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Research Article

Multivalued Variational Inequalities with D_J-Pseudomonotone Mappings in Reflexive Banach Spaces

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This paper is concerned with the study of a class of variational inequalities with multivalued D_J-pseudomonotone mappings in reflexive Banach spaces by using the D_J-antiresolvent technique. An application to the multivalued nonlinear D_J-complementarity problem is also presented. The results coincide with the corresponding results announced by many others for the gradient state.

1. Introduction and Preliminaries

Variational inequalities give a convenient mathematical framework for discussing a large variety of interesting problems appearing in pure and applied sciences. It is well known that the theory of pseudomonotone mappings plays an important part in the study of the above-mentioned variational inequalities.

In recent years, pseudomonotone theory has become an attractive field for many mathematicians (see [1–8]).

In a very recent paper [9], by using the D_J-antiresolvent technique (where J is the duality mapping) devised by the first author, the author introduced a new concept of monotonicity, which is called the D_J-pseudomonotone type.

In the present paper, the concept of multivalued D_J-pseudomonotone mappings in reflexive Banach spaces is used to study a wide class of variational inequalities, called the multivalued D_J-pseudomonotone variational inequalities.

Moreover, the results obtained in this paper can be applied to the multivalued nonlinear D_J-complementarity problem. This problem contains, in particular, a mathematical model arising in the study of the postcritical equilibrium state of a thin plate resting, without friction on a flat rigid support (see [10–12]). The results coincide with the corresponding results (see [2, 13–15]) in the case of gradient mappings.

Unless otherwise stated, V stands for a real reflexive Banach space with norm $\|\cdot\|_V$ and $V^\star$ stands for the uniformly convex dual of V with the dual norm $\|\cdot\|_{V^\star}$. The duality pairing between V and $V^\star$ is denoted by $\langle\cdot,\cdot\rangle$. The set of all nonnegative integers is denoted by $\mathbb{N}$. The field of real (resp., positive real) numbers is denoted by $\mathbb{R}$ (resp., $\mathbb{R}^+$). Notation "$\rightarrow$" stands for strong convergence and "$\rightharpoonup$" for weak convergence.

A mapping $J : V \rightarrow V^\star$ is said to be a duality mapping (see, e.g., [16]) with gauge function Φ (i.e., Φ is continuous strictly increasing real-valued function satisfying $\Phi(0) = 0$ and $\lim_{t\to+\infty}\Phi(t) = +\infty$) if for every $u \in V$, $\langle Ju, u\rangle = \|Ju\|_{V^\star}\|u\|_V = \Phi(\|u\|_V)(\|u\|_V)$. If $V = H$ is a Hilbert space, then $J \equiv I$, the identity mapping.

Assume that $V^\star$ has a weakly sequentially continuous duality mapping J (i.e., if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in V which weakly convergent to a point u, then the sequence $\{J(u_n)\}_{n\in\mathbb{N}}$ converges to $J(u)$ (see, e.g., [17])).

Let $g : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The domain of g is $\operatorname{dom} g = \{u \in V : g(u) < +\infty\}$. When $\operatorname{dom} g \neq \phi$, g is called proper (see, e.g., [18]). The interior of the domain of g is denoted by $\operatorname{int}\operatorname{dom} g$. The function g is said to be Gâteaux differentiable at $u \in \operatorname{int}\operatorname{dom} g$ (see, e.g., [18]), if

$$g'(u,\eta) = \lim_{t\to 0}\frac{g(u+t\eta) - g(u)}{t} \tag{1}$$

exists for all $\eta \in V$.

Let g be proper, convex, lower semicontinuous, and Gâteaux differentiable at $u \in \operatorname{int}\operatorname{dom} g$; then the gradient of g at u is the function $\nabla g(u)$ which is defined by $\langle \nabla g(u), \eta \rangle = g'(u, \eta)$ for any $\eta \in V$. It is known (see, e.g., [19]) that the conjugate $g^\star : V^\star \to \mathbb{R} \cup \{+\infty\}$ is also proper, convex, and lower semicontinuous.

The convex function g is said to be of Legendre type (see, e.g., [20]) if the following conditions hold:

(L_1) $\operatorname{int}\operatorname{dom}(g) \neq \phi$, g is Gâteaux differentiable on $\operatorname{int}\operatorname{dom}(g)$ and $\operatorname{dom} \nabla g = \operatorname{int}\operatorname{dom} g$;

(L_2) $\operatorname{int}\operatorname{dom}(g^\star) \neq \phi$, $g^\star$ is Gâteaux differentiable on $\operatorname{int}\operatorname{dom}(g^\star)$ and $\operatorname{dom} \nabla g^\star = \operatorname{int}\operatorname{dom}(g^\star)$.

