Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 832591, 6 pages
http://dx.doi.org/10.1155/2013/832591

Research Article

Bounds of the Neuman-Sándor Mean Using Power and Identric Means

Yu-Ming Chu[1] and Bo-Yong Long[2]

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou, Zhejiang 313000, China*
[2] *School of Mathematics Science, Anhui University, Hefei, Anhui 230039, China*

Correspondence should be addressed to Yu-Ming Chu; chuyuming2005@yahoo.com.cn

Received 8 November 2012; Revised 4 January 2013; Accepted 11 January 2013

Academic Editor: Wenchang Sun

In this paper we find the best possible lower power mean bounds for the Neuman-Sándor mean and present the sharp bounds for the ratio of the Neuman-Sándor and identric means.

1. Introduction

For $p \in \mathbb{R}$ the pth power mean $M_p(a,b)$, Neuman-Sándor Mean $M(a,b)$ [1], and identric mean $I(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1}$$

$$M(a,b) = \begin{cases} \dfrac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases} \tag{2}$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{3}$$

respectively, where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

The main properties for $M_p(a,b)$ and $I(a,b)$ are given in [2]. It is well known that $M_p(a,b)$ is continuously and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, the power, Neuman-Sándor, and identric means have been a subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [3–26].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of two positive numbers a and b with $a \neq b$, respectively. Then, it is well known that the inequalities

$$\begin{aligned} H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ < P(a,b) < I(a,b) < A(a,b) = M_1(a,b) < M(a,b) \\ < T(a,b) < Q(a,b) = M_2(a,b) < C(a,b), \end{aligned} \tag{4}$$

hold for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for L, I, $(IL)^{1/2}$, and $(I+L)/2$ in terms of power means are presented in [27–32]:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < I^{1/2}(a,b) L^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b), \end{gathered} \tag{5}$$

for all $a, b > 0$ with $a \neq b$.

Pittenger [31] found the greatest value r_1 and the least value r_2 such that the double inequality

$$M_{r_1}(a,b) \le L_p(a,b) \le M_{r_2}(a,b), \quad (6)$$

holds for all $a,b > 0$, where $L_r(a,b)$ is the rth generalized logarithmic means which is defined by

$$L_r(a,b) = \begin{cases} \left[\dfrac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & a\neq b,\ r\neq -1,\ r\neq 0, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\ r=0, \\ \dfrac{b-a}{\log b - \log a}, & a\neq b,\ r=-1, \\ a, & a=b. \end{cases} \quad (7)$$

The following sharp power mean bounds for the first Seiffert mean $P(a,b)$ are given in [10, 33]:

$$M_{\log 2/\log \pi}(a,b) < P(a,b) < M_{2/3}(a,b), \quad (8)$$

for all $a,b>0$ with $a\neq b$.

In [17], the authors answered the question: for $\alpha \in (0,1)$, what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b) < P^{\alpha}(a,b)\,G^{1-\alpha}(a,b) < M_q(a,b) \quad (9)$$

holds for all $a,b>0$ with $a\neq b$?

Neuman and Sándor [1] established that

$$\begin{gathered} A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})}, \\ \frac{\pi}{4}T(a,b) < M(a,b) < T(a,b), \\ M(a,b) < \frac{2A(a,b)+Q(a,b)}{3}, \end{gathered} \quad (10)$$

for all $a,b>0$ with $a\neq b$.

Let $0 < a,b \le 1/2$ with $a\neq b$, $a' = 1-a$ and $b' = 1-b$. Then, the Ky Fan inequalities

$$\begin{aligned} &\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} \\ &\quad < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \end{aligned} \quad (11)$$

were presented in [1].

