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Research Article

On a New Class of Antiperiodic Fractional Boundary Value Problems

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This paper investigates a new class of antiperiodic boundary value problems of higher order fractional differential equations. Some existence and uniqueness results are obtained by applying some standard fixed point principles. Some examples are given to illustrate the results.

1. Introduction

Boundary value problems of fractional differential equations involving a variety of boundary conditions have recently been investigated by several researchers. It has been mainly due to the occurrence of fractional differential equations in a number of disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, andfitting of experimental data. For details and examples, see [1–5]. The recent development of the subject can be found, for example, in papers [6–16].

The mathematical modeling of a variety of physical processes gives rise to a class of antiperiodic boundary value problems. This class of problems has recently received considerable attention; for instance, see [17–24] and the references therein. In [22], the authors studied a Caputo-type antiperiodic fractional boundary value problem of the form

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,T], \ T > 0, \ 1 < q \le 2, \\ x(0) = -x(T), \\ {}^cD^p x(0) = -{}^cD^p x(T), \quad 0 < p < 1. \end{gathered} \tag{1}$$

In this paper, we investigate a new class of antiperiodic fractional boundary value problems given by

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,T], \ T > 0, \ 2 < q \le 3, \\ x(0) = -x(T), \\ {}^cD^p x(0) = -{}^cD^p x(T), \\ {}^cD^{p+1} x(0) = -{}^cD^{p+1} x(T), \quad 0 < p < 1, \end{gathered} \tag{2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q and f is a given continuous function. Some new existence and uniqueness results are obtained for problem (2) by using standard fixed point theorems.

2. Preliminaries

Let us recall some basic definitions [1–3].

Definition 1. The Riemann-Liouville fractional integral of order q for a continuous function $g : [0, +\infty) \to \mathbb{R}$ is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{3}$$

provided the integral exists.

Definition 2. For $(n-1)$ times absolutely continuous function $g : [0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$\begin{gathered} {}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \\ n-1 < q < n, \ n = [q] + 1, \end{gathered} \tag{4}$$

where $[q]$ denotes the integer part of the real number q.

Notice that the Caputo derivative of a constant is zero.

Lemma 3. *For any $y \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{gathered} {}^cD^q x(t) = y(t), \quad 0 < t < T,\ 2 < q \le 3, \\ x(0) = -x(T), \\ {}^cD^p x(0) = -{}^cD^p x(T), \\ {}^cD^{p+1} x(0) = -{}^cD^{p+1} x(T), \quad 0 < p < 1, \end{gathered} \tag{5}$$

is

$$x(t) = \int_0^T G_T(t,s)\, y(s)\, ds, \tag{6}$$

where $G_T(t,s)$ is Green's function (depending on q and p) given by

$$G_T(t,s) = \begin{cases} \dfrac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)} + \dfrac{(\Gamma(2-p))^2 T^{p-1}(T-s)^{q-p-2}}{4\Gamma(q-p-1)\Gamma(3-p)} \\ \quad \times \left\{ \dfrac{(T^2-2t^2)\Gamma(3-p)}{\Gamma(2-q)} - 2T^2 + 4tT \right\}, \quad s \le t, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)} + \dfrac{(\Gamma(2-p))^2 T^{p-1}(T-s)^{q-p-2}}{4\Gamma(q-p-1)\Gamma(3-p)} \\ \quad \times \left\{ \dfrac{(T^2-2t^2)\Gamma(3-p)}{\Gamma(2-q)} - 2T^2 + 4tT \right\}, \quad t < s. \end{cases} \tag{7}$$

Proof. We know that the general solution of equation ${}^cD^q x(t) = y(t)$, $2 < q \le 3$ can be written as [3]

