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Research Article

On the Homomorphisms of the Lie Groups $SU(2)$ and S^3

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We first construct all the homomorphisms from the Heisenberg group to the 3-sphere. Also, defining a topology on these homomorphisms, we regard the set of these homomorphisms as a topological space. Next, using the kernels of homomorphisms, we define an equivalence relation on this topological space. We finally show that the quotient space is a topological group which is isomorphic to $\mathbb{S}^1$.

1. Introduction

Discrete and continuous forms of the Heisenberg group have been studied in mathematics and physics such as analysis [1–3], geometry [4–6], topology [3, 7], and quantum physics [8–14]. An introductory review can be also found in [15].

In [16–18], it was shown that the Heisenberg group $\mathbb{H}$ is nilpotent, and any arbitrary nilpotent subgroup of $SU(2)$ is conjugate to a subgroup of $U(1)$, which is identified with the set of diagonal matrices in $SU(2)$. Since groups can be considered as metric spaces, it leads us to examine if there exists any geometry in these groups.

It is known that the matrices

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \tag{1}$$

form a linear group which is isomorphic to $\mathbb{R}^1$. The Lie groups $\mathbb{R}^1$ and $\mathbb{S}^1$ are related, for the mapping $\rho : \mathbb{R}^1 \to \mathbb{S}^1$ defined by $\rho(x) = e^{2\pi i x}$ is a continuous homomorphism from $\mathbb{R}^1$ onto $\mathbb{S}^1$ [10]. The Lie algebras $\mathbb{R}^1$ and $\mathbb{S}^1$ are trivially isomorphic. In this work, we search if there is a similar relationship between the only other sphere Lie group $\mathbb{S}^3$ and the linear group of matrices which is diffeomorphic to $\mathbb{R}^3$ [17].

The subgroup $\mathbb{H}$ of $\mathbb{GL}(3, \mathbb{R})$ formed by the matrices of type

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \tag{2}$$

is called three-dimensional Heisenberg group. It is convenient to denote the elements of this group by three-tuples of numbers. Using this convention, that is, $\mathbb{H} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, the multiplicative operation of elements can be expressed as

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'). \tag{3}$$

The identity element of this group is $(0, 0, 0)$, and the inverse of an element (x, y, x) is $(-x, -y, -z)$.

This Lie group is diffeomorphic to $\mathbb{R}^3$ [17].

Moreover, the Lie groups $SU(2)$ and $\mathbb{S}^3$ are isomorphic, and they are diffeomorphic as manifolds. Since the respective Lie algebras h and $su(2)$ of $\mathbb{H}$ and $SU(2)$ are three-dimensional real vector spaces, they are isomorphic as real vector spaces. But they are not isomorphic as Lie algebras, because there is no Lie algebra isomorphism between a compact and a noncompact Lie algebra. However, there may be a Lie algebra homomorphism between them. We try to find the relationship between h and $su(2)$ as Lie algebras.

Consider the following matrices:

$$U_1 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad U_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{4}$$

which span the Lie algebra $su(2)$ of the Lie group $SU(2)$, and also consider the following matrices:

$$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

which span the Lie algebra h of the Heisenberg group $\mathbb{H}$.

Using the exponential map from h to $\mathbb{H}$ we may convert our equations in h to equations in $\mathbb{H}$. Letting $\overline{V}_i = \exp(V_i)$ for each i, we see that

$$\overline{V}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \overline{V}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \overline{V}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The standard generating set for the Heisenberg group is $\{\overline{V}_1, \overline{V}_2, \overline{V}_1^{-1}, \overline{V}_2^{-1}\}$, and the group has the relation

$$\begin{aligned} \overline{V}_1\overline{V}_2\overline{V}_1^{-1}\overline{V}_2^{-1} &= \overline{V}_2\overline{V}_1^{-1}\overline{V}_2^{-1}\overline{V}_1 \\ &= \overline{V}_1^{-1}\overline{V}_2^{-1}\overline{V}_1\overline{V}_2 = \overline{V}_2^{-1}\overline{V}_1\overline{V}_2\overline{V}_1^{-1} = \overline{V}_3. \end{aligned} \tag{7}$$

In the above, the notations $\overline{V}_1^{-1}$ and $\overline{V}_2^{-1}$ denote the inverses of the elements $\overline{V}_1$ and $\overline{V}_2$ in the Heisenberg group.

