lambda_0 V(-y, s) U(-z, r) \\
&\quad - \lambda_1 V(-y - y_1, s + s_1) U(-z, r) \\
&\quad - V(-y - z, s + r)) U(x, t) \\
&\quad + (-U(-y, s) V(-z, r) + \lambda_0 U(-y, s) U(-z, r) \\
&\qquad + \lambda_1 U(-y - y_1, s + s_1) U(-z, r) \\
&\qquad + U(-y - z, s + r)) V(x, t) \\
&= -K(x, y, t, s) V(-z, r) + \lambda_0 K(x, y, t, s) U(-z, r) \\
&\quad + \lambda_1 K(x, y + y_1, t, s + s_1) U(-z, r) \\
&\quad - \lambda_1 K(x + y, y_1, t + s, s_1) U(-z, r) \\
&\quad - K(x + y, z, t + s, r) + K(x, y + z, t, s + r).
\end{aligned} \tag{39}$$

Since $K(x, y, t, s)$ is bounded by $\Psi(-y, s)$, if we fix y, z, r, and s, the right hand side of (39) is bounded by a constant M, where

$$
\begin{aligned}
M = {} & \Psi(-y, s) |V(-z, r)| + \Psi(-y, s) |\lambda_0 U(-z, r)| \\
& + \Psi(-y - y_1, s + s_1) |\lambda_1 U(-z, r)| \\
& + \Psi(-y_1, s_1) |\lambda_1 U(-z, r)| + \Psi(-z, r) \\
& + \Psi(-y - z, r + s).
\end{aligned}
\tag{40}
$$

So by our assumption, the left hand side of (39) vanishes, so is the right hand side. Thus we have

$$
\begin{aligned}
& \left(-\lambda_0 K(x, y, t, s) - \lambda_1 K(x, y + y_1, t, s + s_1) \right. \\
& \left. \quad + \lambda_1 K(x + y, y_1, t + s, s_1) \right) U(-z, r) \\
& + K(x, y, t, s) V(-z, r) = K(x, y + z, t, s + r) \\
& - K(x + y, z, t + s, r).
\end{aligned}
\tag{41}
$$

Now by the definition of K we have

$$
\begin{aligned}
& K(x + y, z, t + s, r) - K(x, y + z, t, s + r) \\
& = U(x + y + z, t + s + r) - U(x + y, t + s) V(-z, r) \\
& \quad + V(x + y, t + s) U(-z, r) - U(x + y + z, t + s + r) \\
& \quad + U(x, t) V(-y - z, s + r) - V(x, t) U(-y - z, s + r) \\
& = U(-y - z - x, s + r + t) - U(-y - z, s + r) V(x, t) \\
& \quad + V(-y - z, s + r) U(x, t) - U(-z - x - y, r + t + s) \\
& \quad + U(-z, r) V(x + y, t + s) - V(-z, r) U(x + y, t + s) \\
& = K(-y - z, -x, s + r, t) - K(-z, -x - y, r, t + s).
\end{aligned}
\tag{42}
$$

Hence the left hand side of (41) is bounded by $\Psi(x, t) + \Psi(x + y, t + s)$. So if we fix x, y, t, and s in (41), the left hand side of (41) is a bounded function of z and r. Thus $K(x, y, t, s) \equiv 0$ by our assumption. This completes the proof. $\qquad \square$

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0, \infty) \to [0, \infty)$ is a continuous function such that

$$
\psi(x) := \lim_{t \to 0^+} \Psi(x, t)
\tag{43}
$$

exists and satisfies the conditions $\psi(0) = 0$ and

$$
\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi\left(-2^k x\right) < \infty
\tag{44}
$$

or

$$
\Phi_2(x) := \sum_{k=1}^{\infty} 2^k \psi\left(-2^{-k} x\right) < \infty.
\tag{45}
$$

Lemma 9. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be continuous functions satisfying*

$$
\begin{aligned}
& \left| U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) \right| \\
& \quad \leq \Psi(y, s)
\end{aligned}
\tag{46}
$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$
\left| \lambda_1 U(x, t) - \lambda_2 V(x, t) \right| \leq M.
\tag{47}
$$

Then (U, V) satisfies one of the followings:

 (i) *$U = 0$, V is arbitrary,*

 (ii) *U and V are bounded functions,*

 (iii) *$V(x, t) = \lambda U(x, t) + e^{ic \cdot x - bt}$ for some $\lambda \in \mathbb{C}^n, c(\neq 0) \in \mathbb{R}^n$, and $b \in \mathbb{C}$, and $f(x) := \lim_{t \to 0^+} U(x, t)$ is a continuous function; in particular, there exists $\delta : (0, \infty) \to [0, \infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$
\left| U(x, t) - f(x) e^{-bt} \right| \leq \delta(t)
\tag{48}
$$

for all $x \in \mathbb{R}^n$, $t > 0$, and satisfies the condition; there exists $d \geq 0$ satisfying

$$
|f(x)| \leq \psi(-x) + d
\tag{49}
$$

for all $x \in \mathbb{R}^n$,

 (iv) *$V(x, t) = \lambda U(x, t) + e^{-bt}$ for some $\lambda \in \mathbb{C}^n, b \in \mathbb{C}$, and $f(x) := \lim_{t \to 0^+} U(x, t)$ is a continuous function; in particular, there exists $\delta : (0, \infty) \to [0, \infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$
\left| U(x, t) - f(x) e^{-bt} \right| \leq \delta(t)
\tag{50}
$$

for all $x \in \mathbb{R}^n, t > 0$, and satisfies one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that

$$
\left| f(x) - a_1 \cdot x \right| \leq \Phi_1(x)
\tag{51}
$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$
\left| f(x) - a_2 \cdot x \right| \leq \Phi_2(x)
\tag{52}
$$

for all $x \in \mathbb{R}^n$.

Proof. If $U = 0$, V is arbitrary which is case (i). If U is a nontrivial bounded function, in view of (46) V is also bounded which gives case (ii). If U is unbounded, it follows from (47) that $\lambda_2 \neq 0$ and

$$
V(x, t) = \lambda U(x, t) + R(x, t)
\tag{53}
$$

for some $\lambda \in \mathbb{C}$ and a bounded function R. Putting (53) in (46) we have

$$
\begin{aligned}
& \left| U(x - y, t + s) - U(x, t) R(y, s) + R(x, t) U(y, s) \right| \\
& \quad \leq \Psi(y, s)
\end{aligned}
\tag{54}
$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Replacing y by $-y$ and using the triangle inequality, we have, for some $C > 0$,

$$\left| U(x+y, t+s) - U(x,t) R(-y,s) \right| \\ \leq C \left| U(-y,s) \right| + \Psi(-y,s) \tag{55}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. By Lemma 5, $R(-y,s)$ is an exponential function. If $R = 0$, putting $y = 0, s \to 0^+$ in (54) we have

$$\left| U(x,t) \right| \leq \psi(0) = 0. \tag{56}$$

Thus we have $R \neq 0$ since U is unbounded. Given the continuity of U and V we have

$$R(x,t) = e^{ic \cdot x - bt} \tag{57}$$

for some $c \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \geq 0$. Putting $y = 0$ and $s = 1$ in (54), dividing $R(0,1)$, and using the triangle inequality we have

$$\left| U(x,t) \right| \leq \left| R(0,1) \right|^{-1} \left(\left| U(x, t+1) \right| + C \left| U(0,1) \right| + \Psi(0,1) \right) \tag{58}$$

for all $x \in \mathbb{R}^n$, $t > 0$.

From (58) and the continuity of U it is easy to see that

$$\limsup_{t \to 0^+} U(x,t) := f(x) \tag{59}$$

exists. Putting $x = y = 0$ and replacing s and t by $t/2$ in (54) we have

$$\left| U(0,t) \right| \leq \Psi\left(0, \frac{t}{2}\right) \tag{60}$$

for all $t > 0$.

Fixing x, putting $y = 0$ letting $t \to 0^+$ so that $U(x,t) \to f(x)$ in (54), and using the triangle inequality and (60) we have

$$\left| U(x,s) - f(x) e^{-bs} \right| \leq \Psi\left(0, \frac{s}{2}\right) + \Psi(0,s) := \delta(s) \tag{61}$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (61) we have

$$\lim_{s \to 0^+} U(x,s) = f(x) \tag{62}$$

for all $x \in \mathbb{R}^n$. From (61) the continuity of f can be checked by a usual calculus. Letting $t \to 0^+$ in (60) we see that $f(0) = 0$. Letting $t, s \to 0^+$ in (54) we have

$$\left| f(x-y) - f(x) e^{ic \cdot y} + e^{ic \cdot x} f(y) \right| \leq \psi(y) \tag{63}$$

for all $x, y \in \mathbb{R}^n$. Putting $x = 0$ in (63) and replacing y by $-y$ we have

$$\left| f(-y) + f(y) \right| \leq \psi(-y) \tag{64}$$

for all $y \in \mathbb{R}^n$.