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds - b_0 - b_1 t - b_2 t^2, \tag{8}$$

for some constants b_0, b_1, and $b_2 \in \mathbb{R}$. Using the facts ${}^cD^p b = 0$ (b is a constant),

$$\begin{gathered} {}^cD^p t = \frac{t^{1-p}}{\Gamma(2-p)}, \qquad {}^cD^p t^2 = \frac{2t^{2-p}}{\Gamma(3-p)}, \\ {}^cD^{p+1} t^2 = \frac{2t^{1-p}}{\Gamma(2-p)}, \qquad {}^cD^p I^q y(t) = I^{q-p} y(t), \end{gathered} \tag{9}$$

we get

$$\begin{aligned} {}^cD^p x(t) &= \int_0^t \frac{(t-s)^{q-p-1}}{\Gamma(q-p)} y(s)\, ds - b_1 \frac{t^{1-p}}{\Gamma(2-p)} - \frac{2b_2 t^{2-p}}{\Gamma(3-p)}, \\ {}^cD^{p+1} x(t) &= \int_0^t \frac{(t-s)^{q-p-2}}{\Gamma(q-p-1)} y(s)\, ds - 2b_2 \frac{t^{1-p}}{\Gamma(2-p)}. \end{aligned} \tag{10}$$

Applying the boundary conditions for the problem (5), we find that

$$\begin{aligned} b_0 &= \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} y(s)\, ds - \frac{T^p\Gamma(2-p)}{2} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} y(s)\, ds \\ &\quad + \frac{T^{p+1}\Gamma(2-p)}{2} \left\{ \frac{\Gamma(2-p)}{\Gamma(3-p)} - \frac{1}{2} \right\} \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} y(d)\, ds, \\ b_1 &= \Gamma(2-p) T^{p-1} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} y(s)\, ds - \frac{(\Gamma(2-p))^2 T^p}{\Gamma(3-p)} \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} y(s)\, ds, \\ b_2 &= \frac{\Gamma(2-p)}{2T^{1-p}} \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} y(s)\, ds. \end{aligned} \tag{11}$$

Substituting the values of b_0, b_1, and b_2 in (8), we get the solution (6). This completes the proof. □

Remark 4. For $p = 1$, the solution of the antiperiodic problem

$$\begin{gathered} {}^cD^q x(t) = y(t), \qquad x(0) = -x(T), \\ x'(0) = -x'(T), \qquad x''(0) = -x''(T), \\ 0 < t < T,\ 2 < q \le 3, \end{gathered} \tag{12}$$

is given by [18]

$$x(t) = \int_0^T g(t,s)\, y(s)\, ds, \tag{13}$$

where $g(t,s)$ is

$$g(t,s)=\begin{cases}\dfrac{(t-s)^{q-1}-(1/2)(T-s)^{q-1}}{\Gamma(q)}\\+\dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)}\\+\dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)}, & s\le t,\\-\dfrac{(T-s)^{q-1}}{2\Gamma(q)}+\dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)}\\+\dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)}, & t<s.\end{cases} \tag{14}$$

If we let $p\to 1^-$ in (7), we obtain

$$G_T(t,s)\big|_{p\to 1^-}=\begin{cases}\dfrac{(t-s)^{q-1}-(1/2)(T-s)^{q-1}}{\Gamma(q)}\\+\dfrac{(T-2t)(T-s)^{q-2}}{2\Gamma(q-1)}\\+\dfrac{\left(-2t^2-T^2+4tT\right)(T-s)^{q-3}}{4\Gamma(q-2)}, & s\le t,\\-\dfrac{(T-s)^{q-1}}{2\Gamma(q)}+\dfrac{(T-2t)(T-s)^{q-2}}{2\Gamma(q-1)}\\+\dfrac{\left(-2t^2-T^2+4tT\right)(T-s)^{q-3}}{4\Gamma(q-2)}, & t<s.\end{cases} \tag{15}$$

We note that the solutions given by (14) and (15) are different. As a matter of fact, (15) contains an additional term: $(-t^2-T^2+3tT)(T-s)^{q-3}/4\Gamma(q-2)$. Therefore the fractional boundary conditions introduced in (2) give rise to a new class of problems.