It is well known (see, e.g., [21]) that if g is a proper, convex, lower semicontinuous, and Legendre type, then $\nabla g^\star = (\nabla g)^{-1}$ and range $\nabla g^\star = \operatorname{dom} \nabla g$.

Throughout this paper, the function $g : V \to \mathbb{R}^+ \cup \{+\infty\}$ is proper, convex, and lower semicontinuous which is also Legendre on $\operatorname{int}\operatorname{dom}(g)$.

The Bregman distance (see, e.g., [22]) is the function $D_g : V \times \operatorname{int}\operatorname{dom}(g) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$D_g(\nu, u) = g(\nu) - g(u) - \langle \nabla g(u), \nu - u \rangle, \tag{2}$$

with

$$\operatorname{dom} D_g = (\operatorname{dom} g) \times \operatorname{int}\operatorname{dom} g. \tag{3}$$

It should be pointed out that if $V = H$ is a Hilbert space and $g = (1/2)\|\cdot\|_H^2$, then $\nabla g = I$ (the identity mapping) and $D_J(\nu, u) = (1/2)\|\nu - u\|_H^2$.

For a multivalued mapping $T : V \to 2^{V^\star}$, the associated D_J-antiresolvent (where $J : V \to V^\star$ is the duality mapping) of $J - T$ (see [9]) is the mapping $T^J : V \to 2^V$, defined by

$$T^J = \nabla g^\star (J - T). \tag{4}$$

Such a mapping is known as (see [23]) a D_g-antiresolvent mapping of T when $J = \nabla g$ (in this case, the mapping T^J is denoted by T^g).

In light of the above-mentioned discussion, we note that if $J - T = \nabla g$, then T^J is the identity mapping I.

Following [9], the mapping $J - T : V \to 2^{V^\star}$ is said to be D_J-pseudomonotone, if for every $u, \eta \in \operatorname{dom} J \cap \operatorname{dom} T$, $\nu \in (J - T)(u)$, $\omega \in (J - T)(\eta)$, and every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} J \cap \operatorname{dom} T$ and $\nu_n \in (J - T)(u_n)$ the conditions

$$\begin{gathered} \nabla g^\star(\nu_n) \rightharpoonup \nabla g^\star(\nu), \\ \limsup_{n\to\infty} \langle \nu_n, \nabla g^\star(\nu_n) - \nabla g^\star(\nu) \rangle \le 0 \end{gathered} \tag{5}$$

imply that

$$\liminf_{n\to\infty} \langle \nu_n, \nabla g^\star(\nu_n) - \nabla g^\star(\omega) \rangle \ge \langle \omega, \nabla g^\star(\nu) - \nabla g^\star(\omega) \rangle. \tag{6}$$

As remarked in [9], the D_J-pseudomonotonicity of the mapping $J - T$ coincides with the pseudomonotonicity (or the D_g-pseudomonotonicity in the sense of Bregman distance D_g) of the mapping ∇g, if $J - T = \nabla g$.

The multivalued variational inequality defined by the D_J-mapping (or multivalued D_J-variational inequality) $J - T : V \to 2^{V^\star}$ and the set $K \subset V$ is to find $u \in K$ such that

$$\begin{gathered} \exists \nu \in (J - T)(u), \omega \in (J - T)(\eta) : \\ \langle \nu - f, \nabla g^\star(\omega) - \nabla g^\star(\nu) \rangle \ge 0, \quad \forall \eta \in K, \end{gathered} \tag{7}$$

where $f \in 2^{V^\star}$.

The multivalued nonlinear complementarity problem defined by the D_J-mapping (or multivalued nonlinear D_J-complementarity problem) $J - T : V \to 2^{V^\star}$ and the set K is to find $u \in K$ such that

$$\begin{gathered} \langle \nu - f, \nabla g^\star(\nu) \rangle = 0, \\ \langle \nu - f, \nabla g^\star(\omega) - \nabla g^\star(\nu) \rangle \ge 0, \quad \forall \eta \in K, \end{gathered} \tag{8}$$

where $f \in 2^{V^\star}$, $\nu \in (J - T)(u)$, and $\omega \in (J - T)(\eta)$.

