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Research Article

The Existence and Uniqueness of Solutions for a Class of Nonlinear Fractional Differential Equations with Infinite Delay

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We prove the existence and uniqueness of solutions for two classes of infinite delay nonlinear fractional order differential equations involving Riemann-Liouville fractional derivatives. The analysis is based on the alternative of the Leray-Schauder fixed-point theorem, the Banach fixed-point theorem, and the Arzela-Ascoli theorem in $\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in \mathscr{B}\}$ such that $y|_{[0,b]}$ is continuous and $\mathscr{B}$ is a phase space.

1. Introduction

Fractional derivatives and integrals have been vastly used in different fields, facing a huge development especially during the last few decades (see, e.g., [1–9] and the references therein). The approaches based on fractional calculus establish models of engineering systems better than the ordinary derivatives approaches [1–6].

In particular, fractional differential equations as an important research branch of fractional calculus attracted much more attention (see, e.g., [10–20] and the references therein). Also varieties of schemes for numerical solutions of fractional differential equations are reported (see, e.g., [6, 21–23] and the references therein). We notice that some investigations have been done on the existence and uniqueness of solutions for fractional differential equations with delay (see, e.g., [24, 25] and the references therein).

Having all the aforementioned facts in mind, in this paper we study the existence and uniqueness of solutions for a class of delayed fractional differential equations, namely,

$$\begin{aligned} \mathscr{L}(\mathscr{D})\, y(t) &= f(t, y_t), \quad t \in J = [0, b], \\ y(t) &= \phi(t), \quad t \in (-\infty, 0], \end{aligned} \tag{1}$$

where $\mathscr{L}(\mathscr{D}) = D_{0^+}^{\alpha} - t^n D_{0^+}^{\beta}$, $0 < \beta < \alpha < 1$, n is a positive integer, $f : J \times \mathscr{B} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, $\phi \in \mathscr{B}$ with $\phi(0) = 0$, and $\mathscr{B}$ is called a phase space that will be defined later. $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives. y_t, which is an element $\mathscr{B}$, is defined as any function y on $(-\infty, b]$ as follows:

$$y_t(s) = y(t+s), \quad s \in (-\infty, 0], \ t \in J. \tag{2}$$

Here $y_t(\cdot)$ represents the preoperational state from time $-\infty$ up to time t. We also consider the following nonlinear fractional differential equation:

$$\begin{aligned} \mathscr{L}(\mathscr{D})\{y(t)-g(t,y_t)\} &= f(t,y_t), \quad t\in J, \\ y(t) &= \phi(t), \quad t\in(-\infty,0], \end{aligned} \tag{3}$$

where α, β, f, ϕ, and $\mathscr{L}(\mathscr{D})$ are as (1) and $g: J\times\mathscr{B}\to\mathbb{R}$ is a given function which satisfies $g(0,\phi)=0$.

The notion of the phase space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theories for functional differential equations. A common choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [26].

Our approach is based on the Banach fixed-point theorem and on the nonlinear alternative of Leray-Schauder type [27, 28]. The organization of the paper is as follows.

In Section 2, we present some basic mathematical tools used in the paper. The main results are presented in Section 3. Section 4 is dedicated to our conclusions.

2. Preliminaries

In this section, we present some basic notations and properties which are used throughout this paper. First of all, we will explain the phase space $\mathscr{B}$ introduced by Hale and Kato [26]. Let $\mathbb{R}^{\le 0}=(-\infty,0]$, $\mathbb{R}^{\ge 0}=[0,+\infty)$, $\mathbb{R}=(-\infty,+\infty)$, and let E be a Banach space with norm $|\cdot|_E$. Further, let $\mathscr{B}$ be a linear space of functions mapping $\mathbb{R}^-$ into E with seminorm $|\cdot|_{\mathscr{B}}$ having the following axioms,

