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Research Article

General Split Feasibility Problems in Hilbert Spaces

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Introducing a general split feasibility problem in the setting of infinite-dimensional Hilbert spaces, we prove that the sequence generated by the purposed new algorithm converges strongly to a solution of the general split feasibility problem. Our results extend and improve some recent known results.

1. Introduction

Let H and K be infinite-dimensional real Hilbert spaces, and let $A: H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^p$ and $\{Q_i\}_{i=1}^r$ be the families of nonempty closed convex subsets of H and K, respectively.

(a) The *convex feasibility problem* (CFP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^p C_i. \tag{1}$$

(b) The *split feasibility problem* (SEP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in C, \qquad Ax^* \in C,$$
 (2)

where C and Q are nonempty closed convex subsets of H and K, respectively.

(c) The multiple-set split feasibility problem (MSSFP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^p C_i, \qquad Ax^* \in \bigcap_{i=1}^r Q_i.$$
 (3)

Note that (MSSFP) reduces to (SEP) if we take p = r = 1.

There is a considerable investigation on CFP in view of its applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1]. The split feasibility problem SFP in the setting of finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [2] for modelling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. Since then, a lot of work has been done on finding a solution of SFP and MSSFP; see, for example, [2–25]. Recently, it is found that the SFP can also be applied to study the intensity-modulated radiation therapy; see, for example, [6, 16] and the references therein. Very recently, Xu [8] considered the SFP in the setting of infinite-dimensional Hilbert spaces.

The original algorithm given in [2] involves the computation of the inverse A^{-1} provided it exists. In [8], Xu studied some algorithm and its convergence. In particular, he applied Mann's algorithm to the SFP and purposed an algorithm which is proved to be weakly convergent to a solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained. In [7], Wang and Xu purposed the following cyclic algorithm to solve MSSFP:

$$x_{n+1} = P_{C[n]} (x_n + \gamma A^* (P_{Q[n]} - I) Ax_n),$$
 (4)

where $[n] := n \pmod{p}$, (mod function take values in $\{1, 2, ..., p\}$), and $\gamma \in (0, 2/\|A\|^2)$. They show that the sequence $\{x_n\}$ convergence weakly to a solution of MSSFP provided the solution exists. To study strong convergence to

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a solution of MSSFP, first we introduce a general form of the split feasibility problem for infinite families as follows.

(d) General split feasibility problem (GSFP) is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{\infty} C_i, \qquad Ax^* \in \bigcap_{i=1}^{\infty} Q_i.$$
 (5)

We denote by Ω the solution set of GSFP.

In this paper, using viscosity iterative method defined by Moudafi [21], we propose an algorithm for finding the solutions of the general split feasibility problem in a Hilbert space. We establish the strong convergence of the proposed algorithm to a solution of GSFP.

2. Preliminaries

Throughout the paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \to x$, and strong convergence by $x_n \to x$. Let C be a closed and a convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies

$$||x - P_C x|| \le ||x - y||, \quad \forall y \in C.$$
 (6)

The operator P_C is called the metric projection or the nearest point mapping of H onto C. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \le 0, \quad \forall x \in H, y \in C.$$
 (7)

Recall that a mapping $T: C \to C$ is called nonexpansive if

$$||Tx - Ty|| \le ||x - y||, \quad \forall x, y \in C.$$
 (8)

It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightarrow x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{9}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 1. Let H be a Hilbert space. Then, for all $x, y \in H$

$$||x + y||^2 \le ||x||^2 + 2\langle y, x + y \rangle.$$
 (10)

Lemma 2 (see [22]). Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with i < j,

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{11}$$

Lemma 3 (see [23]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1 - \gamma_n) a_n + \gamma_n \delta_n + \beta_n, \quad n \ge 0,$$
 (12)

where $\{\gamma_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i)
$$\gamma_n \in [0, 1], \sum_{n=1}^{\infty} \gamma_n = \infty$$
,

(ii)
$$\limsup_{n\to\infty} \delta_n \le 0$$
 or $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$,

(iii)
$$\beta_n \ge 0$$
 for all $n \ge 0$ with $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (see [24]). Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$, and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:

$$t_{\tau(n)} \le t_{\tau(n)+1}, \qquad t_n \le t_{\tau(n)+1}.$$
 (13)

In fact

$$\tau(n) = \max\{k \le n : t_k < t_{k+1}\}. \tag{14}$$

Lemma 5 (demiclosedness principle [25]). Let C be a non-empty closed and convex subset of a real Hilbert space H. Let $T:C\to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T)\neq\emptyset$. Then, T is demiclosed on C, that is, if $y_n\to z\in C$, and $(y_n-Ty_n)\to y$, then (I-T)z=y.