Replacing y by $-y$ and using (64) and the triangle inequality we have

$$\left| f(x+y) - f(x) e^{-ic \cdot y} - e^{ic \cdot x} f(y) \right| \leq 2\psi(-y) \tag{65}$$

for all $x, y \in \mathbb{R}^n$. Now we divide (65) into two cases: $c = 0$ and $c \neq 0$. First we consider the case $c \neq 0$. Replacing x by y and y by x in (65) we have

$$\left| f(x+y) - f(y) e^{-ic \cdot x} - e^{ic \cdot y} f(x) \right| \leq 2\psi(-x) \tag{66}$$

for all $x, y \in \mathbb{R}^n$. From (65) and (66), using the triangle inequality and dividing $|e^{ic \cdot y} - e^{-ic \cdot y}|$ we have

$$\left| f(x) \right| \leq \frac{2\left(\psi(-x) + \psi(-y) + \left| f(y) \right| \right)}{\left| e^{ic \cdot y} - e^{-ic \cdot y} \right|} \tag{67}$$

for all $x, y \in \mathbb{R}^n$ such that $c \cdot y \neq 0$. Choosing $y_0 \in \mathbb{R}^n$ so that $c \cdot y_0 = \pi/2$ and putting $y = y_0$ in (67) we have

$$\left| f(x) \right| \leq \psi(-x) + d, \tag{68}$$

where $d = \psi(\pi/2) + |f(\pi/2)|$, which gives (iii). Now we consider the case $c = 0$. It follows from (65) that

$$\left| f(x+y) - f(x) - f(y) \right| \leq 2\psi(-y) \tag{69}$$

for all $x, y \in \mathbb{R}^n$. By the well-known results in [3], there exists a unique additive function $A_1(x)$ given by

$$A_1(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \tag{70}$$

such that

$$\left| f(x) - A_1(x) \right| \leq \Phi_1(x) \tag{71}$$

if $\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi(-2^k x) < \infty$, and there exists a unique additive function $A_2(x)$ given by

$$A_2(x) = \lim_{n \to \infty} 2^n f(2^{-n} x) \tag{72}$$

such that

$$\left| f(x) - A_2(x) \right| \leq \Phi_2(x) \tag{73}$$

if $\Phi_2(x) := \sum_{k=0}^{\infty} 2^k \psi(-2^{-k} x) < \infty$. Now by the continuity of U and inequality (61), it is easy to see that f is continuous. In view of (70) and (72), $A_j(x)$, $j = 1, 2$, are Lebesgue measurable functions. Thus there exist $a_1, a_2 \in \mathbb{C}^n$ such that $A_1(x) = a_1 \cdot x$ and $A_2(x) = a_2 \cdot x$ for all $x \in \mathbb{R}^n$, which gives (iv). This completes the proof. □

In the following we assume that ψ satisfies (44) or (45).

Theorem 10. *Let $u, v \in \mathscr{G}'$ satisfy (19). Then (u, v) satisfies one of the followings:*

(i) *$u = 0$, and v is arbitrary,*

(ii) *u and v are bounded measurable functions,*

(iii) $v(x) = \lambda u(x) + e^{ic \cdot x}$ for some $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, where u is a continuous function satisfying the condition; there exists $d \geq 0$

$$|u(x)| \leq \psi(-x) + d \qquad (74)$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ for some $\lambda \in \mathbb{C}$, where u is a continuous function satisfying one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that

$$|u(x) - a_1 \cdot x| \leq \Phi_1(x) \qquad (75)$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$|u(x) - a_2 \cdot x| \leq \Phi_2(x) \qquad (76)$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$, for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Proof. Convolving in (19) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels in both sides of inequality (19) we have

$$
\begin{aligned}
[u \circ (\xi - \eta) * (E_t(\xi) E_s(\eta))] (x, y) \\
= \left\langle u_\xi, \int E_t(x - \xi - \eta) E_s(y - \eta) d\eta \right\rangle \\
= \left\langle u_\xi, (E_t * E_s)(x - y - \xi) \right\rangle \\
= \left\langle u_\xi, E_{t+s}(x - y - \xi) \right\rangle \\
= U(x - y, t + s).
\end{aligned}
\qquad (77)
$$

Similarly we have

$$
\begin{aligned}
\left[(u \otimes v) * (E_t(\xi) E_s(\eta))\right](x, y) = U(x, t) V(y, s), \\
\left[(v \otimes u) * (E_t(\xi) E_s(\eta))\right](x, y) = V(x, t) U(y, s),
\end{aligned}
\qquad (78)
$$

where U, V are the Gauss transforms of u, v, respectively. Thus we have the following inequality:

$$
\begin{aligned}
\left| U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) \right| \\
\leq \Psi(y, s)
\end{aligned}
\qquad (79)
$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where

$$
\begin{aligned}
\Psi(y, s) = \int \psi(\eta) E_t(x - \xi) E_s(y - \eta) d\xi \, d\eta \\
= \int \psi(\eta) E_s(\eta - y) d\eta = (\psi * E_s)(y).
\end{aligned}
\qquad (80)
$$