- (B_1) If $y:(-\infty,\sigma+b)\to E$, $b>0$ is continuous on $[\sigma,\sigma+b)$ and $y_\sigma\in\mathscr{B}$, then $y_t\in\mathscr{B}$ and y_t are continuous for any $t\in[\sigma,\sigma+b)$.
- (B_2) There exist functions $k(t)>0$ and $m(t)\ge 0$ with the following properties. (i) $k(t)$ is continuous for $t\in\mathbb{R}^{\ge 0}$. (ii) $m(t)$ is locally bounded for $t\in\mathbb{R}^{\ge 0}$. (iii) For every function, y has the properties of (B_1) and $t\in[\sigma,\sigma+b)$, holds that $|y_t|_{\mathscr{B}}\le k(t-\sigma)\sup\{|y(s)|_E : \sigma\le s\le t\}+m(t-\sigma)|y_\sigma|_{\mathscr{B}}$.
- (B_3) There exists a positive constant L such that $|\phi(0)|_E\le L|\phi|_{\mathscr{B}}$ for all $\phi\in\mathscr{B}$.
- (B_4) The quotient space $\widehat{\mathscr{B}}:=\mathscr{B}/|\cdot|_{\mathscr{B}}$ is a Banach space.

We notice that in this paper, we select $\sigma=0$ and $E=\mathbb{R}$; thus (iii) can be converted to $|y_t|_{\mathscr{B}}\le k(t)\sup\{|y(s)|_E : 0\le s\le t\}+m(t)|y_0|_{\mathscr{B}}$, for all $t\in[0,b)$.

See [28] for examples of the phase space $\mathscr{B}$ satisfying all axioms (B_1)–(B_4).

Let $\mathbb{R}^+=(0,+\infty)$ and $C^0(\mathbb{R}^+)$ be the space of all continuous real function on $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R})^{\ge 0}$ of all continuous real functions on $\mathbb{R}^{\ge 0}$ which later identifies with the class of all $f\in C^0(\mathbb{R}^+)$ such that $\lim_{t\to 0^+}f(t)=f(0^+)\in\mathbb{R}$. By $C(J,\mathbb{R})$, we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|y\|_\infty:=\sup\{|y(t)| : t\in J\}$, where $|\cdot|$ is a suitable complete norm on $\mathbb{R}$.

The most common notation for αth order derivative of a real-valued function $y(t)$, which is defined in an interval denoted by (a,b), is $D_a^\alpha y(t)$. Here, the negative value of α corresponds to the fractional integral.

Definition 1. For a function y defined on an interval $[a,b]$, the Riemann-Liouville fractional integral of y of order $\alpha>0$ is defined by [1, 6]

$$I_{a^+}^\alpha y(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)\,ds, \quad t>a, \tag{4}$$

and the Riemann-Liouville fractional derivative of $y(t)$ of order $\alpha>0$ reads as

$$D_{a^+}^\alpha y(t)=\frac{d^n}{dt^n}\{I_{a^+}^{n-\alpha}y(t)\}, \quad n-1<\alpha\le n. \tag{5}$$

Also, we denote $D_{a^+}^\alpha y(t)$ as $D_a^\alpha y(t)$ and $I_{a^+}^\alpha y(t)$ as $I_a^\alpha y(t)$. Further, $D_{0^+}^\alpha y(t)$ and $I_{0^+}^\alpha y(t)$ are referred to as $D^\alpha y(t)$ and $I^\alpha y(t)$, respectively. If the fractional derivative $D_a^\alpha y(t)$ is integrable, then we have [4, page 71]

$$\begin{aligned} &I_a^\alpha\left(D_a^\beta y(t)\right) \\ &\quad = I_a^{\alpha-\beta}y(t)-\left[I_a^{1-\beta}y(t)\right]_{t=a}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}, \\ &\qquad\qquad 0<\beta\le\alpha<1. \end{aligned} \tag{6}$$

If y is continuous on $[a,b]$, then $D_a^\alpha y(t)$ is integrable, $I^{1-\beta}y(t)|_{t=a}=0$, and

$$I_a^\alpha\left(D_a^\beta y(t)\right)=I_a^{\alpha-\beta}y(t), \quad 0<\beta\le\alpha<1. \tag{7}$$