From (4) and (5), we find that

$$\begin{aligned} &[U_1, U_2] = U_3, \qquad [U_2, U_3] = U_1, \qquad [U_3, U_1] = U_2, \\ &[V_1, V_2] = V_3, \qquad [V_2, V_3] = [V_3, V_1] = 0. \end{aligned} \tag{8}$$

It was shown in ([13, 16]) that there is no nontrivial Lie algebra homomorphism from $su(2)$ to h. Here, we study Lie algebra homomorphisms from h to $su(2)$. We observe that there exists a nontrivial Lie algebra homomorphism from h to $su(2)$. Moreover, we describe all Lie algebra homomorphisms from h to $su(2)$.

2. Homomorphisms from $\mathbb{H}$ to $SU(2)$

Let $\varphi : h \to su(2)$ be a Lie algebra homomorphism. Suppose that

$$\varphi(V_i) = \sum_{j=1}^{3} \alpha_{ij} U_j, \quad i = 1, 2, 3 \tag{9}$$

for some real numbers α_{ij}.

By using (8) and the following commutators:

$$\begin{aligned} &\varphi[V_1, V_2] = [\varphi V_1, \varphi V_2], \quad \varphi V_3 = [\varphi V_1, \varphi V_2], \\ &\varphi[V_1, V_3] = [\varphi V_1, \varphi V_3], \quad \varphi[V_1, V_3] = 0, \\ &\varphi[V_2, V_3] = [\varphi V_2, \varphi V_3], \quad \varphi[V_2, V_3] = 0, \end{aligned} \tag{10}$$

we obtain the system of equations of coefficients in the real constants α_{ij}'s

$$\begin{aligned} &\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} = \alpha_{31}, \qquad \alpha_{13}\alpha_{21} - \alpha_{11}\alpha_{23} = \alpha_{32}, \\ &\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \alpha_{33}, \qquad \alpha_{12}\alpha_{33} - \alpha_{13}\alpha_{32} = 0, \\ &\alpha_{13}\alpha_{31} - \alpha_{11}\alpha_{33} = 0, \qquad \alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} = 0, \\ &\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} = 0, \qquad \alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33} = 0, \\ &\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = 0. \end{aligned} \tag{11}$$

By using the Mathematica program, we may solve the system (11). We write below only two of the non-trival solutions, because a group homomorphism induced by any of the other solutions will be equal to a group homomorphism induced by one of the solutions given below. Here, a nontrivial solution means a solution in which at least one of the constants α_{ij} is nonzero.

(1) $\alpha_{22} = \alpha_{12}\alpha_{21}/\alpha_{11}$, $\alpha_{23} = \alpha_{13}\alpha_{21}/\alpha_{11}$, $\alpha_{31} = \alpha_{32} = \alpha_{33} = 0$, $\alpha_{11} \neq 0$, α_{12}, α_{13}, α_{21} arbitrary.

(2) $\alpha_{23} = \alpha_{13}\alpha_{22}/\alpha_{12}$, $\alpha_{11} = \alpha_{21} = \alpha_{31} = \alpha_{32} = \alpha_{33} = 0$, $\alpha_{12} \neq 0$, α_{13}, α_{22} arbitrary.

Also, in the system of (11), if $\alpha_{11} \neq 0$ and $\alpha_{12} = 0$, then the first set of solutions is obtained. If $\alpha_{12} \neq 0$ and $\alpha_{11} = 0$, then the second set of solutions is obtained. Moreover, if α_{11} and α_{12} are both 0, then (11) has trivial solution.

In the set of solutions, we consider α_{22} and α_{23} in terms of the other arbitrary constants α_{11}, α_{12}, α_{13}, and α_{21} and obtain that the group homomorphism is spanned by elements $\{U_1, U_2, U_3\}$ or $\{U_2, U_3\}$. If another configuration, say α_{11}, α_{13}, was chosen, then we see that the homomorphism would be spanned by the same elements.

Since φ is a Lie algebra homomorphism, we observe from solution sets 1 and 2 that the subalgebra of $su(2)$ is only generated by $a_1U_1 + a_2U_2 + a_3U_3$ or $aU_2 + bU_3$ and not by $a_1U_1 + a_2U_3$.

As we will use the constants α_{ij} frequently, to simplify the notations we put a_1, a_2, a_3, and b for the coefficients α_{11}, α_{12}, α_{13}, and α_{21}, respectively, and, a, b, and c for α_{12}, α_{13}, and α_{22}.

Hence, for the first set of solutions, φ has the following form:

$$\begin{aligned} &\varphi(V_1) = a_1U_1 + a_2U_2 + a_3U_3, \\ &\varphi(V_2) = \frac{b}{a_1}\varphi(V_1), \quad a_1 \neq 0, \\ &\varphi(V_3) = 0. \end{aligned} \tag{12}$$

We note that the rank of φ is one, and thus, $\varphi(h)$ is a one-dimensional Lie subalgebra of $su(2)$ generated by $a_1U_1 + a_2U_2 + a_3U_3$.