By Lemma 8 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M, \qquad (81)$$

or else U, V satisfy

$$U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) = 0 \qquad (82)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Assume that (81) holds. Applying Lemma 9, case (i) follows from (i) of Lemma 9. Using (ii) of Lemma 9, it follows from Lemma 7 the initial values u, v of $U(x, t), V(x, t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives (ii). For case (iii), it follows from (50) that, for all $\varphi \in \mathscr{G}(\mathbb{R}^n)$,

$$
\begin{aligned}
&|\langle u, \varphi \rangle - \langle f, \varphi \rangle| \\
&= \left| \lim_{t \to 0^+} \int U(x, t) \varphi(x) \, dx - \int f(x) \varphi(x) \, dx \right| \\
&= \left| \lim_{t \to 0^+} \int \left(U(x, t) - f(x) e^{-bt} \right) \varphi(x) \, dx \right| \\
&\leq \lim_{t \to 0^+} \int \left| U(x, t) - f(x) e^{-bt} \right| |\varphi(x)| \, dx \\
&\leq \lim_{t \to 0^+} \delta(t) \int |\varphi(x)| \, dx = 0.
\end{aligned}
\qquad (83)
$$

Thus we have $u = f$ in $\mathscr{G}'(\mathbb{R}^n)$. Letting $t \to 0^+$ in (iii) of Lemma 9 we get case (iii). Finally we assume that (82) holds. Letting $t, s \to 0^+$ in (82) we have

$$u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y = 0. \qquad (84)$$

By Lemma 6 the solutions of (84) satisfy (i), (iv), or (v). This completes the proof. $\qquad \square$

Let $\psi(x) = \epsilon |x|^p$, $p > 0$. Then ψ satisfies the conditions assumed in Theorem 10. In view of (44) and (45) we have

$$\Phi_1(x) = \frac{2\epsilon |x|^p}{2 - 2^p} \qquad (85)$$

if $0 < p < 1$, and

$$\Phi_2(x) = \frac{2\epsilon |x|^p}{2^p - 2} \qquad (86)$$

if $p > 1$. Thus as a direct consequence of Theorem 10 we have the following.

Corollary 11. *Let $0 < p < 1$ or $p > 1$. Suppose that $u, v \in \mathscr{G}'$ satisfy*

$$\left\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \right\| \leq \epsilon |y|^p. \qquad (87)$$

Then (u, v) satisfies one of the followings:

(i) $u = 0$, and v is arbitrary,

(ii) u and v are bounded measurable functions,

(iii) $v(x) = \lambda u(x) + e^{ic \cdot x}$ for some $\lambda \in \mathbb{C}$, $c (\neq 0) \in \mathbb{R}^n$, where u is a continuous function satisfying the condition; there exists $d \geq 0$

$$|u(x)| \leq \epsilon |x|^p + d \qquad (88)$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ for some $\lambda \in \mathbb{C}$, where u is a continuous function satisfying the conditions; there exists $a \in \mathbb{C}^n$ such that

$$|u(x) - a \cdot x| \leq \frac{2\epsilon |x|^p}{|2^p - 2|} \qquad (89)$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$, for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Now we prove the stability of (20). For the proof we need the following.

Lemma 12. Let $U, V : G \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a $\Psi : G \times (0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} &\left| V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) \right| \\ &\leq \Psi(y, s) \end{aligned} \qquad (90)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$\left| \lambda_1 U(x, t) - \lambda_2 V(x, t) \right| \leq M, \qquad (91)$$

or else

$$V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) = 0 \qquad (92)$$

for all $x, y \in G$, $t, s > 0$.

Proof. As in Lemma 9, suppose that $\lambda_1 U(x, t) - \lambda_2 V(x, t)$ is bounded only when $\lambda_1 = \lambda_2 = 0$, and let

$$\begin{aligned} L(x, y, t, s) = &V(x + y, t + s) - V(x, t) V(-y, s) \\ &- U(x, t) U(-y, s). \end{aligned} \qquad (93)$$

Since we may assume that U is nonconstant, we can choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily got that

$$\begin{aligned} U(x, t) = &\lambda_0 V(x, t) + \lambda_1 V(x + y_1, t + s_1) \\ &- \lambda_1 L(x, y_1, t, s_1), \end{aligned} \qquad (94)$$

where $\lambda_0 = -V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = 1/U(-y_1, s_1)$. From the definition of L and the use of (94), we have the following two equations:

$$\begin{aligned} &V((x + y) + z, (t + s) + r) \\ &\quad = V(x + y, t + s) V(-z, r) + U(x + y, t + s) U(-z, r) \\ &\qquad + L(x + y, z, t + s, r) \\ &\quad = (V(x, t) V(-y, s) + U(x, t) U(-y, s) \\ &\qquad + L(x, y, t, s)) V(-z, r) \\ &\qquad + (\lambda_0 V(x + y, t + s) + \lambda_1 V(x + y + y_1, t + s + s_1) \\ &\qquad\quad - \lambda_1 L(x + y, y_1, t + s, s_1)) U(-z, r) \\ &\qquad + L(x + y, z, t + s, r) \\ &\quad = (V(x, t) V(-y, s) + U(x, t) U(-y, s) \\ &\qquad + L(x, y, t, s)) V(-z, r) \\ &\qquad + \lambda_0 (V(x, t) V(-y, s) + U(x, t) U(-y, s) \\ &\qquad\quad + L(x, y, t, s)) U(-z, r) \\ &\qquad + \lambda_1 (V(x, t) V(-y - y_1, s + s_1) \\ &\qquad\quad + U(x, t) U(-y - y_1, s + s_1) \\ &\qquad\quad + L(x, y + y_1, t, s + s_1)) U(-z, r) \\ &\qquad - \lambda_1 L(x + y, y_1, t + s, s_1) U(-z, r) \\ &\qquad + L(x + y, z, t + s, r), \end{aligned} \qquad (95)$$

$$\begin{aligned} &V(x + (y + z), t + (s + r)) \\ &\quad = V(x, t) V(-y - z, s + r) + U(x, t) U(-y - z, s + r) \\ &\qquad + L(x, y + z, t, s + r). \end{aligned} \qquad (96)$$

By equating (95) and (96), we have

$$\begin{aligned} &V(x, t) (V(-y, s) V(-z, r) + \lambda_0 V(-y, s) U(-z, r) \\ &\qquad + \lambda_1 V(-y - y_1, s + s_1) U(-z, r) \\ &\qquad - V(-y - z, s + r)) \\ &\qquad + U(x, t) (U(-y, s) V(-z, r) + \lambda_0 U(-y, s) U(-z, r) \\ &\qquad\quad + \lambda_1 U(-y - y_1, s + s_1) U(-z, r) \\ &\qquad\quad - U(-y - z, s + r)) \end{aligned}$$

$$= -L(x, y, t, s) V(-z, r) - \lambda_0 L(x, y, t, s) U(-z, r)$$
$$\quad - \lambda_1 L(x, y + y_1, t, s + s_1) U(-z, r)$$
$$\quad + \lambda_1 L(x + y, y_1, t + s, s_1) U(-z, r)$$
$$\quad - L(x + y, z, t + s, r) + L(x, y + z, t, s + r).$$

$$(97)$$

In (97), when y, s, z, and r are fixed, the right hand side is bounded; so by our assumption we have

$$L(x, y, t, s) V(-z, r)$$
$$\quad + (\lambda_0 L(x, y, t, s) + \lambda_1 L(x, y + y_1, t, s + s_1)$$
$$\quad - \lambda_1 L(x + y, y_1, t + s, s_1)) U(-z, r) \quad (98)$$
$$\quad = L(x, y + z, t, s + r) - L(x + y, z, t + s, r).$$

Here, we have

$$L(x, y + z, t, s + r) - L(x + y, z, t + s, r)$$
$$\quad = V(x + y + z, t + s + r) - V(x, t) V(-y - z, s + r)$$
$$\quad - U(x, t) U(-y - z, s + r) - V(x + y + z, t + s + r)$$
$$\quad + V(x + y, t + s) V(-z, r) + U(x + y, t + s) U(-z, r)$$
$$\quad = L(-y - z, -x, s + r, t) - L(-z, -x - y, r, t + s)$$
$$\quad \leq \Psi(x, t) + \Psi(x + y, t + s).$$

$$(99)$$

Considering (98) as a function of z and r for all fixed x, y, t, and s again, we have $L(x, y, t, s) \equiv 0$. This completes the proof. □

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0, \infty) \to [0, \infty)$ is a continuous function such that

$$\psi(x) := \lim_{t \to 0^+} \Psi(x, t) \quad (100)$$

exists and satisfies the condition $\psi(0) = 0$.

Lemma 13. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be continuous functions satisfying*

$$\left| V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) \right|$$
$$\leq \Psi(y, s) \quad (101)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$\left| \lambda_1 U(x, t) - \lambda_2 V(x, t) \right| \leq M. \quad (102)$$

Then (U, V) satisfies one of the followings:

(i) *U and V are bounded functions in $\mathbb{R}^n \times (0, 1)$,*

(ii) *$\pm i U(x, t) = V(x, t) - e^{ia \cdot x - bt}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$, and $g(x) := \lim_{t \to 0^+} V(x, t)$ is a continuous function; in particular, there exists $\delta : (0, \infty) \to [0, \infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$\left| V(x, t) - g(x) e^{-bt} \right| \leq \delta(t) \quad (103)$$

for all $x \in \mathbb{R}^n$, $t > 0$, and g satisfies

$$\left| g(x) - \cos(a \cdot x) \right| \leq \frac{1}{2} \psi(x) \quad (104)$$

for all $x \in \mathbb{R}^n$.