Proposition 2. *Let y be continuous on $[0,b]$ and n a nonnegative integer, then*

$$\begin{aligned} (i)\ I^\alpha(t^n y(t)) &= \sum_{k=0}^n \binom{-\alpha}{k}\left[D^k t^n\right]\left[I^{\alpha+k}y(t)\right] \\ &= \sum_{k=0}^n \binom{-\alpha}{k}\frac{n!t^{n-k}}{(n-k)!}I^{\alpha+k}y(t), \end{aligned} \tag{8}$$

$$(ii)\ I^\alpha\left(t^n D^\beta y(t)\right)=\sum_{k=0}^n\binom{-\alpha}{k}\frac{n!t^{n-k}}{(n-k)!}I^{\alpha-\beta+k}y(t), \tag{9}$$

where

$$\begin{aligned} \binom{-\alpha}{k} &= (-1)^k\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha)}=(-1)^k\binom{\alpha+k-1}{k} \\ &= \frac{\Gamma(1-\alpha)}{\Gamma(k+1)\Gamma(1-\alpha-k)}. \end{aligned} \tag{10}$$

Proof. (i) can be found in [6, page 53], and (ii) is an immediate consequence of (7), and (i). □

Lemma 3 (see [29]). *Let $v:[0,b]\to[0,\infty)$ be a real function and $w(\cdot)$ a nonnegative, locally integrable function on*

$[0,b]$. *If there exist positive constants a and $\alpha \in (0,1)$ such that $v(t) \le w(t) + a\int_0^t (t-s)^{-\alpha} v(s)ds$, then there exists a constant $K = K(\alpha)$ such that $v(t) \le w(t) + Ka\int_0^t w(s)(t-s)^{-\alpha}ds$, for all $t \in [0,b]$.*

In this paper we use the alternative Leray-Schauder's theorem and Banach's contraction principle for getting the main results. These theorems can be found in [27, 28].

3. Existence and Uniqueness

In this section, we prove the existence results for (1) and (3) by using the alternative of Leray-Schauder's theorem. Further, our results for the unique solution is based on the Banach contraction principle. Let us start by defining what we mean by a solution of (1). Let the space

$$\Omega = \{y : (-\infty, b] \longrightarrow \mathbb{R} : y|_{(-\infty,0]} \in \mathscr{B} \text{ and } y|_{[0,b]} \text{ is continuous}\}. \tag{11}$$

A function $y \in \Omega$ is said to be a solution of (1) if y satisfies (1).

For the existence results on (1), we need the following lemma.

Lemma 4. *Equation* (1) *is equivalent to the Volterra integral equation*

$$y(t) = \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k} y(t) + I^{\alpha} f(t, y_t), \quad t \in J. \tag{12}$$

Proof. The proof is an immediate consequence of Proposition 2. □

To study the existence and uniqueness of solutions for (1), we transform (1) into a fixed-point problem. Consider the operator $P : \Omega \to \Omega$ defined by

$$Py(t) = \begin{cases} \mathscr{L}(I) y(t) + I^{\alpha} f(t, y_t), & t \in [0,b], \\ \phi(t), & t \in (-\infty, 0], \end{cases} \tag{13}$$

where,

$$\mathscr{L}(I) = \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k}. \tag{14}$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be the function defined as

$$x(t) = \begin{cases} 0, & \text{if } t \in [0,b], \\ \phi(t), & \text{if } t \in (-\infty, 0]. \end{cases} \tag{15}$$

Then, we get $x_0 = \phi$. For each $z \in C([0,b], \mathbb{R})$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined as follows:

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0,b], \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases} \tag{16}$$

If $y(\cdot)$ satisfies the integral equation $y(t) = \mathscr{L}(I)y(t) + I^{\alpha} f(t, y_t)$, then we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t)$, $-\infty < t \le b$, which implies $y_t = \bar{z}_t + x_t$ for every $0 \le t \le b$, and the function $z(\cdot)$ satisfies

$$z(t) = \mathscr{L}(I) z(t) + I^{\alpha} f(t, \bar{z}_t + x_t), \tag{17}$$

set $C_0 = \{z \in C([0,b], \mathbb{R}) : z(0) = 0\}$, and let $\|\cdot\|_b$ be the seminorm in C_0 defined by $\|z\|_b = \|z_0\|_{\mathscr{B}} + \sup\{|z(t)| : 0 \le t \le b\} = \sup\{|z(t)| : 0 \le t \le b\}$, $z \in C_0$. C_0 is a Banach space with norm $\|\cdot\|_b$. Let the operator $F : C_0 \to C_0$ be defined by

$$Fz(t) = \mathscr{L}(I) z(t) + I^{\alpha} f(t, \bar{z}_t + x_t), \tag{18}$$

where $t \in [0,b]$. The operator P has a fixed point equivalent to F that has a fixed point too.