Remark 5. When the phenomenon of antiperiodicity occurs at an intermediate point $\eta\in(0,T)$, the parametric-type antiperiodic fractional boundary value problem takes the form

$$\begin{gathered}{}^cD^q x(t)=f(t,x(t)),\quad t\in[0,T],\ 2<q\le 3,\\ x(0)=-x(\eta),\qquad {}^cD^p(0)=-{}^cD^p(\eta),\\ {}^cD^{p+1}(0)=-{}^cD^{p+1}(\eta),\end{gathered} \tag{16}$$

whose solution is

$$x(t)=\int_0^T G_\eta(t,s)\,f(s,x(s))\,ds, \tag{17}$$

where $G_\eta(t,s)$ is given by (7). Notice that $G_\eta(t,s)\to G_T(t,s)$ when $\eta\to T^-$.

3. Existence Results

Let $\mathfrak{C}=C([0,T],R)$ denotes a Banach space of all continuous functions defined on $[0,T]$ into R endowed with the usual supremum norm.

In relation to (2), we define an operator $\mathcal{F}:\mathfrak{C}\to\mathfrak{C}$ as

$$\begin{aligned}(\mathcal{F}x)(t)={}&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\\&-\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\\&+\frac{\Gamma(2-p)(T-2t)}{2T^{1-p}}\\&\times\int_0^T\frac{(T-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))\,ds\\&+\Gamma(2-p)T^{p-1}\\&\times\left(T^2-2t^2-\frac{2T^2\Gamma(2-p)}{\Gamma(3-p)}+\frac{4tT\Gamma(2-p)}{\Gamma(3-p)}\right)\\&\times(4)^{-1}\\&\times\int_0^T\frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)}f(s,x(s))\,ds,\quad t\in[0,T].\end{aligned} \tag{18}$$

Observe that the problem (2) has a solution if and only if the operator $\mathcal{F}$ has a fixed point.

For the sequel, we need the following known fixed point theorems.

Theorem 6 (see [25]). *Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X\mid u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 7 (see [25]). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator such that*

$$\|Tu\|\le\|u\|,\quad\forall u\in\partial\Omega. \tag{19}$$

Then T has a fixed point in $\overline{\Omega}$.

Now we are in a position to present the main results of the paper.

Theorem 8. *Assume that there exists a positive constant L_1 such that $|f(t,x(t))|\le L_1$ for $t\in[0,T]$, $x\in\mathfrak{C}$. Then the problem (2) has at least one solution.*

Proof. First, we show that the operator $\mathcal{F}$ is completely continuous. Clearly continuity of the operator $\mathcal{F}$ follows from the continuity of f. Let $\Omega\subset\mathfrak{C}$ be bounded. Then, for all $x \in \Omega$ together with the assumption $|f(t,x(t))| \le L_1$, we get

$$
\begin{aligned}
&|(\mathscr{F}x)(t)| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \frac{\Gamma(2-p)|T-2t|}{2T^{1-p}} \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s))| ds \\
&\quad + \Gamma(2-p) T^{p-1} \\
&\quad \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| (4)^{-1} \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s,x(s))| ds \\
&\le L_1 \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right. \\
&\qquad + \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds \\
&\qquad + \Gamma(2-p) T^{p-1} \\
&\qquad \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| (4)^{-1} \\
&\qquad \left. \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} ds \right\} \\
&\le L_1 \left\{ \frac{3T^q}{2\Gamma(q+1)} + \frac{\Gamma(2-p)T^q}{2\Gamma(q-p+1)} + \frac{\Gamma(2-p)T^q}{4\Gamma(q-p)} \right. \\
&\qquad \left. \times \left(1 - \frac{2\Gamma(2-p)}{\Gamma(3-p)} + 2\left(\frac{\Gamma(2-p)}{\Gamma(3-p)}\right)^2 \right) \right\} = M_1,
\end{aligned} \tag{20}
$$

which implies that $\|(\mathscr{F}x)(t)\| \le M_1$.