The multivalued D_J-variational inequality and multivalued nonlinear D_J-complementarity problem are very general in the sense that they include, as special cases, multivalued variational inequality and multivalued nonlinear complementarity problem.

The following definition and results will be used in the sequel.

Definition 1 (see, e.g., [15, p. 84]). The mapping $A : V \to 2^{V^\star}$ is continuous on finite dimensional subspaces if for any finite dimensional subspace $V_0 \subset V$, the restriction of A to $V_0 \cap \operatorname{dom}(A)$ is weakly continuous.

Corollary 2 (see [24]). *Let $j : V_0 \subset V \to V$ be the injection mapping. Let $j^\star : V^\star \to V_0^\star$ be its dual mapping. Then, $j^\star A j : \operatorname{dom}(A) \cap jV_0 \to 2^{V_0^\star}$ is continuous.*

Corollary 3 (see [25]). *Let K be a nonempty compact convex set of $\mathbb{R}^n$ and let $S : K \to 2^K$ be continuous. Then S admits a fixed point.*

2. Main Results

Theorem 4. *Let K be a closed convex set in H and let $T : K \to 2^H$ be a multivalued mapping. Then the following are equivalent:*

(1) $$\begin{aligned} \nabla g^\star(\nu') \in Pr_K(x) &= \arg\min\Big\{ D_I(\nabla g^\star(\omega'), x) \\ &= \frac{1}{2}\|\nabla g^\star(\omega') - x\|_H^2 : \nabla g^\star(\omega') \\ &\in K, \omega' \in (I - T)(\eta), \eta \in K \Big\}, \end{aligned} \tag{9}$$

the multivalued projection for K;

$$(2)\ \nabla g^{\star}(\nu') \in K : \langle \nabla g^{\star}(\nu') - x, \nabla g^{\star}(\omega') - \nabla g^{\star}(\nu') \rangle \geq 0 \quad \forall \nabla g^{\star}(\omega') \in K, \tag{10}$$
$$\text{where } \omega' \in (I-T)(\eta), \quad \nu' \in (I-T)(y), \quad y, \eta \in K.$$

Proof. Assume that (1) holds. Let $x \in 2^{H}$ and $\nabla g^{\star}(\nu') \in Pr_K(x) \subset K$. For every $\nabla g^{\star}(\omega') \in K$ and $t \in (0,1]$, we have

$$\begin{aligned} D_I(x, \nabla g^{\star}(\nu')) &\leq D_I(x, (1-t)\nabla g^{\star}(\nu') + t\nabla g^{\star}(\omega')) \\ &= D_I(x, \nabla g^{\star}(\nu')) \\ &\quad - t\langle x - \nabla g^{\star}(\nu'), \nabla g^{\star}(\omega') - \nabla g^{\star}(\nu') \rangle \\ &\quad + t^2 D_I(\nabla g^{\star}(\omega'), \nabla g^{\star}(\nu')). \end{aligned} \tag{11}$$

This implies

$$\begin{aligned} &\langle x - \nabla g^{\star}(\nu'), \nabla g^{\star}(\omega') - \nabla g^{\star}(\nu') \rangle \\ &\quad \leq t D_I(\nabla g^{\star}(\omega'), \nabla g^{\star}(\nu')). \end{aligned} \tag{12}$$

Hence, $t \to 0^+$ implies (2).

On the other hand, assume that (2) holds. For every $\nabla g^{\star}(\omega') \in K$, we have

$$\begin{aligned} D_I(\nabla g^{\star}(\omega'), x) &= D_I(\nabla g^{\star}(\omega'), \nabla g^{\star}(\nu')) \\ &\quad + \langle \nabla g^{\star}(\omega') - x, \nabla g^{\star}(\omega') - \nabla g^{\star}(\nu') \rangle \\ &\quad + D_I(\nabla g^{\star}(\nu'), x) \geq D_I(\nabla g^{\star}(\nu'), x). \end{aligned} \tag{13}$$

This implies (1). □

Remark 5. In the particular situation when $I - T = \nabla g$ Theorem 4 coincides (in gradient setting) with Theorem 2.3 in [15] and also with Proposition 2.1 (1) and (2) in [2].

