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Research Article

Sufficiency Criteria for a Class of p-Valent Analytic Functions of Complex Order

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In the present paper, we consider a subclass of p-valent analytic functions and obtain certain simple sufficiency criteria by using three different methods for the functions belonging to this class. Many known results appear as special consequences of our work.

1. Introduction

Let $A_p(n)$ be the class of functions $f(z)$ analytic and p-valent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p \in \mathbb{N}). \tag{1}$$

In particular, $A_p(1) = A_p$, $A_1(n) = A(n)$, and $A_1(1) = A$. By $\mathcal{S}_p^*(n,b)$ and $\mathcal{C}_p(n,b)$, $n, p \in \mathbb{N}$ and $b \in \mathbb{C} \setminus \{0\}$, we mean the subclasses of $A_p(n)$ which are defined, respectively, by

$$\begin{gathered}\operatorname{Re}\left\{1 + \frac{1}{p+b-1}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > 0, \quad (z \in \mathbb{U}), \\ \operatorname{Re}\left\{1 + \frac{1}{p+b-1}\left(\frac{zf''(z)}{f'(z)} - p + 1\right)\right\} > 0, \quad (z \in \mathbb{U}).\end{gathered} \tag{2}$$

For $b = 1$, $p = 1$, $n = 1$, the previous two classes defined in (2) reduce to the well-known classes of starlike and convex, respectively.

For functions $f(z), g(z) \in A_p(n)$ of the form (1), we define the convolution (Hadamard product) of $f(z)$ and $g(z)$ by

$$(f \star g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U}). \tag{3}$$

Now we define the subclass $\mathcal{M}_p(n, b; g(z))$ of $A_p(n)$ by

$$\operatorname{Re}\left\{1 + \frac{1}{p+b-1}\left(\frac{z(f(z) \star g(z))'}{f(z) \star g(z)} - p\right)\right\} > 0, \quad (z \in \mathbb{U}). \tag{4}$$

Sufficient conditions were studied by various authors for different subclasses of analytic and multivalent functions, for some of the related work see [1–8]. The object of the present paper is to obtain sufficient conditions for the subclass $\mathcal{M}_p(n, b; g(z))$ of $A_p(n)$. We also consider some special cases of our results which lead to various interesting corollaries and relevances of some of these with other known results also being mentioned.

We will assume throughout our discussion, unless otherwise stated, that $n \in \mathbb{N}$, $p \in \mathbb{N}$, $b \in \mathbb{C} \setminus \{0\}$.

2. Preliminary Results

To obtain our main results, we need the following Lemma's.

Lemma 1 (see [9]). *If $q(z) \in A(n)$ with $n \geq 1$ and satisfies the condition*

$$\left|q'(z) - 1\right| < \frac{n+1}{\sqrt{(n+1)^2 + 1}} \quad (z \in \mathbb{U}), \tag{5}$$

then

$$q(z) \in \mathcal{S}^*(n,1). \quad (6)$$

Lemma 2 (see [10]). *If $q(z) \in A(n)$ satisfing the condition*

$$\left|\arg q'(z)\right| < \frac{\pi}{2}\delta_n \quad (z \in \mathbb{U}), \quad (7)$$

where δ_n is the unique root of the equation

$$2\tan^{-1}\left[n\left(1-\delta_n\right)\right] + \pi\left(1-2\delta_n\right) = 0, \quad (8)$$

then

$$q(z) \in \mathcal{S}^*(n,1). \quad (9)$$

Lemma 3 (see [11]). *Let Ω be a set in the complex plane $\mathbb{C}$, and suppose that Ψ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Psi(ix, y, z) \notin \Omega$ for $z \in \mathbb{U}$ and for all real x, y such that $y \le (-n/2)(1+x^2)$. If $q(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{U}$ and $\Psi(q(z), zq'(z), z) \in \Omega$ for all $z \in \mathbb{U}$, then* $\operatorname{Re} q(z) > 0$.