3. Main Result

In the following result, we propose an algorithm and prove that the sequence generated by the proposed method converges strongly to a solution of the GSFP.

Theorem 6. Let H and K be real Hilbert spaces, and let $A: H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be the families of nonempty closed convex subsets of H and K, respectively. Assume that GSFP (5) has a nonempty solution set Ω . Suppose that f is a self k-contraction mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ as

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n)$$

$$+ \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n, \quad n \ge 0,$$
(15)

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i)
$$\lim_{n\to\infty}\beta_n = 0$$
 and $\sum_{n=0}^{\infty}\beta_n = \infty$,

(ii) for each $i \in \mathbb{N}$, $\liminf_{n} \alpha_n \gamma_{n,i} > 0$,

(iii) for each
$$i \in \mathbb{N}$$
, $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ and $0 < \liminf_{n \to \infty} \lambda_{n,i} \le \limsup_{n \to \infty} \lambda_{n,i} < 2/\|A\|^2$,

then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_{\Omega} f(x^*)$.

Proof. First, we show that $\{x_n\}$ is bounded. In fact, let $z \in \Omega$. Since $\{\lambda_{n,i}\} \subset (0,2/\|A\|^2)$, the operators $P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)$ are nonexpansive, and hence we have

$$\|x_{n+1} - z\|$$

$$= \|\alpha_{n}x_{n} + \beta_{n}f(x_{n}) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_{i}} (I - \lambda_{n,i}A^{*} (I - P_{Q_{i}}) A) x_{n} - z\|$$

$$\leq \alpha_{n} \|x_{n} - z\| + \beta_{n} \|f(x_{n}) - z\|$$

$$+ \sum_{i=1}^{\infty} \gamma_{n,i} \|P_{C_{i}} (I - \lambda_{n,i}A^{*} (I - P_{Q_{i}}) A) x_{n} - z\|$$

$$\leq \alpha_{n} \|x_{n} - z\| + \beta_{n} \|f(x_{n}) - z\|$$

$$+ \sum_{i=1}^{\infty} \gamma_{n,i} \|x_{n} - z\|$$

$$\leq (1 - \beta_{n}) \|x_{n} - z\| + \beta_{n} \|f(x_{n}) - z\|$$

$$\leq (1 - \beta_{n}) \|x_{n} - z\| + \beta_{n} \|f(x_{n}) - f(z)\|$$

$$+ \beta_{n} \|f(z) - z\|$$

$$\leq (1 - \beta_{n}) \|x_{n} - z\| + \beta_{n}k \|x_{n} - z\|$$

$$+ \beta_{n} \|f(z) - z\|$$

$$\leq (1 - (1 - k)) \beta_{n} \|x_{n} - z\|$$

$$+ (1 - k) \frac{\beta_{n}}{1 - k} \|f(z) - z\|$$

$$\leq \max \left\{ \|x_{n} - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\}$$

$$\vdots$$

$$\leq \max \left\{ \|x_{0} - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\},$$

which implies that $\{x_n\}$ is bounded, and we also obtain that $\{f(x_n)\}$ is bounded. Next, we show that for each $i \in \mathbb{N}$,

$$\lim_{n \to \infty} \left\| x_n - P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n \right\| = 0.$$
 (17)