Theorem 6. *In addition to conditions on V, $V^{\star}$, $g^{\star}$, and J, one assumes that V is separable, $K \subset V$ is nonempty closed and convex, $T : K \to 2^{V^{\star}}$, $\nabla g^{\star}$ are weakly continuous mappings, and either T or $\nabla g^{\star}$ is continuous. Moreover, assume that the mapping $J - T : K \to 2^{V^{\star}}$ is a bounded D_J-pseudomonotone mapping and that, for each $\eta \in K$, there exist $\eta_0 \in K, \omega_0 \in (J-T)(\eta_0)$, $\omega \in (J-T)(\eta)$, and $r > 0$ such that*

$$\begin{aligned} &\langle \omega, \nabla g^{\star}(\omega) - \nabla g^{\star}(\omega_0) \rangle > \langle f, \nabla g^{\star}(\omega) - \nabla g^{\star}(\omega_0) \rangle, \\ &f \in 2^{V^{\star}}, \quad \|\nabla g^{\star}(\omega)\|_V \geq r. \end{aligned} \tag{14}$$

Then there exists a solution to the multivalued D_J-variational inequality (7).

Proof. Suppose that $\{x_1, x_2, x_3, \ldots\}$ is an infinite dense set in K and $V_m, m \in \mathbb{N}$, is the linear span of $\{x_1, x_2, x_3, \ldots, x_m\}$.

Let $K_m = \operatorname{con}\{x_1, x_2, x_3, \ldots, x_m\} = \{\sum_{i=1}^{m} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}$. Let $j_m : V_m \to V$ be the injection mapping and let $j_m^{\star} : V^{\star} \to V_m^{\star}$ be its restriction dual. Observe that $\cup K_m$ is dense in K, for $m \in \mathbb{N}$.

Now, fix an integer $m \geq 1$ and consider the finite dimensional problem.

Find $u_m \in K_m$ such that for each $\eta \in K_m$, there exist $s_m \in j_m^{\star}(J-T)j_m(u_m)$, $\nu_m \in (J-T)(u_m)$, and $\omega \in (J-T)(\eta)$ such that

$$\langle s_m - j_m^{\star} f, \nabla g^{\star}(\omega) - \nabla g^{\star}(\nu_m) \rangle \geq 0. \tag{15}$$

The equivalent form of problem (15) is to find $u_m \in K_m$ such that for each $\eta \in K_m$, there exist $s_m \in j_m^{\star}(J-T)j_m(u_m)$, $\nu_m \in (J-T)(u_m)$, and $\omega \in (J-T)(\eta)$ such that

$$\begin{aligned} &\langle \nabla g^{\star}(\nu_m), \nabla g^{\star}(\omega) - \nabla g^{\star}(\nu_m) \rangle \\ &\quad \geq \langle \nabla g^{\star}(\nu_m) + j_m^{\star} f - s_m, \nabla g^{\star}(\omega) - \nabla g^{\star}(\nu_m) \rangle. \end{aligned} \tag{16}$$

Using the identification of V_m with $\mathbb{R}^m$ and $V_m^{\star}$ and Theorem 4 (with $\mathbb{R}^m = H$ and $J = I$), we see that (16) is equivalent to $\nabla g^{\star}(\nu_m) \in Pr_{K_m}(\nabla g^{\star}(\nu_m) + j_m^{\star} f - s_m)$.

Let $\overline{B}_r(0)$ be any closed ball containing K_m. It is well known (see, e.g., [26, p. 54, 224]) that $\overline{B}_r(0)$ is compact and convex in K_m; thus it is weakly closed.

From Corollary 2, $j^{\star}(J-T)j : \overline{B}_r(0) \to 2^{\overline{B}_r(0)}$ is continuous; hence the function $\nabla g^{\star}(\nu) \mapsto Pr_{\overline{B}_r(0)}(\nabla g^{\star}(\nu) + j_m^{\star} f - s_m)$ is continuous from $\overline{B}_r(0)$ into $2^{\overline{B}_r(0)}$.

Hence, by Corollary 3, this equation admits a solution. If the closed convex set K is assumed to be bounded, then by the reflexivity of V it is weakly compact (by employing the Banach -Alaoglu theorem (see, e.g., [16, p. 3]).

Then we have a subsequence denoted by $\{u_m\}_{m \in \mathbb{N}}$ such that $u_m \rightharpoonup u \in K$. Since $J - T$ is bounded, we have $\|\nu_m\|_{V^{\star}} \leq M$ for all $m \in \mathbb{N}$. Since $J - T$ is weakly continuous and since either $J - T$ or $\nabla g^{\star}$ is continuous by hypothesis, it follows that T^J is weakly continuous by [27, Lemma 1]. So, we have $\nabla g^{\star}(\nu_m) \rightharpoonup \nabla g^{\star}(\nu)$.