For the second set of the solutions, φ is of the following form:

$$\begin{gathered}\varphi(V_1) = aU_2 + bU_3,\\ \varphi(V_2) = \frac{c}{a}\varphi(V_1), \quad a \neq 0, \\ \varphi(V_3) = 0.\end{gathered} \tag{13}$$

Here, we note again that the rank of φ is one, and thus, $\varphi(h)$ is a one-dimensional Lie subalgebra of $su(2)$ generated by $aU_2 + bU_3$.

It is known from ([8, 18]) that if $\Phi : \mathbb{H} \to SU(2)$ is a homomorphism, then the following diagram commutes:

$$\begin{array}{ccc} h & \xrightarrow{d\Phi} & su(2) \\ \text{exp}\downarrow & & \downarrow\text{exp} \\ \mathbb{H} & \xrightarrow{\Phi} & SU(2), \end{array} \tag{14}$$

where $d\Phi$ is the differential of Φ, and it is a Lie algebra homomorphism. It is also known that for the matrix groups the exponential map is given by the exponentiation of matrices. In our notation, $\varphi = d\Phi$ for some Φ, and we will determine Φ.

By using (5), for any element of $V = c_1V_1 + c_2V_2 + c_3V_3 \in h$ we have

$$\begin{gathered} V = \begin{pmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V^2 = \begin{pmatrix} 0 & 0 & c_1c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ V^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}. \end{gathered} \tag{15}$$

By using (15), we obtain $\exp(V)$ as

$$\exp(V) = 1 + V + \frac{1}{2}V^2 + \frac{1}{3!}V^3 + \cdots = \begin{pmatrix} 1 & c_1 & c_3 + \dfrac{c_1c_2}{2} \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

To guarantee that

$$\exp(V) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (x, y, z) \tag{17}$$

we must take $V = xV_1 + yV_2 + (z - xy/2)V_3 \in h$. Here, we put (x, y, z) for the matrix (2) for any real x, y, and z.

For the first set of solutions, we obtain

$$\begin{aligned}\varphi(V) &= \varphi\left(xV_1 + yV_2 + \left(z - \frac{xy}{2}\right)V_3\right) \\ &= \left(x + \frac{b}{a_1}y\right)(a_1U_1 + a_2U_2 + a_3U_3).\end{aligned} \tag{18}$$

Thus, the kernel of φ is the plane $x + (b/a_1)y = 0$ in $\mathbb{R}^3$.

For $V \notin \operatorname{Ker}\varphi$, we obtain

$$\begin{aligned}\exp\varphi(V) = {}& I\cos A\left(x + \frac{b}{a_1}y\right) \\ &+ \frac{1}{A(x + (b/a_1)y)}\sin\left(A\left(x + \frac{b}{a_1}y\right)\right)\varphi(V),\end{aligned} \tag{19}$$

where I is the 2×2 identity matrix, and $A = (1/2)\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

In this case, Φ is of the following form:

$$\begin{aligned}(x, y, z) \longrightarrow {}& I\cos A\left(x + \frac{b}{a_1}y\right) \\ &+ \frac{1}{A}\sin\left(A\left(x + \frac{b}{a_1}y\right)\right)(a_1U_1 + a_2U_2 + a_3U_3).\end{aligned} \tag{20}$$

For the second set of solutions, we obtain

$$\varphi(V) = \left(x + \frac{c}{a}y\right)(aU_2 + bU_3). \tag{21}$$

From (21), the kernel of φ is the plane $x + (c/a)y = 0$ in $\mathbb{R}^3$.

For $V \notin \operatorname{Ker}\varphi$, we have

$$\begin{aligned}\exp\varphi(V) = {}& I\cos A\left(x + \frac{c}{a}y\right) \\ &+ \frac{1}{A(x + (c/a)y)}\sin\left(A\left(x + \frac{c}{a}y\right)\right)\varphi(V),\end{aligned} \tag{22}$$

where I is the 2×2 identity matrix, and $A = (1/2)\sqrt{a^2 + b^2}$.

In this case, Φ is of the form

$$\begin{aligned}(x, y, z) \longrightarrow {}& I\cos A\left(x + \frac{c}{a}y\right) \\ &+ \frac{1}{A}\sin\left(A\left(x + \frac{c}{a}y\right)\right)(aU_2 + bU_3).\end{aligned} \tag{23}$$

Hence, we can state the following theorem.

Theorem 1. *Any nontrivial homomorphism from the Heisenberg group to the 3-sphere is one of* (20) *and* (23).