By using Lemma 2, for every $z \in \Omega$ and $i \in \mathbb{N}$, we have that

$$\begin{aligned} \left\| x_{n+1} - z \right\|^2 \\ &= \left\| \alpha_n x_n + \beta_n f\left(x_n\right) \right. \\ &+ \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j} \left(I - \lambda_{n,j} A^* \left(I - P_{Q_j} \right) A \right) x_n - z \right\|^2 \end{aligned}$$

$$\leq \alpha_{n} \|x_{n} - z\|^{2} + \beta_{n} \|f(x_{n}) - z\|^{2}
+ \sum_{j=1}^{\infty} \gamma_{n,j} \|P_{C_{j}} (I - \lambda_{n,j} A^{*} (I - P_{Q_{j}}) A) x_{n} - z\|^{2}
- \alpha_{n} \gamma_{n,i} \|P_{C_{i}} (I - \lambda_{n,i} A^{*} (I - P_{Q_{i}}) A) x_{n} - x_{n}\|^{2}
\leq \alpha_{n} \|x_{n} - z\|^{2} + \beta_{n} \|f(x_{n}) - z\|^{2}
+ \sum_{j=1}^{\infty} \gamma_{n,j} \|x_{n} - z\|^{2}
- \alpha_{n} \gamma_{n,i} \|P_{C_{i}} (I - \lambda_{n,i} A^{*} (I - P_{Q_{i}}) A) x_{n} - x_{n}\|^{2}
\leq (1 - \beta_{n}) \|x_{n} - z\|^{2} + \beta_{n} \|f(x_{n}) - z\|^{2}
- \alpha_{n} \gamma_{n,i} \|P_{C_{i}} (I - \lambda_{n,i} A^{*} (I - P_{Q_{i}}) A) x_{n} - x_{n}\|^{2}.$$
(18)

Hence, for each $i \in \mathbb{N}$, we have

$$\alpha_{n} \gamma_{n,i} \| P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} - x_{n} \|^{2}$$

$$\leq \| x_{n} - z \|^{2} - \| x_{n+1} - z \|^{2} + \beta_{n} \| f \left(x_{n} \right) - z \|^{2}.$$

$$(19)$$

Next, we show that there exists a unique $x^* \in \Omega$ such that $x^* = P_{\Omega} f(x^*)$. We observe that for each $n \ge 0$, $x^* \in \Omega$ solves the GSFP (5) if and only if x^* solves the fixed point equation

$$x^{*} = P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x^{*}, \quad i \in \mathbb{N},$$
 (20)

that is, the solution sets of fixed point equation (20) and GSFP (5) are the same (see for details [8]). Note that if $\{\lambda_{n,i}\}$ \subset $(0,2/\|A\|^2)$, then the operators $P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)$ are nonexpansive. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set Ω is well defined whenever $\Omega \neq \emptyset$. We observe that $P_{\Omega}(f)$ is a contraction of H into itself. Indeed, since P_{Ω} is nonexpansive,

$$\left\|P_{\Omega}\left(f\right)\left(x\right) - P_{\Omega}\left(f\right)\left(y\right)\right\| \le \left\|f\left(x\right) - f\left(y\right)\right\| \le k \left\|x - y\right\|. \tag{21}$$

Hence, there exists a unique element $x^* \in \Omega$ such that $x^* = P_{\Omega} f(x^*)$.

In order to prove that $x_n \to x^*$ as $n \to \infty$, we consider two possible cases.

Case 1. Assume that $\{\|x_n - x^*\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - x^*\|\}_{n \ge n_0}$ is either nondecreasing or nonincreasing. Since $\|x_n - x^*\|$ is bounded we have $\|x_n - x^*\|$ is convergent. Since $\lim_{n \to \infty} \beta_n = 0$ and $\{f(x_n)\}$ is bounded, from (19) we get that

$$\lim_{n \to \infty} \alpha_n \gamma_{n,i} \left\| P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n - x_n \right\|^2 = 0.$$
(22)

By assuming that $\liminf_{n} \alpha_n \gamma_{n,i} > 0$, we obtain

$$\lim_{n \to \infty} \left\| P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n - x_n \right\| = 0, \quad \forall i \in \mathbb{N}.$$
(23)

Now, we show that

$$\lim_{n \to \infty} \sup \left\langle f\left(x^{\star}\right) - x^{\star}, x_{n} - x^{\star}\right\rangle \le 0. \tag{24}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \to \infty} \langle f(x^*) - x^*, x_{n_k} - x^* \rangle$$

$$= \lim_{n \to \infty} \sup_{x \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle.$$
(25)