In [24], Li et al. found the best possible bounds for the Neuman-Sándor mean $M(a,b)$ in terms of the generalized logarithmic mean $L_r(a,b)$. Neuman [25] and Zhao et al. [26] proved that the inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha)A(a,b) \\ &\quad < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b), \\ &\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) \\ &\quad < \mu C(a,b) + (1-\mu)A(a,b), \\ &\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) \\ &\quad < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) \\ &\quad < \beta_2 G(a,b) + (1-\beta_2)Q(a,b) \end{aligned} \quad (12)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]$, $\beta \ge 1/3$, $\lambda \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$, $\mu \ge 1/6$, $\alpha_1 \ge 2/9$, $\beta_1 \le 1-1/[\sqrt{2}\log(1+\sqrt{2})]$, $\alpha_2 \ge 1/3$, and $\beta_2 \le 1-1/[\sqrt{2}\log(1+\sqrt{2})]$.

In [7], Sándor and Trif proved that the inequalities

$$\begin{aligned} &e^{((a-b)^2/6(a+b)^2)} < \frac{A(a,b)}{I(a,b)} < e^{((a-b)^2/24ab)}, \\ &e^{((a-b)^2/3(a+b)^2)} < \frac{I(a,b)}{G(a,b)} < e^{((a-b)^2/12ab)}, \\ &e^{((a-b)^4/30(a+b)^4)} < \frac{I(a,b)}{A^{2/3}(a,b)\,G^{1/3}(a,b)} \\ &\quad < e^{((a-b)^4/120ab(a+b)^4)} \end{aligned} \quad (13)$$

hold for all $a,b>0$ with $a\neq b$.

Neuman and Sándor [15] and Gao [20] proved that $\alpha_1 = 1$, $\beta_1 = e/2$, $\alpha_2 = 1$, $\beta_2 = 2\sqrt{2}/e$, $\alpha_3 = 1$, $\beta_3 = 3/e$, $\alpha_4 = e/\pi$, $\beta_4 = 1$, $\alpha_5 = 1$, and $\beta_5 = 2e/\pi$ are the best possible constants such that the double inequalities $\alpha_1 < A(a,b)/I(a,b) < \beta_1$, $\alpha_2 < I(a,b)/M_{2/3}(a,b) < \beta_2$, $\alpha_3 < I(a,b)/He(a,b) < \beta_3$, $\alpha_4 < P(a,b)/I(a,b) < \beta_4$, and $\alpha_5 < T(a,b)/I(a,b) < \beta_5$ hold for all $a,b>0$ with $a\neq b$, where $He(a,b) = (a+\sqrt{ab}+b)/3 = (2A(a,b)+G(a,b))/3$ is the Heronian mean of a and b.

In [34], Sándor established that

$$He(a,b) < M_{2/3}(a,b), \quad (14)$$

for all $a,b>0$ with $a\neq b$.

It is not difficult to verify that the inequality

$$\frac{2A(a,b)+Q(a,b)}{3} < \left[He\left(a^2,b^2\right)\right]^{1/2} \quad (15)$$

holds for all $a,b>0$ with $a\neq b$.

From inequalities (10), (14), and (15), one has

$$M(a,b) < \left[M_{2/3}\left(a^2,b^2\right)\right]^{1/2} = M_{4/3}(a,b), \quad (16)$$

for all $a,b>0$ with $a\neq b$.

It is the aim of this paper to find the best possible lower power mean bound for the Neuman-Sándor mean $M(a,b)$ and to present the sharp constants α and β such that the double inequality

$$\alpha < \frac{M(a,b)}{I(a,b)} < \beta \tag{17}$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Results

Theorem 1. $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ *is the greatest value such that the inequality*

$$M(a,b) > M_{p_0}(a,b) \tag{18}$$

holds for all $a, b > 0$ with $a \neq b$.

Proof. From (1) and (2), we clearly see that both $M(a,b)$ and $M_p(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$.