3. Main Results

Theorem 4. *If $f(z) \in A_p(n)$ satisfies*

$$\left|\left(\frac{f(z)*g(z)}{z^p}\right)^{1/(p+b-1)}\left\{\frac{z(f(z)*g(z))'}{f(z)*g(z)} + b - 1\right\} - (p+b-1)\right| < \frac{n+1}{\sqrt{(n+1)^2+1}}\left|p+b-1\right| \quad (z \in \mathbb{U}), \quad (10)$$

then $f(z) \in \mathcal{M}_p(n, b; g(z))$.

Proof. Let us set a function $p(z)$ by

$$p(z) = z\left(\frac{f(z)*g(z)}{z^p}\right)^{1/(p+b-1)} = z + \frac{a_{n+p}b_{n+p}}{(p+b-1)}z^n + \cdots \quad (11)$$

for $f(z) \in A_p(n)$. Then clearly (11) shows that $p(z) \in A(n)$.

Differentiating (11) logarithmically, we have

$$\frac{p'(z)}{p(z)} = \frac{1}{p+b-1}\left[\frac{f(z)*g(z)'}{f(z)*g(z)} - \frac{p}{z}\right] + \frac{1}{z} \quad (12)$$

which gives

$$\left|p'(z) - 1\right| = \left|\left(\frac{f(z)*g(z)}{z^p}\right)^{1/(p+b-1)}\frac{1}{p+b-1} \times \left\{\frac{z(f(z)*g(z))'}{f(z)*g(z)} + b - 1\right\} - 1\right|. \quad (13)$$

Thus using (10), we have

$$\left|p'(z) - 1\right| \le \frac{n+1}{\sqrt{(n+1)^2+1}}, \quad (z \in \mathbb{U}). \quad (14)$$

Hence, using Lemma 1, we have $p(z) \in \mathcal{S}^*(n,1)$.

From (12), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{p+b-1}\left[\frac{z(f(z)*g(z))'}{f(z)*g(z)} - p\right] + 1. \quad (15)$$

Since $p(z) \in \mathcal{S}^*(n,1)$, it implies that $\operatorname{Re}(zp'(z)/p(z)) > 0$. Therefore, we get

$$\operatorname{Re}\left\{1 + \frac{1}{p+b-1}\left(\frac{z(f(z)*g(z))'}{f(z)*g(z)} - p\right)\right\} = \operatorname{Re}\frac{zp'(z)}{p(z)} > 0, \quad (16)$$

and this implies that $f(z) \in \mathcal{M}_p(n, b; g(z))$.

Setting $n = p = 1$ and $g(z) = z/(1-z)$ in Theorem 4, we get the following. ◻

Corollary 5. *If $f(z) \in A$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^{1/b}\left\{\frac{zf'(z)}{f(z)} + b - 1\right\} - b\right| < \frac{2|b|}{\sqrt{5}} \quad (z \in \mathbb{U}), \quad (17)$$

then $f(z) \in \mathcal{S}^*(b)$, the class of starlike functions of complex order b.

Putting $n = p = 1$ and $g(z) = z/(1-z)^2$ in Theorem 4, we have the following.

Corollary 6. *If $f(z) \in A$ satisfies*

$$\left|\left(f'(z)\right)^{(1-b)/b}\left\{zf''(z) + bf'(z)\right\} - b\right| < \frac{2|b|}{\sqrt{5}} \quad (z \in \mathbb{U}), \quad (18)$$

then $f(z) \in \mathcal{C}(b)$, the class of convex functions of complex order b.

Remark 7. If we put $b = 1-\alpha$ in Corollaries 5 and 6, we get the results proved by Uyanık et al. [1]. Furthermore, for $b = 1$, we obtain the results studied by Mocanu [2] and Nunokawa et al. [3], respectively. Also if we set $b = 1-\alpha$ with $g(z) = z^p/(1-z^n)$ and $g(z) = z^p/(1-z^n)^{2p}$ in Theorem 4, we obtain the results due to Goyal et al. [4].