Now, we prove that

$$\limsup_{m \to \infty} \langle \nu_m, \nabla g^{\star}(\nu_m) - \nabla g^{\star}(\nu) \rangle \leq 0. \tag{17}$$

For any $\epsilon > 0$, choose N so large and $\tilde{u} \in K_N$ such that

$$\|\nabla g^{\star}(\nu) - \nabla g^{\star}(\tilde{\nu})\|_V < \epsilon \quad \text{for } \tilde{\nu} \in (J-T)(\tilde{u}). \tag{18}$$

Therefore, we have

$$\langle \nu_m - f, \nabla g^{\star}(\nu_m) - \nabla g^{\star}(\tilde{\nu}) \rangle \leq 0 \quad \text{for } m \geq N. \tag{19}$$

Since $K_N \subset K_m$, we have

$$\begin{aligned}
\limsup_{m\to\infty} &\left\langle \nu_m, \nabla g^\star(\nu_m) - \nabla g^\star(\nu)\right\rangle \\
&= \limsup_{m\to\infty} \left[\left\langle \nu_m, \nabla g^\star(\nu_m) - \nabla g^\star(\tilde{\nu})\right\rangle \right. \\
&\qquad \left. + \left\langle \nu_m, \nabla g^\star(\tilde{\nu}) - \nabla g^\star(\nu)\right\rangle\right] \\
&\leq \left(\|f\|_{V^\star} + M\right)\epsilon.
\end{aligned} \tag{20}$$

Since ϵ is arbitrary, this shows the desired inequality.

By the D_J-pseudomonotonicity of $J-T$, it follows that

$$\liminf_{m\to\infty} \left\langle \nu_m, \nabla g^\star(\nu_m) - \nabla g^\star(\omega)\right\rangle \geq \left\langle \nu, \nabla g^\star(\nu) - g^\star(\omega)\right\rangle \tag{21}$$

for all $\eta \in \operatorname{dom} J \cap \operatorname{dom} T$ and $\omega \in (J-T)(\eta)$.

If $\eta \in K_n$, $m \geq n$, we have

$$\left\langle \nu_m, \nabla g^\star(\nu_m) - \nabla g^\star(\omega)\right\rangle \leq \left\langle f, \nabla g^\star(\nu_m) - \nabla g^\star(\omega)\right\rangle. \tag{22}$$

Hence

$$\left\langle \nu, \nabla g^\star(\nu) - \nabla g^\star(\omega)\right\rangle \leq \left\langle f, \nabla g^\star(\nu) - \nabla g^\star(\omega)\right\rangle \tag{23}$$

for every η in K_n, $n \in \mathbb{N}$, $\omega \in (J-T)(\eta)$.

Since $\cup_n K_n$ is dense in K_n, so we have that u is a solution to (7).

Now, to complete the proof, we consider the case when K is unbounded.

In this case we consider the set $K_\rho = \{\eta \in K : \|\nabla g^\star(\omega)\|_V \leq \rho\}$, where $\rho = \max\{\|\nabla g^\star(\omega_0)\|_V, r\}$.

Since K_ρ is bounded, there exists at least one $u_\rho \in K_\rho$:

$$\left\langle \nu_\rho - f, \nabla g^\star(\omega) - \nabla g^\star(\nu_\rho)\right\rangle \geq 0 \tag{24}$$

for $\nu_\rho \in (J-T)(u_\rho)$ and $\eta \in K_\rho$.

Since $\eta_0 \in K_\rho$, we have

$$\left\langle \nu_\rho - f, \nabla g^\star(\nu_\rho) - \nabla g^\star(\omega_0)\right\rangle \leq 0 \quad \text{for } \omega_0 \in (J-T)(\eta_0). \tag{25}$$

This, together with (14), implies that $\|\nabla g^\star(\nu_\rho)\|_V < \rho$.