We now observe the following properties for the first type of homomorphisms. It can be shown that same observations are valid for the second set of solutions.

(a) By considering Φ as a map from $\mathbb{R}^3$ to $\mathbb{R}^4$, we can write

$$\Phi : (\mathrm{x}, y, z) \longrightarrow (u_1, u_2, u_3, u_4), \tag{24}$$

where

$$\begin{gathered} u_1 = \cos At, \qquad u_2 = -\frac{a_3}{2A}\sin At, \qquad u_3 = -\frac{a_2}{2A}\sin At, \\ u_4 = -\frac{a_1}{2A}\sin At, \qquad t = x + \frac{b}{a_1}y. \end{gathered} \tag{25}$$

Then, we find that the image of the planes P_t in $\mathbb{H}$, with the equation $x + b/a_1 = t$, is a periodic (closed) curve in $\mathbb{S}^3$.

(b) In the first case, every homomorphism Φ from $\mathbb{H}$ to $SU(2)$ depends on four parameters, namely, a_1, a_2, a_3, and b with $a_1 \neq 0$.

We now concentrate on the parameters a_1 and b, since they are involved in the kernel of Φ. To each kernel $x + (b/a_1)y = 0$, we associate a point $(b, -a_1)$ in the xy-plane. Furthermore, we normalize the vector $(b, -a_1)$ as $(b/\sqrt{b^2 + a_1^2}, -a_1/\sqrt{b^2 + a_1^2})$ which can also be considered as a point of $\mathbb{S}^1$ in the xy-plane.

By considering the principal value of $\arctan(-b/a_1)$, we define a topology on the set $\mathbb{H}'$ of all homomorphisms from $\mathbb{H}$ to $SU(2)$ as follows.

For any $\Phi_1(a_1, a_2, a_3, b) \in \mathbb{H}'$, $\Phi_2(a_1', a_2', a_3', b') \in \mathbb{H}'$ is in the ϵ-neighborhood of Φ_1, if and only if $|\arctan(-b/a_1) - \arctan(-b'/a_1')| < \epsilon$, $|a_2 - a_2'| < \epsilon$, and $|a_3 - a_3'| < \epsilon$.

Now let us define an equivalence relation on $\mathbb{H}'$. For $\Phi_1, \Phi_2 \in \mathbb{H}'$, Φ_1 is equivalent to Φ_2 if and only if Φ_1 and Φ_2 have the same kernel. Denote the set of equivalence classes by $\widetilde{\mathbb{H}}$. We define a multiplication on $\widetilde{\mathbb{H}}$ such that, for any $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ in $\widetilde{\mathbb{H}}$, $\widetilde{\Phi}_1 . \widetilde{\Phi}_2$ denotes the element of $\widetilde{\mathbb{H}}$ whose kernel is the plane obtained by the product of the elements of $\mathbb{S}^1$ corresponding to Φ_1 and Φ_2. This multiplication makes $\widetilde{\mathbb{H}}$ into a group which is isomorphic to $\mathbb{S}^1$.

(c) The set of the kernels of all the homomorphisms from $\mathbb{H}$ to $\mathbb{S}^3$ is a subset of the Grassmann manifold of 2 planes in $\mathbb{R}^3$. It is known that the Grassmann manifold of 2 planes in $\mathbb{R}^3$ is diffeomorphic to $\mathbb{S}^1$. Any point p of $\mathbb{S}^1$ corresponds to the plane which is orthogonal to the normal of $\mathbb{S}^1$ at p and which contains the z-axis. Hence, there exists a 1-1 correspondence between the set of the kernels of all the homomorphisms from $\mathbb{H}$ to $\mathbb{S}^3$ and the equator $\mathbb{S}^1$ of $\mathbb{S}^2$.

Thus, we state the following theorem.

Theorem 2. *The set of all homomorphisms from the Heisenberg group to the 3-sphere is isomorphic (up to a certain equivalence relation concerning kernels) with the topological group* $\mathbb{S}^1$.

3. Conclusion

In this paper, our aim is to construct all homomorphisms between the Heisenberg group and the 3-sphere which is isomorphic to $SU(2)$. In the literature, it has been shown that no nontrivial Lie algebra homomorphism from $su(2)$ to h exists ([8, 11, 17]). So, a natural question arises: is there a homomorphism from h to $su(2)$? Here, we answer this question completely.

We here observe that there are nontrivial homomorphisms from the Heisenberg group to the 3-sphere, and they can be only in the form of (20) or (23).

Also, we use these maps to define a topology in order to construct an equivalence relation, and we show that quotient space is isomorphic to $\mathbb{S}^1$.

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