Proof. If U is bounded, then in view of inequality (100), for each y, s, $V(x + y, t + s) - V(x, t) V(-y, s)$ is also bounded. It follows from Lemma 5 that V is (101). If V is bounded, case (i) follows. If V is a nonzero exponential function, then by the continuity of V we have

$$V(x, t) = e^{c \cdot x + bt} \quad (105)$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. Putting (105) in (101) and using the triangle inequality we have for some $d \geq 0$

$$\left| e^{c \cdot x} e^{b(t+s)} \left(e^{-c \cdot y} - e^{c \cdot y} \right) \right| \leq \Psi(y, s) + d \quad (106)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. In view of (106) it is easy to see that $c = ia$, $a \in \mathbb{R}^n$. Thus $V(x, t)$ is bounded on $\mathbb{R}^n \times (0, 1)$. If U is unbounded; then in view of (101) V is also unbounded, hence $\lambda_1 \lambda_2 \neq 0$ and

$$U(x, t) = \lambda V(x, t) + R(x, t) \quad (107)$$

for some $\lambda \neq 0$ and a bounded function R. Putting (107) in (101), replacing y by $-y$, and using the triangle inequality we have

$$\left| V(x + y, t + s) - V(x, t) \left(\left(\lambda^2 + 1 \right) V(-y, s) + \lambda R(-y, s) \right) \right|$$
$$\leq \left| \left(\lambda V(-y, s) + R(-y, s) \right) R(x, t) \right| + \Psi(-y, s).$$

$$(108)$$

From Lemma 5 we have

$$\left(\lambda^2 + 1 \right) V(y, s) + \lambda R(y, s) = m(y, s) \quad (109)$$

for some exponential function m. From (107) and (109), m is continuous, and we have

$$m(x, t) = e^{c \cdot x + bt} \quad (110)$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. If $\lambda^2 \neq -1$, we have

$$U = \frac{\lambda m + R}{\lambda^2 + 1}, \qquad V = \frac{m - \lambda R}{\lambda^2 + 1}. \quad (111)$$

Putting (111) in (101), multiplying $|\lambda^2 + 1|$ in the result, and using the triangle inequality we have, for some $d \geq 0$,

$$\left| m(x, t) \left(m(-y, s) - m(y, s) \right) \right| \leq \left| \lambda^2 + 1 \right| \Psi(y, s) + d \quad (112)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is unbounded, we have

$$m(y, s) = m(-y, s) \qquad (113)$$

for all $y \in \mathbb{R}$ and $s > 0$. Thus it follows that $m(x, t) = e^{bt}$ and that U, V are bounded in $\mathbb{R}^n \times (0, 1)$. If $\lambda^2 = -1$, we have

$$U = \pm i(V - m), \qquad (114)$$

where m is a bounded exponential function. Putting (114) in (101) we have

$$\begin{aligned} &\left| V(x - y, t + s) - V(x, t) m(y, s) - V(y, s) m(x, t) \right. \\ &\left. + m(x, t) m(y, s) \right| \leq \Psi(y, s) \end{aligned} \qquad (115)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is a bounded continuous function, we have

$$m(x, t) = e^{ia \cdot x - bt} \qquad (116)$$

for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \geq 0$.

Similarly as in the proof of Lemma 9, by (101) and the continuity of V, it is easy to see that

$$\limsup_{t \to 0^+} V(x, t) := g(x) \qquad (117)$$

exists. Putting $x = y = 0$ in (115), multiplying $|e^{bt}|$ in both sides of the result, and using the triangle inequality we have

$$\left| V(0, s) - e^{-bs} \right| \leq \left| e^{bt} \right| \left(\left| V(0, t + s) - V(0, t) e^{-bs} \right| + \Psi(0, s) \right) \qquad (118)$$

for all $t, s > 0$. Letting $s \to 0^+$ in (118) we have

$$\lim_{t \to 0^+} V(0, t) = 1. \qquad (119)$$

Putting $y = 0$, fixing x, letting $t \to 0^+$ in (115) so that $V(x, t) \to g(x)$, and using the triangle inequality we have

$$\left| V(x, s) - g(x) e^{-bs} \right| \leq \left| V(0, s) - e^{-bs} \right| + \Psi(0, s) \qquad (120)$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (120) we have

$$\lim_{s \to 0^+} V(x, s) = g(x) \qquad (121)$$

for all $x \in \mathbb{R}^n$. The continuity of g follows from (120). Letting $t, s \to 0^+$ in (115) we have

$$\left| g(x - y) - g(x) e^{ia \cdot y} - g(y) e^{ia \cdot x} + e^{ia \cdot (x + y)} \right| \leq \psi(y) \qquad (122)$$

for all $x, y \in \mathbb{R}^n$. Replacing y by x in (122) and dividing the result by $2e^{ia \cdot x}$ we have

$$\left| g(x) - \cos(a \cdot x) \right| \leq \frac{1}{2} \psi(x). \qquad (123)$$

From (114), (116), (120) and (123) we get (ii). This completes the proof. $\qquad \square$