To clarify that u_ρ is also a solution to original problem on K, for any $\eta \in K$, set $\nabla g^\star(\omega_t) = (1-t)\nabla g^\star(\nu_\rho) + t\nabla g^\star(\omega)$ for $t > 0$ is sufficiently small, where $\omega_t \in (J-T)(\eta_t)$ and $\eta_t \in K_\rho$. Consequently

$$\begin{aligned}
u_\rho \in K_\rho \subset K : 0 &\leq \left\langle \nu_\rho - f, \nabla g^\star(\omega_t) - \nabla g^\star(\nu_\rho)\right\rangle \\
&= t\left\langle \nu_\rho - f, \nabla g^\star(\omega) - \nabla g^\star(\nu_\rho)\right\rangle \\
&\qquad \text{for } \eta \in K.
\end{aligned} \tag{26}$$

This completes the proof. □

Remark 7. In the particular situation when $J - T = \nabla g$, Theorem 6 coincides with the Brezis Theorem (see, e.g., [13, 14]) for the case of gradient mapping.

We are now in a position to state and prove the following theorem.

Theorem 8. *Let all assumptions of Theorem 6 hold, except for condition* (14) *let it be replaced by the D_J-coercive condition: for $\omega \in (J-T)(\eta)$,*

$$\begin{gathered}
\lim_{\|\nabla g^\star(\omega)\|_V\to\infty} \left[\frac{\left\langle \omega, \nabla g^\star(\omega) - \nabla g^\star(\omega_0)\right\rangle}{\|\nabla g^\star(\omega)\|_V}\right] = +\infty, \\
\omega_0 \in (J-T)(\eta_0), \quad \eta, \eta_0 \in K.
\end{gathered} \tag{27}$$

Suppose further that K has the following property (W): $\alpha\nabla g^\star(\omega) \in K$ for all $\nabla g^\star(\omega) \in K$ and $\alpha \geq 0$.

Then for every $f \in 2^{V^\star}$ there exist $u \in K$, $\nu \in (J-T)(u)$ such that

$$\left\langle \nu - f, \nabla g^\star(\nu)\right\rangle = 0, \qquad \left\langle \nu - f, \nabla g^\star(\omega) - \nabla g^\star(\nu)\right\rangle \geq 0 \tag{28}$$

for all $\eta \in K, \omega \in (J-T)(\eta)$.

Proof. Let $f \in 2^{V^\star}$ satisfy $\|f\|_{V^\star} < M$ and

$$\|\nabla g^\star(\omega_0)\|_V < \frac{2M - \|f\|_{V^\star}}{\|f\|_{V^\star}}\rho. \tag{29}$$

The D_J-coercivity of $J-T$ implies that there exists $\rho > 0$ such that

$$\left\langle \omega, \nabla g^\star(\omega) - \nabla g^\star(\omega_0)\right\rangle \geq 2M\|\nabla g^\star(\omega)\|_V \tag{30}$$

for $\omega \in (J-T)(\eta)$, $\eta \in K$ with $\|\nabla g^\star(\omega)\|_V \geq \rho$.

So we conclude

$$\begin{aligned}
&\left\langle \omega - f, \nabla g^\star(\omega) - \nabla g^\star(\omega_0)\right\rangle \\
&\quad \geq 2M\|\nabla g^\star(\omega)\|_V - \|f\|_{V^\star}\|\nabla g^\star(\omega)\|_V - \|f\|_{V^\star}\|\nabla g^\star(\omega_0)\|_V \\
&\quad \geq \left(2M - \|f\|_{V^\star}\right)\rho - \|f\|_{V^\star}\|\nabla g^\star(\omega_0)\|_V \\
&\quad > 0 \text{ for } \eta \in K, \omega \in (J-T)(\eta) \ \textit{with} \ \|\nabla g^\star(\omega)\|_V \geq \rho.
\end{aligned} \tag{31}$$

The second part of (28) thus follows from Theorem 6.

To prove the first part of (28), observe that we can choose a point η in K and $\omega \in (J-T)(\eta)$ and assume that $\nabla g^\star(\omega) = 0$.

Therefore, from Theorem 6, we have

$$\left\langle \nu - f, \nabla g^\star(\nu)\right\rangle \leq 0 \tag{32}$$

for all $\nu \in (J-T)(u)$, $u \in K$.

On the other hand, setting $\nabla g^\star(\omega) = \alpha\nabla g^\star(\nu)$, where $\alpha > 1$, we get

$$0 \leq \left\langle \nu - f, \nabla g^\star(\omega) - \nabla g^\star(\nu)\right\rangle = (\alpha - 1)\left\langle \nu - f, \nabla g^\star(\nu)\right\rangle. \tag{33}$$

This implies

$$\left\langle \nu - f, \nabla g^\star(\nu)\right\rangle \geq 0. \tag{34}$$

So,

$$\left\langle \nu - f, \nabla g^\star(\nu)\right\rangle = 0. \tag{35}$$

This completes the proof of (28). □

The following proposition gives a characterization of the sum of two D_J-Pseudomonotone mappings.