Theorem 8. *If $f(z) \in A_p(n)$ satisfies*

$$\left|\arg\left(\frac{f(z)*g(z)}{z^p}\right)^{1/(p+b-1)} + \arg\left\{\frac{1}{p+b-1}\left(\frac{z(f(z)*g(z))'}{f(z)*g(z)} + b - 1\right)\right\}\right| < \frac{\pi}{2}\delta_n \quad (z \in \mathbb{U}), \quad (19)$$

where δ_n is the unique root of (8), *then $f(z) \in \mathcal{M}_p(n, b; g(z))$.*

Proof. Let $p(z)$ be given by (11), which clearly belongs to the class $A(n)$.

Now differentiating (11), we have

$$p'(z) = \left(\frac{f(z) * g(z)}{z^p}\right)^{1/(p+b-1)} \times \frac{1}{p+b-1}\left\{\frac{z(f(z) * g(z))'}{f(z) * g(z)} + b - 1\right\} \tag{20}$$

which gives

$$\begin{aligned}&\left|\arg p'(z)\right| \\ &= \left|\arg\left(\frac{f(z) * g(z)}{z^p}\right)^{1/(p+b-1)} \right.\\ &\quad \left. + \arg\left\{\frac{1}{p+b-1}\left(\frac{z(f(z) * g(z))'}{f(z) * g(z)} + b - 1\right)\right\}\right|.\end{aligned} \tag{21}$$

Thus using (19), we have

$$\left|\arg p'(z)\right| \leq \frac{\pi}{2}\delta_n \quad (z \in \mathbb{U}), \tag{22}$$

where δ_n is the root of (8). Hence, using Lemma 2, we have $p(z) \in \mathcal{S}^*(n, 1)$.

From (20), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{p+b-1}\left[\frac{z(f(z) * g(z))'}{f(z) * g(z)} - p\right] + 1. \tag{23}$$

Since $p(z) \in \mathcal{S}^*(n, 1)$, it implies that $\mathrm{Re}(zp'(z)/p(z)) > 0$. Therefore, we get (16), and hence $f(z) \in \mathcal{M}_p(n, b; g(z))$.

Making $n = 1$, $b = 1 - \alpha$ with $0 \leq \alpha < p$ and $g(z) = z^p/(1 - z)$, we have the following. □

Corollary 9. *If $f(z) \in A_p$ satisfies*

$$\begin{aligned}&\left|\arg\left(\frac{f(z)}{z^p}\right) + (p - \alpha)\arg\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}\right| \\ &\quad < \frac{\pi}{2}\delta_1(p - \alpha) \quad (z \in \mathbb{U}),\end{aligned} \tag{24}$$

where δ_1 is the unique root of (8) *with $n = 1$, then $f(z) \in \mathcal{S}_p^*(\alpha)$, the class of p-valent starlike functions of order α.*

Also if we take $n = 1$, $b = 1 - \alpha$ with $0 \leq \alpha < p$ and $g(z) = z^p/(1 - z)^{2p}$ in Theorem 8, we obtain the following result.

Corollary 10. *If $f(z) \in A_p$ satisfies*

$$\begin{aligned}&\left|\arg\left(\frac{f'(z)}{pz^{p-1}}\right) + (p - \alpha)\arg\left\{\frac{zf''(z)}{f'(z)} + 1 - \alpha\right\}\right| \\ &\quad < \frac{\pi}{2}\delta_1(p - \alpha) \quad (z \in \mathbb{U}),\end{aligned} \tag{25}$$

where δ_1 is the unique root of (8) *with $n = 1$, then $f(z) \in \mathcal{C}_p(\alpha)$, the class of p-valent convex functions of order α.*

Remark 11. For putting $p = 1$, $\alpha = 0$ in Corollary 10 and $p = 1$ in Corollary 9, we obtain the results proved by Mocanu [10] and Uyanık et al. [1], respectively.