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup w$ and $\lambda_{n,i} \rightarrow \lambda_i \in (0,2/\|A\|^2)$ for each $i \in \mathbb{N}$. From (23), we have

$$\begin{aligned} & \| P_{C_{i}} \left(I - \lambda_{i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} - x_{n} \| \\ & \leq & \| P_{C_{i}} \left(I - \lambda_{i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} \\ & - P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} \| \\ & + \| P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} - x_{n} \| \\ & \leq & \| \left(I - \lambda_{i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} \\ & - \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} \| \\ & + \| P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} - x_{n} \| \\ & \leq & | \lambda_{i} - \lambda_{n,i} | \| A^{*} \left(I - P_{Q_{i}} \right) A x_{n} \| \\ & + \| P_{C_{i}} \left(I - \lambda_{n,i} A^{*} \left(I - P_{Q_{i}} \right) A \right) x_{n} - x_{n} \| \\ & \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Notice that for each $i \in \mathbb{N}$, $P_{C_i}(I - \lambda_i A^*(I - P_{Q_i})A)$ is nonexpansive. Thus, from Lemma 5, we have $w \in \Omega$. Therefore, it follows that

$$\lim_{n \to \infty} \sup \left\langle f\left(x^{\star}\right) - x^{\star}, x_{n} - x^{\star} \right\rangle$$

$$= \lim_{k \to \infty} \left\langle f\left(x^{\star}\right) - x^{\star}, x_{n_{k}} - x^{\star} \right\rangle$$

$$= \left\langle f\left(x^{\star}\right) - x^{\star}, w - x^{\star} \right\rangle \le 0.$$
(27)

Finally, we show that $x_n \to P_\Omega f(x^*)$. Applying Lemma 1, we have that

$$\|x_{n+1} - x^*\|^2$$

$$= \|\alpha_n (x_n - x^*)$$

$$+ \sum_{i=1}^{\infty} \gamma_{n,i} \left(P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n - x^* \right) \|^2$$

$$+ 2\beta_n \left\langle f (x_n) - x^*, x_{n+1} - x^* \right\rangle$$

$$\leq (1 - \beta_n)^2 \|x_n - x^*\|^2$$

$$+ 2\beta_n \left\langle f (x_n) - f (x^*), x_{n+1} - x^* \right\rangle$$

$$+ 2\beta_n \left\langle f (x^*) - x^*, x_{n+1} - x^* \right\rangle$$

$$\leq (1 - \beta_n)^2 \|x_n - x^*\|^2$$

$$+ 2\beta_n k \|x_n - x^*\| \|x_{n+1} - x^*\|$$

$$+ 2\beta_n \left\langle f (x^*) - x^*, x_{n+1} - x^* \right\rangle$$

$$\leq (1 - \beta_n)^2 \|x_n - x^*\|^2$$

$$+ \beta_n k \left\{ \|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right\}$$

$$+ 2\beta_n \left\langle f (x^*) - x^*, x_{n+1} - x^* \right\rangle.$$

This implies that

$$\|x_{n+1} - x^*\|^2$$

$$\leq \frac{(1 - \beta_n)^2 + \beta_n k}{1 - \beta_n k} \|x_n - x^*\|^2$$

$$+ \frac{2\beta_n}{1 - \beta_n k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$= \frac{1 - 2\beta_n + \beta_n k}{1 - \beta_n k} \|x_n - x^*\|^2$$

$$+ \frac{\beta_n^2}{1 - \beta_n k} \|x_n - x^*\|^2$$

$$+ \frac{2\beta_n}{1 - \beta_n k} \langle f(z) - x^*, x_{n+1} - x^* \rangle$$

$$\leq \left(1 - \frac{2(1 - k)\beta_n}{1 - \beta_n k}\right) \|x_n - x^*\|^2$$

$$+ \frac{2(1 - k)\beta_n}{1 - \beta_n k} \left\{\frac{\beta_n M}{2(1 - k)}\right\}$$

$$+ \frac{1}{1 - k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$\leq (1 - \eta_n) \|x_n - x^*\|^2 + \eta_n \delta_n,$$
(29)

where

$$\delta_n = \frac{\beta_n M}{2(1-k)} + \frac{1}{1-k} \left\langle f(x^*) - x^*, x_{n+1} - x^* \right\rangle, \quad (30)$$