Proposition 9. *Let V, $V^\star$, and J be as above and let $T_i : V \to 2^{V^\star}$, $i = 1, 2$, and $\nabla g^\star$ be weakly continuous mappings. If $J - T_i : V \to 2^{V^\star}$, $i = 1, 2$, are D_J-pseudomonotone mappings such that $\operatorname{dom} J \cap \operatorname{dom} T_i \neq \phi$, $i = 1, 2$, then $\sum_{i=1}^{2}(J - T_i)$ is D_J-pseudomonotone.*

Proof. Let $\tilde{y}_n \in \sum_{i=1}^{2}(J-T_i)(\eta_n)$, $\tilde{y} \in \sum_{i=1}^{2}(J-T_i)(\eta)$, η_n, $\eta \in \operatorname{dom} J \cap \operatorname{dom} T_i$, $i = 1, 2$, with $\nabla g^\star(\tilde{y}_n) \rightharpoonup \nabla g^\star(\tilde{y})$ and

$$\limsup_{n\to\infty} \langle \tilde{y}_n, \nabla g^\star(\tilde{y}_n) - \nabla g^\star(\tilde{y}) \rangle \le 0. \tag{36}$$

Now, we prove for $y_n^{(i)} \in (J - T_i)(\eta_n)$, $y^{(i)} \in (J - T_i)(\eta)$, $i = 1, 2$, that

$$\begin{aligned} &\limsup_{n\to\infty} \left\langle y_n^{(1)}, \nabla g^\star\left(y_n^{(1)}\right) - \nabla g^\star\left(y^{(1)}\right)\right\rangle \le 0, \\ &\limsup_{n\to\infty} \left\langle y_n^{(2)}, \nabla g^\star\left(y_n^{(2)}\right) - \nabla g^\star\left(y^{(2)}\right)\right\rangle \le 0. \end{aligned} \tag{37}$$

If

$$\limsup_{n\to\infty} \left\langle y_n^{(2)}, \nabla g^\star\left(y_n^{(2)}\right) - \nabla g^\star\left(y^{(2)}\right)\right\rangle = \epsilon > 0, \tag{38}$$

(note that otherwise, by symmetry), then there exists a subsequence $\{y_{n_k}^{(2)}\}_{k\in\mathbb{N}} \subset \{y_n^{(2)}\}_{n\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty} \left\langle y_{n_k}^{(2)}, \nabla g^\star\left(y_{n_k}^{(2)}\right) - \nabla g^\star\left(y^{(2)}\right)\right\rangle = \epsilon. \tag{39}$$

This implies that

$$\begin{aligned} \limsup_{k\to\infty} &\left\langle y_{n_k}^{(1)}, \nabla g^\star\left(y_{n_k}^{(1)}\right) - \nabla g^\star\left(y^{(1)}\right)\right\rangle \\ &= \limsup_{k\to\infty} \Big[\left\langle \tilde{y}_{n_k}, \nabla g^\star\left(\tilde{y}_{n_k}\right) - \nabla g^\star(\tilde{y})\right\rangle \\ &\qquad - \left\langle y_{n_k}^{(2)}, \nabla g^\star\left(y_{n_k}^{(2)}\right) - \nabla g^\star\left(y^{(2)}\right)\right\rangle \Big] \\ &\le 0 - \epsilon. \end{aligned} \tag{40}$$

From the D_J-pseudomonotonicity of $J-T_1$, we get for all $y' \in (J - T_1)(\eta')$, $\eta' \in \operatorname{dom} J \cap \operatorname{dom} T_1$

$$\begin{aligned} &\left\langle y^{(1)}, \nabla g^\star\left(y^{(1)}\right) - \nabla g^\star\left(y'\right)\right\rangle \\ &\quad \le \liminf_{k\to\infty} \left\langle y_{n_k}^{(1)}, \nabla g^\star\left(y_{n_k}^{(1)}\right) - \nabla g^\star\left(y'\right)\right\rangle. \end{aligned} \tag{41}$$

Letting $y' = y^{(1)}$, we obtain

$$0 \le \liminf_{k\to\infty} \left\langle y_{n_k}^{(1)}, \nabla g^\star\left(y_{n_k}^{(1)}\right) - \nabla g^\star\left(y^{(1)}\right)\right\rangle \le 0 - \epsilon, \tag{42}$$

a contradiction.