Let $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})]$, then from (1) and (2) one has

$$\log M(x,1) - \log M_{p_0}(x,1) = \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}. \tag{19}$$

Let

$$f(x) = \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}. \tag{20}$$

Then, simple computations lead to

$$\lim_{x\to 1^+} f(x) = 0, \tag{21}$$

$$\lim_{x\to+\infty} f(x) = \frac{1}{p_0}\log 2 - \log\left[2\sinh^{-1}(1)\right] = 0, \tag{22}$$

$$f'(x) = \frac{\left(1+x^{p_0-1}\right) f_1(x)}{(x-1)(x^{p_0}+1)\sinh^{-1}((x-1)/(x+1))}, \tag{23}$$

where

$$\begin{gathered} f_1(x) = -\frac{\sqrt{2}(x-1)(x^{p_0}+1)}{(x+1)(x^{p_0-1}+1)\sqrt{1+x^2}} + \sinh^{-1}\left(\frac{x-1}{x+1}\right), \\ f_1(1) = 0, \end{gathered} \tag{24}$$

$$\lim_{x\to+\infty} f_1(x) = -\sqrt{2} + \sinh^{-1}(1) = -0.5328\cdots < 0, \tag{25}$$

$$f_1'(x) = \frac{\sqrt{2}(x-1) f_2(x)}{(x+1)^2(x^{p_0-1}+1)^2(1+x^2)^{3/2}}, \tag{26}$$

where

$$\begin{aligned} f_2(x) = {} & 1 + x + 2x^2 + (p_0-1)x^{p_0-2} - x^{p_0-1} + x^{p_0+1} \\ & - (p_0-1)x^{p_0+2} - 2x^{2p_0-2} - x^{2p_0-1} - x^{2p_0}, \end{aligned} \tag{27}$$

$$\begin{gathered} f_2(1) = 0, \\ \lim_{x\to+\infty} f_2(x) = -\infty, \end{gathered} \tag{28}$$

$$\begin{aligned} f_2'(x) = {} & 1 + 4x + (p_0-1)(p_0-2)x^{p_0-3} - (p_0-1)x^{p_0-2} \\ & + (p_0+1)x^{p_0} - (p_0-1)(p_0+2)x^{p_0+1} \\ & - 4(p_0-1)x^{2p_0-3} - (2p_0-1)x^{2p_0-2} - 2p_0x^{2p_0-1}, \\ & f_2'(1) = 4(4-3p_0) > 0, \end{aligned} \tag{29}$$

$$\lim_{x\to+\infty} f_2'(x) = -\infty, \tag{30}$$

$$\begin{aligned} f_2''(x) = {} & 4 + (p_0-1)(p_0-2)(p_0-3)x^{p_0-4} \\ & - (p_0-1)(p_0-2)x^{p_0-3} + p_0(p_0+1)x^{p_0-1} \\ & - (p_0-1)(p_0+2)(p_0+1)x^{p_0} \\ & - 4(p_0-1)(2p_0-3)x^{2p_0-4} \\ & - 2(2p_0-1)(p_0-1)x^{2p_0-3} \\ & - 2p_0(2p_0-1)x^{2p_0-2}, \\ & f_2''(1) = 4(2p_0-1)(4-3p_0) > 0, \end{aligned} \tag{31}$$

$$\lim_{x\to+\infty} f_2''(x) = -\infty, \tag{32}$$

$$f_2'''(x) = (p_0-1)x^{p_0-5} f_3(x), \tag{33}$$

where

$$\begin{aligned} f_3(x) = {} & -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x \\ & + p_0(p_0+1)x^3 - p_0(p_0+1)(p_0+2)x^4 \\ & - 8(3-2p_0)(2-p_0)x^{p_0} + 2(2p_0-1)(3-2p_0)x^{p_0+1} \\ & - 4p_0(2p_0-1)x^{p_0+2} \\ < {} & -(2-p_0)(3-p_0)(4-p_0) \\ & - (2-p_0)(3-p_0)x + p_0(p_0+1)x^4 \\ & - p_0(p_0+1)(p_0+2)x^4 - 8(3-2p_0)(2-p_0)x^{p_0} \\ & + 2(2p_0-1)(3-2p_0)x^{p_0+2} - 4p_0(2p_0-1)x^{p_0+2} \\ = {} & -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x \\ & - p_0(p_0+1)^2x^4 - 8(3-2p_0)(2-p_0)x^{p_0} \\ & - 2(2p_0-1)(4p_0-3)x^{p_0+2} < 0, \end{aligned} \tag{34}$$

for $x > 1$.