Theorem 5. *Assume that f is a continuous function, and there exist $p, q \in C(J, \mathbb{R}^+)$ such that $|f(t,u)| \le p(t) + q(t)\|u\|_{\mathscr{B}}$, $t \in J$, $u \in \mathscr{B}$. Then,* (1) *has at least one solution on $(-\infty, b]$.*

Proof. It is enough to show that the operator $F : C_0 \to C_0$ defined as (18) satisfies the following: (i) F is continuous, (ii) F maps bounded sets into bounded sets in C_0, (iii) F maps bounded sets into equicontinuous sets of C_0, and (iv) F is completely continuous.

(i) Let $\{z_n\}$ converges to z in C_0, then

$$\begin{aligned} &\|Fz_n(t) - Fz(t)\| \\ &\quad \le \sum_{k=0}^{n} \frac{|\binom{-\alpha}{k}| n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k} |z_n(t) - z(t)| \\ &\qquad + I^{\alpha} |f(t, (\bar{z}_n)_t + x_t) - f(t, \bar{z}_t + x_t)| \\ &\quad \le \sum_{k=0}^{n} \frac{|\binom{-\alpha}{k}| n!\, b^{n-k} \|z_n - z\|}{(n-k)!\Gamma(\alpha-\beta+k+1)} \\ &\qquad + \frac{b^{\alpha} \|f(t, (\bar{z}_n)_t + x_t) - f(t, \bar{z}_t + x_t)\|}{\Gamma(\alpha+1)}. \end{aligned} \tag{19}$$

Hence, $\|Fz_n(t) - Fz(t)\| \to 0$ as $z_n \to z$, and thus f is continuous.

(ii) For any $\lambda > 0$, let $\mathscr{B}_{\lambda} = \{z \in C_0 : \|z\|_b \le \lambda\}$ be a bounded set. We show that there exists a positive constant μ such that $\|Fz\|_\infty \leq \mu$. Let $z \in \mathscr{B}_\lambda$, since f is a continuous function, we have for each $t \in [0,b]$,

$$\begin{aligned}
|Fz(t)| \leq & \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| n! t^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k)} \\
& \times \int_0^b (t-s)^{\alpha-\beta+k-1} z(s) \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s)\, ds \\
\leq & \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| n! b^{n+\alpha-\beta}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b + \frac{1}{\Gamma(\alpha)} \\
& \times \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|\overline{z}_s + x_s\|_{\mathscr{B}}\right] ds \\
\leq & \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| n! b^{n+\alpha-\beta}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b \\
& + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} \left\{\|\overline{z}_s\|_{\mathscr{B}} + \|x_s\|_{\mathscr{B}}\right\} \\
\leq & \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| n! b^{n+\alpha-\beta}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b \\
& + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + k_b \lambda + m_b \|\phi\|_{\mathscr{B}} := \mu,
\end{aligned} \tag{20}$$

where $m_b = \sup\{|m(t)| : t \in [0,b]\}$, and $k_b = \sup\{|k(t)| : t \in [0,b]\}$. Hence, we obtain $\|Fz\|_\infty \leq \mu$.