Furthermore,

$$
\begin{aligned}
&\left|(\mathscr{F}x)'(t)\right| \\
&\le \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| ds \\
&\quad + \Gamma(2-p) T^{p-1} \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s))| ds \\
&\quad + \frac{\Gamma(2-p)T^{p-1}}{4} \left| \frac{4T\Gamma(2-p)}{\Gamma(3-p)} - 4t \right| \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s,x(s))| ds \\
&\le L_1 T^{q-1} \left| \left\{ \frac{1}{\Gamma(q)} + \frac{\Gamma(2-p)}{\Gamma(q-p+1)} \right. \right. \\
&\qquad \left. \left. \times \left\{ 1 + \frac{(q-p)\Gamma(2-p)}{\Gamma(3-p)} \right\} \right\} \right| = M_2.
\end{aligned} \tag{21}
$$

Hence, for $t_1, t_2 \in [0,T]$, we have

$$
\begin{aligned}
|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| &\le \int_{t_1}^{t_2} \left|(\mathscr{F}x)'(s)\right| ds \\
&\le M_2 (t_2 - t_1).
\end{aligned} \tag{22}
$$

This implies that $\mathscr{F}$ is equicontinuous on $[0,T]$, by the Arzela-Ascoli theorem, the operator $\mathscr{F} : \mathfrak{C} \to \mathfrak{C}$ is completely continuous.

Next, we consider the set

$$
V = \{x \in \mathfrak{C} \mid x = \mu \mathscr{F}x, 0 < \mu < 1\} \tag{23}
$$

and show that the set V is bounded. Let $x \in V$, then $x = \mu\mathscr{F}x$, $0 < \mu < 1$. For any $t \in [0,T]$, we have

$$
\begin{aligned}
x(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s)) ds \\
&- \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s)) ds \\
&+ \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \\
&\times \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s)) ds \\
&+ \Gamma(2-p) T^{p-1} \\
&\times \left(T^2 - 2t^2 - \frac{2T^2\Gamma(2-p)}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right) (4)^{-1} \\
&\times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} f(s,x(s)) ds,
\end{aligned}
$$

$$
\begin{aligned}
|x(t)| &= \mu |(\mathscr{F}x)(t)| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \frac{\Gamma(2-p)|T-2t|}{2T^{1-p}} \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s))| ds \\
&\quad + \Gamma(2-p) T^{p-1} \\
&\quad \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| (4)^{-1} \\
&\quad \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s,x(s))| ds \\
&\leq L_1 \max_{t\in[0,T]} \left\{ \frac{2t^q + T^q}{2\Gamma(q+1)} \right. \\
&\qquad + \frac{\Gamma(2-p)|T-2t| T^{q-1}}{2\Gamma(q-p+1)} \\
&\qquad + \Gamma(2-p) T^{q-2} \\
&\qquad \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} \right. \\
&\qquad \left. \left. + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| (4)^{-1} \right\} = M_3.
\end{aligned}
\tag{24}
$$

Thus, $\|x\| \leq M_3$ for any $t \in [0,T]$. So, the set V is bounded. Thus, by the conclusion of Theorem 6, the operator $\mathscr{F}$ has at least one fixed point, which implies that (2) has at least one solution. □

Theorem 9. *Let there exists a positive constant r such that $|f(t,x)| \leq \delta|x|$ with $0 < |x| < r$, where δ is a positive constant satisfying*

$$
\begin{aligned}
&\max_{t\in[0,T]} \left\{ \frac{2|t|^q + T^q}{2\Gamma(q+1)} + \frac{\Gamma(2-p)T^{q-1}|T-2t|}{2\Gamma(q-p+1)} \right. \\
&\quad + \frac{\Gamma(2-p)T^{q-2}}{4\Gamma(q-p)} \\
&\quad \left. \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| \right\} \delta \leq 1.
\end{aligned}
\tag{25}
$$

Then the problem (2) *has at least one solution.*

Proof. Define $\Omega_1 = \{x \in \mathfrak{C} : \|x\| < r\}$ and take $x \in \mathfrak{C}$ such that $\|x\| = r$; that is, $x \in \partial\Omega$. As before, it can be shown that $\mathscr{F}$ is completely continuous and that

$$
\begin{aligned}
&|\mathscr{F}x(t)| \\
&\leq \max_{t\in[0,T]} \left\{ \frac{2|t|^q + T^q}{2\Gamma(q+1)} \right. \\
&\qquad + \frac{\Gamma(2-p)T^{q-1}|T-2t|}{2\Gamma(q-p+1)} \\
&\qquad + \frac{\Gamma(2-p)T^{q-2}}{4\Gamma(q-p)} \\
&\qquad \left. \times \left| T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4tT\Gamma(2-p)}{\Gamma(3-p)} \right| \right\} \\
&\qquad \times \delta \|x\| \leq \|x\|
\end{aligned}
\tag{26}
$$

for $x \in \partial\Omega$, where we have used (25). Therefore, by Theorem 7, the operator $\mathscr{F}$ has at least one fixed point which in turn implies that the problem (2) has at least one solution. □