 $M=\sup\{\|x_n-x^\star\|^2:n\geq 0\}$ and $\eta_n=2(1-k)\beta_n/(1-\beta_nk)$. It is easy to see that $\eta_n\to 0$, $\sum_{n=1}^\infty \eta_n=\infty$ and $\limsup_{n\to\infty}\delta_n\leq 0$. Hence, by Lemma 3, the sequence $\{x_n\}$ converges strongly to $x^\star=P_\Omega f(x^\star)$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max \left\{ k \in \mathbb{N}; k \le n : \left\| x_k - x^* \right\| < \left\| x_{k+1} - x^* \right\| \right\}. \tag{31}$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$,

$$||x_{\tau(n)} - x^*|| < ||x_{\tau(n)+1} - x^*||.$$
 (32)

From (19), we obtain that

$$\lim_{n \to \infty} \| P_{C_i} \left(I - \lambda_{\tau(n),i} A^* \left(I - P_{Q_i} \right) A \right) x_{\tau(n)} - x_{\tau(n)} \| = 0.$$
(33)

Following an argument similar to that in Case 1, we have

$$\lim_{n \to \infty} \sup_{x \to \infty} \left\langle f\left(x^{\star}\right) - x^{\star}, x_{\tau(n)+1} - x^{\star} \right\rangle \le 0. \tag{34}$$

And by similar argument, we have

$$\|x_{\tau(n)+1} - x^*\|^2 \le (1 - \eta_{\tau(n)}) \|x_{\tau(n)} - x^*\|^2 + \eta_{\tau(n)} \delta_{\tau(n)},$$
(35)

where $\eta_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty} \eta_{\tau(n)} = \infty$ and $\limsup_{n \to \infty} \delta_{\tau(n)} \le 0$. Hence, by Lemma 3, we obtain $\lim_{n \to \infty} \|x_{\tau(n)} - x^*\| = 0$ and $\lim_{n \to \infty} \|x_{\tau(n)+1} - x^*\| = 0$. Now, from Lemma 4, we have

$$0 \le \|x_n - x^*\|$$

$$\le \max\{\|x_{\tau(n)} - x^*\|, \|x_n - x^*\|\}$$

$$\le \|x_{\tau(n)+1} - x^*\|.$$
(36)

Therefore, $\{x_n\}$ converges strongly to $x^* = P_{\Omega} f(x^*)$.

For finite collections we have the following consequence of Theorem 6.

Theorem 7. Let H and K be real Hilbert spaces, and let $A: H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^p$ be a family of nonempty closed convex subsets in H, and let $\{Q_i\}_{i=1}^p$ be a family of nonempty closed convex subsets in K. Assume that MSSFP has a nonempty solution set Ω . Let u be an arbitrary element in H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and

$$x_{n+1} = \alpha_n x_n + \beta_n u + \sum_{i=1}^{p} \gamma_{n,i} P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n, \quad n \ge 0,$$
(37)

where $\alpha_n + \beta_n + \sum_{i=1}^p \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i)
$$\lim_{n\to\infty}\beta_n = 0$$
 and $\sum_{n=0}^{\infty}\beta_n = \infty$,

(ii) for all
$$i \in \{1, 2, ..., p\}$$
, $\liminf_{n} \alpha_n \gamma_{n,i} > 0$,

(iii) for all
$$i \in \{1, 2, ..., p\}$$
, $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ and

$$0 < \liminf_{n \to \infty} \lambda_{n,i} \le \limsup_{n \to \infty} \lambda_{n,i} < \frac{2}{\|A\|^2}, \tag{38}$$

then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_{\Omega}u$.

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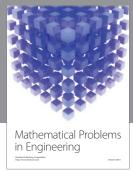
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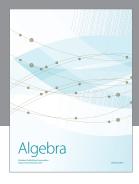
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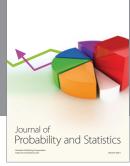
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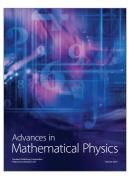


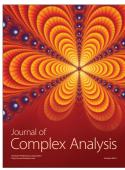




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