(iii) Let $t_1, t_2 \in [0,b]$ and $t_1 < t_2$. Let $\mathscr{B}_\lambda$ be a bounded set of C_0 as in (ii) and $z \in \mathscr{B}_\lambda$, then given $\epsilon > 0$ choose

$$\delta = \min\left\{\frac{1}{2\Lambda_1}\epsilon^{1/\alpha}, \frac{1}{2(n+1)\Lambda_2}\epsilon^{1/(\alpha-\beta+k)} : k = 0, 1, \ldots, n\right\}, \tag{21}$$

where

$$\begin{aligned}
\Lambda_1 &= 2\frac{\|p\|_\infty + \Lambda\|q\|_\infty}{\Gamma(\alpha+1)}, \\
\Lambda_2 &= \sum_{k=0}^{n} \frac{2\left|\binom{-\alpha_n}{k}\right| k! b^{n-k} \|z\|_b}{(n-k)!\Gamma(\alpha-\beta+k+1)},
\end{aligned} \tag{22}$$

and $\Lambda = k_b\lambda + m_b\|\phi\|_{\mathscr{B}}$. If $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
& |Fz(t_2) - Fz(t_1)| \\
& \leq \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k)} \|z\|_b \\
& \quad \times \left|\int_0^{t_1} \left\{(t_2-s)^{\alpha-\beta+k-1} - (t_1-s)^{\alpha-\beta+k-1}\right\} ds\right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta+k-1} ds\right| \\
& \quad + \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_1} \left\{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right\} f(s, \overline{z}_s + x_x)\, ds\right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, \overline{z}_s + x_x)\, ds\right| \\
& \leq \sum_{k=0}^{n} \frac{2\left|\binom{-\alpha_n}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b (t_2-t_1)^{\alpha-\beta+k} \\
& \quad + \frac{\|p\|_\infty + \Lambda\|q\|_\infty}{\Gamma(\alpha+1)} \left\{\int_0^{t_1} \left\{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right\} ds\right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\right\} \\
& \leq \sum_{k=0}^{n} \frac{2\left|\binom{-\alpha_n}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b (t_2-t_1)^{\alpha-\beta+k} \\
& \quad + 2\frac{\|p\|_\infty + \Lambda\|q\|_\infty}{\Gamma(\alpha+1)} (t_2-t_1)^\alpha \\
& = \Lambda_2 \delta^{\alpha-\beta+k} + \Lambda_1 \delta^\alpha < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned} \tag{23}$$

where $\|\overline{z}_s + x_s\|_{\mathscr{B}} \leq \|\overline{z}_s\|_{\mathscr{B}} + \|x_s\|_{\mathscr{B}} \leq k_b\lambda + m_b\|\phi\|_{\mathscr{B}} := \Lambda$. Hence, $F(\mathscr{B}_\lambda)$ is equicontinuous.

(iv) It is an immediate consequence from (i)–(iii), together with the Arzela-Ascoli theorem.

We show in the following that there exists an open set $U \subseteq C_0$ with $z \neq \gamma F(z)$ for $\gamma \in (0,1)$ and $z \in \partial U$. Let $z \in C_0$ and $z = \gamma F(z)$ for some $0 < \gamma < 1$. Then, for each $t \in [0,b]$, we have $z(t) = \lambda\{\mathscr{L}(I)z(t) + I^\alpha f(t, \overline{z}_t + x_t)\}$. It follows by assumption of the theorem

$$\begin{aligned}
|z(t)| \leq & \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k)} \int_0^t (t-s)^{\alpha-\beta+k-1} |z(s)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s, \overline{z}_s + x_x)\right| ds
\end{aligned}$$

$$\leq \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k} \|z\|_b}{(n-k)!\Gamma(\alpha-\beta+k+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \|\bar{z}_s + x_s\|_{\mathscr{B}} ds + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}. \tag{24}$$

On other hand, we have

$$\begin{aligned}\|\bar{z}_s + x_s\|_B &\leq \|\bar{z}_s\|_{\mathscr{B}} + \|x_s\|_{\mathscr{B}} \\ &\leq k(t) \sup\{|z(s)| : 0 \leq s \leq t\} \\ &\quad + m(t) \|z_0\|_{\mathscr{B}} \\ &\quad + k(t) \sup\{|x(s)| : 0 \leq s \leq t\} \\ &\quad + m(t) \|x_0\|_{\mathscr{B}} \\ &\leq k_b \sup\{|z(s)| : 0 \leq t \leq t\} \\ &\quad + m_b \|\phi\|_{\mathscr{B}}.\end{aligned} \tag{25}$$