Theorem 10. *Assume that $f : [0,T]\times\mathscr{R} \to \mathscr{R}$ is a continuous function satisfying the condition*

$$
|f(t,x) - f(t,y)| \leq L|x-y|, \quad \forall t \in [0,T], \ x, y \in \mathscr{R}, \tag{27}
$$

with $L\kappa < 1$, where

$$
\begin{aligned}
\kappa = T^q \Bigg[&\frac{3}{2\Gamma(q+1)} + \frac{\Gamma(2-p)}{2\Gamma(q-p+1)} + \frac{\Gamma(2-p)}{4\Gamma(q-p)} \\
&\times \left\{ 1 - \frac{2\Gamma(2-p)}{\Gamma(3-p)} + 2\left(\frac{\Gamma(2-p)}{\Gamma(3-p)}\right)^2 \right\} \Bigg].
\end{aligned}
\tag{28}
$$

Then the problem (2) *has a unique solution.*

Proof. Let us fix $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and select

$$
r \geq \frac{M\kappa}{1 - L\kappa}, \tag{29}
$$

where κ is given by (28). Then we show that $\mathscr{F}B_r \subset B_r$, where $B_r = \{x \in \mathfrak{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$
\begin{aligned}
&|(\mathscr{F}x)(t)| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,0) + f(s,0)| ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left|f(s,x(s))-f(s,0)+f(s,0)\right|ds \\
&+\frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \\
&\times\int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)}\left|f(s,x(s))-f(s,0)+f(s,0)\right|ds \\
&+\Gamma(2-p)T^{p-1} \\
&\times\left(T^2-2t^2-\frac{2T^2\Gamma(2-p)}{\Gamma(3-p)}+\frac{4tT\Gamma(2-p)}{\Gamma(3-p)}\right)(4)^{-1} \\
&\times\int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)}\left|f(s,x(s))-f(s,0)+f(s,0)\right|ds \\
\le{}&(Lr+M) \\
&\times\left\{\frac{2|t|^q+T^q}{2\Gamma(q+1)}+\frac{\Gamma(2-p)T^{q-1}|T-2t|}{2\Gamma(q-p+1)}\right. \\
&\quad+\frac{\Gamma(2-p)T^{q-2}}{4\Gamma(q-p)} \\
&\quad\left.\times\left|T^2-2t^2-\frac{2\Gamma(2-p)T^2}{\Gamma(3-p)}+\frac{4tT\Gamma(2-p)}{\Gamma(3-p)}\right|\right\} \\
\le{}&(Lr+M) \\
&\times T^q\left[\frac{3}{2\Gamma(q+1)}+\frac{\Gamma(2-p)}{2\Gamma(q-p+1)}+\frac{\Gamma(2-p)}{4\Gamma(q-p)}\right. \\
&\quad\left.\times\left\{1-\frac{2\Gamma(2-p)}{\Gamma(3-p)}+2\left(\frac{\Gamma(2-p)}{\Gamma(3-p)}\right)^2\right\}\right] \\
={}&(Lr+M)\kappa\le r.
\end{aligned}
\tag{30}
$$