Hence,

$$\begin{aligned} &\limsup_{k\to\infty} \left\langle y_{n_k}^{(1)}, \nabla g^\star\left(y_{n_k}^{(1)}\right) - \nabla g^\star\left(y^{(1)}\right)\right\rangle \le 0, \\ &\limsup_{k\to\infty} \left\langle y_{n_k}^{(2)}, \nabla g^\star\left(y_{n_k}^{(2)}\right) - \nabla g^\star\left(y^{(2)}\right)\right\rangle \le 0. \end{aligned} \tag{43}$$

This holds for any subsequence, so (37) holds and the proof follows immediately by the superadditivity of the lim inf. □

3. Application to Multivalued Nonlinear D_J-Complementarity Problem

As applications of Theorem 8 we consider the multivalued nonlinear D_J-complementarity problem (8) with $T = \lambda T_1 + T_2 - \lambda J$, where T_i, $i = 1, 2$ are two nonlinear multivalued mappings from K to $2^{V^\star}$, and $\lambda \in (0, \infty)$.

Theorem 10. *Let V, $V^\star$, $g^\star$, J, and K be the same as in Theorem 6, and suppose that K has the property (W). Let the mappings $\nabla g^\star$ and T_i, $i = 1, 2$ be weakly continuous and let either $\nabla g^\star$ or both T_i, $i = 1, 2$ be continuous. Let $J - T_i : K \to 2^{V^\star}$, $i = 1, 2$ be two bounded D_J-pseudomonotone mappings.*

Let

$$\begin{aligned} \frac{1}{\rho_1} &= \inf_{\substack{n\in K \\ \|\nabla g^\star(\lambda w_1 + w_2)\|_V \neq 0}} \Big(\big\langle w_1, \nabla g^\star(\lambda w_1 + w_2) \\ &\qquad - \nabla g^\star\left(\lambda w_0^{(1)} + w_0^{(2)}\right)\big\rangle \\ &\qquad \times \left(\|\nabla g^\star(\lambda w_1 + w_2)\|_V^2\right)^{-1}\Big), \\ a &= \liminf_{\substack{n\in K \\ \|\nabla g^\star(\lambda w_1 + w_2)\|_V \to \infty}} \Big(\big\langle w_2, \nabla g^\star(\lambda w_1 + w_2) \\ &\qquad - \nabla g^\star\left(\lambda w_0^{(1)} + w_0^{(2)}\right)\big\rangle \\ &\qquad \times \left(\|\nabla g^\star(\lambda w_1 + w_2)\|_V^2\right)^{-1}\Big), \\ b &= \liminf_{t\to\infty} \frac{\Phi(t)}{t} \end{aligned} \tag{44}$$

Be such that $a < b$, where $w_i \in (J - T_i)(\eta)$, $w_0^{(i)} \in (J - T_i)(\eta_0)$, $i = 1, 2$, η, $\eta_0 \in K$, $\rho_1 > 0$ and Φ is the gauge function. Then for every $\lambda > \rho_1(b - a)$ problem (8) with $T = \lambda T_1 + T_2 - \lambda J$ has a solution in K.

Proof. By Proposition 9, $J - T = \lambda(J - T_1) + (J - T_2)$ is D_J-pseudomonotone for every $\lambda \ge 0$. Set $\lambda > \rho(b - a)$. Then

$$\begin{aligned} \liminf_{\substack{\eta\in K \\ \|\nabla g^\star(\lambda w_1 + w_2)\|_V \to \infty}} &\Big(\big\langle \lambda w_1 + w_2, \nabla g^\star(\lambda w_1 + w_2) \\ &\qquad - \nabla g^\star\left(\lambda w_0^{(1)} + w_0^{(2)}\right)\big\rangle \\ &\qquad \times \left(\|\nabla g^\star(\lambda w_1 + w_2)\|_V^2\right)^{-1}\Big) \\ &\ge \frac{\lambda}{\rho_1} + a > b > 0. \end{aligned} \tag{45}$$

This implies that the mapping $J - T = \lambda(J - T_1) + (J - T_2)$ is D_J-coercive. □

The conclusion follows from Theorem 8.

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