If we let $\delta(t)$ the right-hand side of (25), then $\|\bar{z}_s + x_s\|_{\mathscr{B}} \leq \delta(t)$ and, therefore,

$$|z(t)| \leq \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k} \|z\|_b}{(n-k)!\Gamma(\alpha-\beta+k+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \delta(s) ds + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}. \tag{26}$$

Using the aforementioned inequality and the definition of δ, we get

$$\delta(t) \leq \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k} \|z\|_b k_b}{(n-k)!\Gamma(\alpha-\beta+k+1)} + m_b \|\phi\|_{\mathscr{B}} + \frac{k_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{k_b \|q\|_\infty}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} \delta(s) ds. \tag{27}$$

Then, using Lemma 3, there exists a constant Δ such that

$$|\delta(t)| \leq \frac{1}{2} k_b \Lambda_2 + m_b \|\phi\|_{\mathscr{B}} + \frac{k_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \Delta \frac{k_b \|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R\, ds, \tag{28}$$

where Λ_2 is mentioned in (22), and

$$R = \frac{1}{2} k_b \Lambda_2 + m_b \|\phi\|_{\mathscr{B}} + \frac{k_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}. \tag{29}$$

Hence,

$$\|\delta\|_\infty \leq R + \frac{R \Delta b^\alpha k_b \|q\|_\infty}{\Gamma(\alpha+1)} := \widetilde{M}, \tag{30}$$

and then $\|z\|_\infty \leq \Lambda_2 + \widetilde{M} \|I^\alpha q\|_\infty + b^\alpha \|p\|_\infty / \Gamma(\alpha+1)$. Therefore,

$$\|z\|_\infty \leq \frac{\widetilde{M} \|I^\alpha q\|_\infty + b^\alpha \|p\|_\infty / \Gamma(\alpha+1)}{1-\Lambda_2} := \Delta^*. \tag{31}$$

Set $U = \{z \in C_0 : \|z\|_b < \Delta^* + 1\}$. Then, $F : \overline{U} \to C_0$ is continuous and completely continuous. From the choice of U, there is no $z \in \partial U$ such that $z = \gamma F(z)$, for $\gamma \in (0,1)$; therefore, by the nonlinear alternative of the Leray-Schauder theorem, the proof is complete. □

Theorem 6. *Let $f : J \times B \to \mathbb{R}$ be a continuous function. If there exists a positive constant l such that $|f(t,u) - f(t,v)| \leq l\|u-v\|_{\mathscr{B}}$, $t \in J$, $u, v \in \mathscr{B}$, and $0 < T + l k_b b^\alpha / \Gamma(\alpha+1) := L < 1$ then* (1) *has a unique solution in the interval $(-\infty, b]$, where,*

$$T = \sum_{k=0}^{n} \frac{\left|\binom{-\alpha_n}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k+1)}. \tag{32}$$

Proof. The solution of (1) is equivalent to the solution of the integral equation (17). Hence, it is enough to show that the operator $F : C_0 \to C_0$, satisfies the Banach fixed-point theorem. Consider $u, v \in C_0$ and for each $t \in [0,b]$, we have

$$\begin{aligned}&|F(z)(t) - F(u)(t)| \\ &\leq T\|u-v\|_b + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l \|\bar{u}_s - \bar{v}_s\|_{\mathscr{B}} ds \\ &\leq T\|u-v\|_b + \frac{l}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\bar{u}_s - \bar{v}_s\|_{\mathscr{B}} ds \\ &\leq T\|u-v\|_b + \frac{l}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times k_b \sup \|u(s) - v(s)\| ds \\ &\leq \left\{T + \frac{l k_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l\, ds\right\} |u-v|_b \\ &\leq \left\{T + \frac{l k_b b^\alpha}{\Gamma(\alpha+1)}\right\} \|u-v\|_b = L\|u-v\|_b.\end{aligned} \tag{33}$$

Hence, $\|F(z) - F(v)\|_b \leq L \|z - z^*\|_b$, and then F is a contraction. Therefore, F has a unique fixed point by Banach's contraction principle. □