Thus we get $\mathscr{F}x\in B_r$. Now, for $x,y\in\mathfrak{C}$ and for each $t\in[0,T]$, we obtain

$$
\begin{aligned}
&\left|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\right| \\
\le{}&\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left\|f(s,x(s))-f(s,y(s))\right\|ds\right. \\
&+\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left\|f(s,x(s))-f(s,y(s))\right\|ds \\
&+\frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \\
&\times\int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)}\left\|f(s,x(s))-f(s,y(s))\right\|ds \\
&+\Gamma(2-p)T^{p-1} \\
&\times\left(T^2-2t^2-\frac{2T^2\Gamma(2-p)}{\Gamma(3-p)}+\frac{4tT\Gamma(2-p)}{\Gamma(3-p)}\right)(4)^{-1} \\
&\left.\times\int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)}\left\|f(s,x(s))-f(s,y(s))\right\|ds\right\} \\
\le{}&L\|x-y\| \\
&\times T^q\left[\frac{3}{2\Gamma(q+1)}+\frac{\Gamma(2-p)}{2\Gamma(q-p+1)}+\frac{\Gamma(2-p)}{4\Gamma(q-p)}\right. \\
&\quad\left.\times\left\{1-\frac{2\Gamma(2-p)}{\Gamma(3-p)}+2\left(\frac{\Gamma(2-p)}{\Gamma(3-p)}\right)^2\right\}\right] \\
={}&\kappa L\|x-y\|,
\end{aligned}
\tag{31}
$$

which, in view of the condition $\kappa L<1$ (κ is given by (28)), implies that the operator $\mathscr{F}$ is a contraction. Hence, by Banach's contraction mapping principle, the problem (2) has a unique solution. □

Example 11. Consider the following antiperiodic fractional boundary value problem:

$$
\begin{aligned}
&{}^cD^q x(t)=\left(t^2+1\right)e^{-x^2(t)}\ln\left(4+3\sin^2 x(t)\right), \\
&\qquad 0<t<1,\ 2<q\le 3, \\
&x(0)=-x(1), \\
&{}^cD^p x(0)=-{}^cD^p x(1),\qquad {}^cD^{p+1}x(0)=-{}^cD^{p+1}x(1), \\
&\qquad 0<p<1.
\end{aligned}
\tag{32}
$$

Clearly $|f(t,x(t))|\le(3\ln 7)$. So, the hypothesis of Theorem 8 holds. Therefore, the conclusion of Theorem 8 applies to antiperiodic fractional boundary value problem (32).

Example 12. Consider the following antiperiodic fractional boundary value problem:

$$
\begin{aligned}
&{}^cD^{5/2}x(t)=\frac{L}{2}\left(x(t)+\tan^{-1}x(t)\right)+\sqrt{1+\sin^3 t}, \\
&\qquad L>0,\ t\in[0,2], \\
&x(0)=-x(2) \\
&{}^cD^{3/4}x(0)=-{}^cD^{3/4}x(2),\qquad {}^cD^{7/4}x(0)=-{}^cD^{7/4}x(2),
\end{aligned}
\tag{33}
$$

where $q=5/2$, $p=3/4$, $p+1=7/4$ $f(t,x)=L(x+\tan^{-1}x)/2+\sqrt{1+\sin^3 t}$, and $T=2$. Clearly,

$$
\begin{aligned}
\left|f(t,x)-f(t,\overline{x})\right|&\le\frac{L}{2}\left(|x-\overline{x}|+\left|\tan^{-1}x-\tan^{-1}\overline{x}\right|\right) \\
&\le L\left(|x-\overline{x}|\right),
\end{aligned}
\tag{34}
$$

where we have used the fact that $|(\tan^{-1} y)'| = 1/(1+y^2) < 1$. Further,

$$\kappa = T^q \left[\frac{3}{2\Gamma(q+1)} + \frac{\Gamma(2-p)}{2\Gamma(q-p+1)} + \frac{\Gamma(2-p)}{4\Gamma(q-p)} \times \left\{ 1 - \frac{2\Gamma(2-p)}{\Gamma(3-p)} + 2\left(\frac{\Gamma(2-p)}{\Gamma(3-p)}\right)^2 \right\} \right] = \frac{16}{5}\sqrt{\frac{2}{\pi}} + \frac{526\Gamma(1/4)}{525\Gamma(3/4)}. \quad (35)$$

With $L < 1/\kappa$, all the assumptions of Theorem 10 are satisfied. Hence, the fractional boundary value problem (33) has a unique solution on $[0, 2]$.

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