Theorem 12. *If $f(z) \in A_p(n)$ satisfies*

$$\begin{aligned}&\mathrm{Re}\left[\frac{1}{p+b-1}\left\{\frac{z(f(z)*g(z))'}{f(z)*g(z)}\left(\rho\frac{z(f(z)*g(z))''}{(f(z)*g(z))'} + 1\right)\right\}\right. \\ &\quad \left. + b - 1\right] > \frac{M^2}{4L} + N,\end{aligned} \tag{26}$$

where $0 \leq \rho \leq 1$ and

$$\begin{aligned}L &= \rho\left(p + \mathrm{Re}\, b - 1 + \frac{n}{2}\right), \\ M &= 2\rho\ \mathrm{Im}\, b, \\ N &= \rho\Bigg(\Big(\big((\mathrm{Re}\, b)^2 - (\mathrm{Im}\, b)^2 - \mathrm{Re}\, b\big) \\ &\qquad \times (p + \mathrm{Re}\, b - 1) + (\mathrm{Im}\, b)^2 (2\ \mathrm{Re}\, b - 1) \\ &\qquad \times \left((p + \mathrm{Re}\, b - 1)^2 + (\mathrm{Im}\, b)^2\right)^{-1}\Big) - \frac{n}{2}\Bigg),\end{aligned} \tag{27}$$

then $f(z) \in \mathcal{M}_p(n, b; g(z))$.

Proof. Let us set

$$\frac{z(f(z) * g(z))'}{f(z) * g(z)} = (p + b - 1)\, p(z) - b + 1. \tag{28}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

Taking logarithmic differentiation of (28) and then by simple computation, we obtain

$$\begin{aligned}&\frac{1}{p+b-1}\left\{\frac{z(f(z) * g(z))'}{f(z) * g(z)}\right. \\ &\qquad \left. \times \left(\rho\frac{z(f(z) * g(z))''}{(f(z) * g(z))'} + 1\right) + b - 1\right\} \\ &\quad = Azp'(z) + Bp^2(z) + Cp(z) + D \\ &\quad = \Psi\left(p(z), zp'(z), z\right)\end{aligned} \tag{29}$$

with

$$\begin{aligned}A &= \rho, \qquad B = \rho(p + b - 1), \\ C &= -2\rho b + \rho + 1, \qquad D = \frac{\rho b^2 - \rho b}{p + b - 1}.\end{aligned} \tag{30}$$

Now for all real x and y satisfying $y \leq -(n/2)(1 + x^2)$, we have

$$\Psi(ix, y, z) = Ay - Bx^2 + C(ix) + D. \tag{31}$$

Reputing the values of A, B, C, D and then taking real part, we obtain

$$\begin{aligned} \operatorname{Re} \Psi(ix, y, z) &\le -Lx^2 + Mx + N \\ &= -\left(\sqrt{L}x - \frac{M}{2\sqrt{L}}\right)^2 + \frac{M^2}{4L} + N \\ &< \frac{M^2}{4L} + N, \end{aligned} \tag{32}$$

where L, M, N are given in (27).

Let $\Omega = \{w : \operatorname{Re} w > (M^2/4L) + N\}$. Then $\Psi(h(z), zh'(z), z) \in \Omega$ and $\Psi(ix, y, z) \notin \Omega$, for all real x and y satisfying $y \le -(n/2)(1 + x^2)$, $z \in \mathbb{U}$. Using Lemma 3, we have $\operatorname{Re}\ p(z) > 0$. This implies that

$$\operatorname{Re}\left\{1 + \frac{1}{p+b-1}\left(\frac{z(f(z) * g(z))'}{f(z) * g(z)} - p\right)\right\} > 0, \tag{33}$$

and hence $f(z) \in \mathscr{M}_p(n, b; g(z))$.

If we put $p = n = 1$, $b = 1-\alpha$ and $g(z) = z/(1-z)^{2(1-\alpha)}$ in Theorem 12, we obtain the following result proved in [12]. □

Corollary 13. *If $f(z) \in A$ satisfies*

$$\begin{aligned} \operatorname{Re}&\left\{\frac{zf'(z)}{f(z)}\left(\rho\frac{zf''(z)}{f'(z)} + 1\right)\right\} \\ &> \alpha\rho\left(\alpha - \frac{1}{2}\right) + \left(\alpha - \frac{\rho}{2}\right), \end{aligned} \tag{34}$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Furthermore, for $\alpha = 0$ in Corollary 13, we have the following result proved in [13].

Corollary 14. *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\rho\frac{zf''(z)}{f'(z)} + 1\right)\right\} > -\frac{\rho}{2}, \tag{35}$$

then $f(z) \in \mathcal{S}^*$.

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