Theorem 7. *Let $f : J \times \mathscr{B} \to \mathbb{R}$ be a continuous function, and let the following assumptions hold.*

(H1) *There exist $p, q \in C(J, \mathscr{R}^{\geq 0})$ such that $|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathscr{B}}$ for each $t \in J$, $u \in \mathscr{B}$ and and $\|I^\alpha p\| < +\infty$.*

(H2) *The function g is continuous and completely continuous. For any bounded set $\mathscr{D}$ in Ω, the set $\{t \to g(t, y_t) : y \in \mathscr{D}\}$ is equicontinuous in $C([0,b], \mathbb{R})$. There exist* *positive constants d_1 and d_2 such that $|g(t,u)| \le d_1\|u\|_{\mathscr{B}} + d_2$ for each $t \in [0,b]$ and $u \in \mathscr{B}$.*

If $k_b d_1 \in (0,1)$, then (3) *has at least one solution on $(-\infty, b]$, where $k_b = \sup\{|k(t)| : t \in [0,b]\}$.*

Proof. Consider the operator $P^* : \Omega \to \Omega$ defined by

$$P^*(y)(t) = \begin{cases} \mathscr{L}(I)\,y(t) + I^{\alpha} f(t, y_t) + g(t, y_t), & t \in [0,b], \\ \phi(t), & t \in (-\infty, 0], \end{cases} \tag{34}$$

where

$$\mathscr{L}(I) = \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k}. \tag{35}$$

In analog to Theorem 5, we consider the operator $F^* : C_0 \to C_0$ defined by

$$F^* z(t) = \mathscr{L}(I)\,z(t) + I^{\alpha} f(t, \overline{z}_t + x_t) + g(t, \overline{z}_t + x_t). \tag{36}$$

By using (H2) and Theorem 5, the operator F^* is continuous and completely continuous. Now, it is sufficient to show that there exists an open set $U^* \subseteq C_0$ with $z \ne \lambda F^*(z)$ for $\gamma \in (0,1)$ and $z \in \partial U^*$.

Let $z \in C_0$ and $z = \gamma F^*(z)$ for some $\gamma \in (0,1)$. Then, for each $t \in [0,b]$, $z(t) = \gamma[g(t, \overline{z}_t + x_t) + \mathscr{L}(I)z(t) + I^{\alpha} f(t, \overline{z}_t + x_t)]$. Hence,

$$\begin{aligned} |z(t)| &\le d_1\|\overline{z}_t + x_t\|_{\mathscr{B}} + d_2 \\ &\quad + \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b \\ &\quad + \frac{b^{\alpha_n}\|p\|_{\infty}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha_n - 1} q(s) \|\overline{z}_s + x_s\|_{\mathscr{B}} ds, \\ &\le d_1 \delta(t) + d_2 + \frac{b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s)\,\delta(s)\, ds \\ &\quad + \sum_{k=0}^{n} \frac{\left|\binom{-\alpha}{k}\right| k! b^{n-k}}{(n-k)!\Gamma(\alpha-\beta+k+1)} \|z\|_b, \end{aligned} \tag{37}$$

where $\delta(t)$ is named the in right-hand side of (25) such that $\|\overline{z}_s - x_s\| \le \delta(t)$. Since $0 < k_b d_1 < 1$, if we let $T^* = \sum_{k=0}^{n}(|\binom{-\alpha}{k}| k! b^{n-k} \|z\|_b\, k_b/(n-k)!\Gamma(\alpha-\beta+k+1))$, then

$$\begin{aligned} \delta(t) &\le k_b d_1 \delta(t) + k_b d_2 + m_b\|\phi\|_{\mathscr{B}} + T^* + m_b\|\phi\|_{\mathscr{B}} \\ &\quad + \frac{k_b b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} + \frac{k_b\|q\|_{\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\delta(s)\, ds \\ &\le \frac{1}{1-k_b d_1} \Bigg\{ k_b d_2 + m_b\|\phi\|_{\mathscr{B}} + T^* + m_b\|\phi\|_{\mathscr{B}} \\ &\quad + \frac{k_b b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} + \frac{k_b\|q\|_{\infty}}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1}\delta(s)\, ds \Bigg\}. \end{aligned} \tag{38}$$