Theorem 14. *Let $u, v \in \mathscr{G}'$ satisfy (20). Then (u, v) satisfies one of the followings:*

   (i) *u and v are bounded measurable functions,*

  (ii) *$v(x) = \cos(a \cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$\left| r(x) \right| \leq \frac{1}{2} \psi(x) \qquad (124)$$

    for all $x \in \mathbb{R}^n$,

 (iii) *$v(x) = \cos(c \cdot x)$ and $u(x) = \sin(c \cdot x)$ for some $c \in \mathbb{C}^n$.*

Proof. Similarly as in the proof of Theorem 10 convolving in (20) the tensor product $E_t(x) E_s(y)$ we obtain the inequality

$$\begin{aligned} &\left| V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) \right| \\ &\leq \Psi(y, s) \end{aligned} \qquad (125)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, and

$$\begin{aligned} \Psi(y, s) &= \int \psi(\eta) E_t(x - \xi) E_s(y - \eta) \, d\xi \, d\eta \\ &= \int \psi(\eta) E_s(\eta - y) \, d\eta = (\psi * E_s)(y). \end{aligned} \qquad (126)$$

By Lemma 12 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$\left| \lambda_1 U(x, t) - \lambda_2 V(x, t) \right| \leq M, \qquad (127)$$

or else U, V satisfy

$$V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) = 0 \qquad (128)$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Firstly we assume that (127) holds. Letting $t \to 0^+$ in (i) of Lemma 13, by Lemma 6, the initial values u, v of $U(x, t), V(x, t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives case (i). Using the same approach of the proof of case (iii) of Theorem 10, we have $v = g$ in $\mathscr{G}'$. It follows from (104) that

$$v(x) = \cos(a \cdot x) + r(x), \qquad (129)$$

where $r(x)$ is a continuous function satisfying

$$\left| r(x) \right| \leq \frac{1}{2} \psi(x) \qquad (130)$$

for all $x \in \mathbb{R}^n$. Letting $t \to 0^+$ in (ii) of Lemma 13 we have

$$\pm i u(x) = v(x) - e^{ia \cdot x}. \qquad (131)$$

Putting (129) in (131) we have

$$\pm u(x) = \sin(a \cdot x) + ir(x). \qquad (132)$$

Secondly we assume that (128) holds. Letting $t, s \to 0^+$ in (127) we have

$$v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0. \qquad (133)$$

By Lemma 7 the solution of (133) satisfies (i) or (iii). This completes the proof. □

Every infraexponential function f of order 2 defines an element of $\mathscr{G}'(\mathbb{R}^n)$ via the correspondence

$$\langle f, \varphi \rangle = \int f(x) \varphi(x) \, dx \qquad (134)$$

for $\varphi \in \mathscr{G}$. Thus as a direct consequence of Corollary 11 and Theorem 14 we have the followings.

Corollary 15. *Let $0 < p < 1$ or $p > 1$. Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$\left| f(x - y) - f(x) g(y) + g(x) f(y) \right| \leq \epsilon |x|^p \qquad (135)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the following:

   (i) *$f(x) = 0$, almost everywhere $x \in \mathbb{R}^n$, and g is arbitrary,*

  (ii) *f and g are bounded in almost everywhere,*

 (iii) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + e^{ic \cdot x}$ for almost everywhere $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, and f_0 is a continuous function satisfying the condition; there exists $d \geq 0$*

$$\left| f_0(x) \right| \leq \epsilon |x|^p + d \qquad (136)$$

  for all $x \in \mathbb{R}^n$,

 (iv) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + 1$ for a.e. $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$ and f_0 is a continuous function satisfying the condition; there exists $a \in \mathbb{C}^n$ such that*

$$\left| f_0(x) - a \cdot x \right| \leq \frac{2\epsilon |x|^p}{|2^p - 2|} \qquad (137)$$

  for all $x \in \mathbb{R}^n$,

  (v) *$f(x) = \lambda \sin(c \cdot x)$, $g(x) = \cos(c \cdot x) + \lambda \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.*

Corollary 16. *Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$\left| g(x - y) - g(x) g(y) - f(x) f(y) \right| \leq \epsilon |y|^p \qquad (138)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the followings:

   (i) *f and g are bounded in almost everywhere,*

  (ii) *there exists $a \in \mathbb{R}^n$ such that*

$$\left| g(x) - \cos(a \cdot x) \right| \leq \frac{1}{2} \epsilon |x|^p, \qquad (139)$$

$$\left| f(x) \pm \sin(a \cdot x) \right| \leq \frac{1}{2} \epsilon |x|^p \qquad (140)$$

  for almost every $x \in \mathbb{R}^n$,

 (iii) *$g(x) = \cos(c \cdot x)$ and $f(x) = \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$.*