Then, using Lemma 3, there exists a constant Δ^* such that

$$\begin{aligned} \delta(t) &\le k_b d_1 \delta(t) + k_b d_2 + m_b\|\phi\|_{\mathscr{B}} \\ &\quad + T^* + m_b\|\phi\|_{\mathscr{B}} + \frac{k_b b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} \\ &\quad + \frac{k_b\|q\|_{\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\delta(s)\, ds \\ &\le \frac{1}{1-k_b d_1} \\ &\quad \times \Bigg\{ k_b\, d_2 + m_b\|\phi\|_{\mathscr{B}} + T^* + m_b\|\phi\|_{\mathscr{B}} + \frac{k_b b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} \\ &\quad + \Delta^* \frac{k_b\|q\|_{\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\delta(s)\, ds \Bigg\}, \end{aligned} \tag{39}$$

and, therefore, $\|w\|_{\infty} \le R + R\Delta^* k_b \|q^*\|_{\infty}/\Gamma(\alpha+1) := L^*$, where $\|q^*\|_{\infty} = \|q\|_{\infty}/(1-k_b d_1)$ and $R = 1/(1-k_b d_1)[k_b d_2 + m_b\|\phi\|_{\mathscr{B}} + (k_b b^{\alpha}\|p\|_{\infty})/\Gamma(\alpha+1) + T^*]$. Then,

$$\|z\|_{\infty} \le d_1 L^* + d_2 + \frac{b^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} + L\|I^{\alpha} q\|_{\infty} + T^*, \tag{40}$$

and, hence,

$$\|z\|_{\infty} \le \frac{d_1 L^* + d_2 + b^{\alpha}\|p\|_{\infty}/\Gamma(\alpha+1) + L^*\|I^{\alpha} q\|_{\infty}}{1 - \|z\|_{\infty} T^*} := M^*. \tag{41}$$

Set $U^* = \{z \in C_0 : \|z\|_b < M^* + 1\}$. From the choice of U^*, there is no $z \in \partial U^*$ such that $z = \gamma F^*(z)$ for $\gamma \in (0,1)$. As a consequence of the nonlinear alternative of the Leray-Schauder theorem, we deduce that F^* has a fixed-point z^* in U^*, which is a solution of (3). □

The unique solution of (3), under some conditions, is studied in the following theorem which is the result of the Banach contraction mapping.

Theorem 8. *Let $f : J \times \mathscr{B} \to \mathbb{R}$ be a continuous function, and there exist positive constants l, μ, such that*

$$\begin{aligned} |f(t,u) - f(t,v)| &\le l\|u - v\|_{\mathscr{B}}, \\ |g(t,u) - g(t,v)| &\le \mu\|u - v\|_{\mathscr{B}}, \end{aligned} \tag{42}$$

where $t \in J$ and $u, v \in \mathscr{B}$. Then, (3) *with the following conditions has a unique solution in the interval $(-\infty, b]$*

$$0 < T + \frac{lk_b b^\alpha}{\Gamma(\alpha+1)} < 1, \qquad 0 < k_b\mu + T + \frac{k_b l b^\alpha}{\Gamma(\alpha+1)} < 1, \tag{43}$$

such that T is defined in Theorem 6.

Proof. The proof is a similar process Theorem 6. □

4. Conclusions

In this paper, the existence and the uniqueness of solutions for the nonlinear fractional differential equations with infinite delay comprising standard Riemann-Liouville derivatives have been discussed in the phase space. Leray-Schauder's alternative theorem and the Banach contraction principle were used to prove the obtained results. Further generalizations can be developed to some other class of fractional differential equations such as $\mathscr{L}(D)y(t) = f(t, y_t)$, where $\mathscr{L}(D) = D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(t) D^{\alpha_{n-j}}$, $0 < \alpha_1 < \cdots < \alpha_n < 1$, $p_j(t) = \sum_{k=0}^{N_j} a_{jk} t^k$, and N_j is nonnegative integer.

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