Remark 17. Taking the growth of $u = e^{c \cdot x}$ as $|x| \to \infty$ into account, $u \in \mathscr{S}'(\mathbb{R}^n)$ only when $c = ia$ for some $a \in \mathbb{R}^n$. Thus Theorems 10 and 14 are reduced to the following:

Corollary 18. *Let $u, v \in \mathscr{S}'$ satisfy (19). Then (u, v) satisfies one of the followings:*

   (i) *$u = 0$, and v is arbitrary,*

  (ii) *u and v are bounded measurable functions,*

 (iii) *$v(x) = \lambda u(x) + e^{ic \cdot x}$ for some $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, where u is a continuous function satisfying the condition; there exists $d \geq 0$*

$$\left| u(x) \right| \leq \psi(-x) + d \qquad (141)$$

  for all $x \in \mathbb{R}^n$,

 (iv) *$v(x) = \lambda u(x) + 1$ for some $\lambda \in \mathbb{C}$, where u is a continuous function satisfying one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that*

$$\left| u(x) - a_1 \cdot x \right| \leq \Phi_1(x) \qquad (142)$$

  for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$\left| u(x) - a_2 \cdot x \right| \leq \Phi_2(x) \qquad (143)$$

  for all $x \in \mathbb{R}^n$.

Corollary 19. *Let $u, v \in \mathscr{S}'$ satisfy (20). Then (u, v) satisfies one of the followings:*

   (i) *u and v are bounded measurable functions,*

  (ii) *$v(x) = \cos(a \cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$\left| r(x) \right| \leq \frac{1}{2} \psi(x) \qquad (144)$$

  for all $x \in \mathbb{R}^n$.

Acknowledgments

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] D. G. Bourgin, "Multiplicative transformations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 36, pp. 564–570, 1950.

[5] D. G. Bourgin, "Approximately isometric and multiplicative transformations on continuous function rings," *Duke Mathematical Journal*, vol. 16, pp. 385–397, 1949.

[6] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[7] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[8] J. A. Baker, "On a functional equation of Aczél and Chung," *Aequationes Mathematicae*, vol. 46, no. 1-2, pp. 99–111, 1993.

[9] J. A. Baker, "The stability of the cosine equation," *Proceedings of the American Mathematical Society*, vol. 80, no. 3, pp. 411–416, 1980.

[10] L. Székelyhidi, "The stability of d'Alembert-type functional equations," *Acta Scientiarum Mathematicarum*, vol. 44, no. 3-4, pp. 313–320, 1982.

[11] J. Chung, "A distributional version of functional equations and their stabilities," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 62, no. 6, pp. 1037–1051, 2005.

[12] J. Chang and J. Chung, "Stability of trigonometric functional equations in generalized functions," *Journal of Inequalities and Applications*, vol. 2012, Article ID 801502, 12 pages, 2010.

[13] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Fla, USA, 2003.

[14] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[15] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser Boston Publisher, Boston, Mass, USA, 1998.

[16] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[17] K.-W. Jun and H.-M. Kim, "Stability problem for Jensen-type functional equations of cubic mappings," *Acta Mathematica Sinica (English Series)*, vol. 22, no. 6, pp. 1781–1788, 2006.

[18] G. H. Kim, "On the stability of the Pexiderized trigonometric functional equation," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 99–105, 2008.

[19] G. H. Kim and Y. H. Lee, "Boundedness of approximate trigonometric functional equations," *Applied Mathematics Letters*, vol. 31, pp. 439–443, 2009.

[20] C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Bulletin des Sciences Mathématiques*, vol. 132, no. 2, pp. 87–96, 2008.

[21] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[22] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques. 2e Série*, vol. 108, no. 4, pp. 445–446, 1984.

[23] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.

[24] L. Székelyhidi, "The stability of the sine and cosine functional equations," *Proceedings of the American Mathematical Society*, vol. 110, no. 1, pp. 109–115, 1990.

[25] I. Tyrala, "The stability of d'Alembert's functional equation," *Aequationes Mathematicae*, vol. 69, no. 3, pp. 250–256, 2005.

[26] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 2. Spaces of Fundamental and Generalized Functions*, Academic Press, New York, NY, USA, 1968.

[27] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, vol. 256 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1983.

[28] T. Matsuzawa, "A calculus approach to hyperfunctions. III," *Nagoya Mathematical Journal*, vol. 118, pp. 133–153, 1990.

[29] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, France, 1966.

[30] D. V. Widder, *The Heat Equation. Pure and Applied Mathematics*, vol. 67, Academic Press, New York, NY, USA, 1975.

[31] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[32] I. Fenyö, "Über eine Lösungsmethode gewisser Funktionalgleichungen," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 7, pp. 383–396, 1956.

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