Equation (33) and inequality (34) imply that $f_2''(x)$ is strictly decreasing on $[1,+\infty)$. Then, the inequality (31) and (32) lead to the conclusion that there exists $x_1 > 1$, such that $f_2'(x)$ is strictly increasing on $[1,x_1]$ and strictly decreasing on $[x_1,+\infty)$.

From (29) and (30) together with the piecewise monotonicity of $f_2'(x)$, we clearly see that there exists $x_2 > x_1 > 1$, such that $f_2(x)$ is strictly increasing on $[1,x_2]$ and strictly decreasing on $[x_2,+\infty)$.

It follows from (26)–(28) and the piecewise monotonicity of $f_2(x)$ that there exists $x_3 > x_2 > 1$, such that $f_1(x)$, is strictly increasing on $[1,x_3]$ and strictly decreasing on $[x_3,+\infty)$.

From (23)–(25) and the piecewise monotonicity of $f_1(x)$ we see that there exists $x_4 > x_3 > 1$, such that $f(x)$ is strictly increasing on $(1,x_4]$ and strictly decreasing on $[x_4,+\infty)$.

Therefore, $M(x,1) > M_{p_0}(x,1)$ for $x > 1$ follows easily from (19)–(22) and the piecewise monotonicity of $f(x)$.

Next, we prove that $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ is the greatest value such that $M(x,1) > M_{p_0}(x,1)$ for all $x > 1$.

For any $\varepsilon > 0$ and $x > 1$, from (1) and (2), one has

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{M_{p_0+\varepsilon}(x,1)}{M(x,1)}\\
&=\lim_{x\to+\infty}\left[\left(\frac{1+x^{p_0+\varepsilon}}{2}\right)^{1/(p_0+\varepsilon)}\frac{2\sinh^{-1}((x-1)/(x+1))}{x-1}\right]\\
&=2^{-1/(p_0+\varepsilon)}\times 2\sinh^{-1}(1)\\
&=2^{\varepsilon/p_0(p_0+\varepsilon)}>1.
\end{aligned}\tag{35}$$

Inequality (35) implies that for any $\varepsilon > 0$, there exists $X = X(\varepsilon) > 1$, such that $M(x,1) < M_{p_0+\varepsilon}(x,1)$ for $x \in (X,+\infty)$. □

Remark 2. 4/3 is the least value such that inequality (16) holds for all $a,b > 0$ with $a \neq b$, namely, $M_{4/3}(a,b)$ is the best possible upper power mean bound for the Neuman-Sándor mean $M(a,b)$.

In fact, for any $\varepsilon \in (0,4/3)$ and $x > 0$, one has

$$\begin{aligned}
&M_{4/3-\varepsilon}(1+x,1)-M(1+x,1)\\
&\quad=\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}.
\end{aligned}\tag{36}$$

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned}
&\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}\\
&=\left[1+\frac{4-3\varepsilon}{6}x+\frac{(4-3\varepsilon)(1-3\varepsilon)}{36}x^2+o(x^2)\right]^{1/(4/3-\varepsilon)}\\
&\quad-\frac{x}{x-(1/2)x^2+(5/24)x^3+o(x^3)}\\
&=\left[1+\frac{1}{2}x+\frac{1-3\varepsilon}{24}x^2+o(x^2)\right]\\
&\quad-\left[1+\frac{1}{2}x+\frac{1}{24}x^2+o(x^2)\right]=-\frac{\varepsilon}{8}x^2+o(x^2).
\end{aligned}\tag{37}$$

Equations (36) and (37) imply that for any $\varepsilon \in (0,4/3)$ there exists $\delta = \delta(\varepsilon) > 0$, such that $M(1+x,1) > M_{(4/3)-\varepsilon}(1+x,1)$ for $x \in (0,\delta)$.

Theorem 3. *For all $a,b > 0$ with $a \neq b$, one has*

$$1<\frac{M(a,b)}{I(a,b)}<\frac{e}{2\log(1+\sqrt{2})},\tag{38}$$

with the best possible constants 1 *and* $e/[2\log(1+\sqrt{2})] = 1.5419\ldots$.

Proof. From (2) and (3), we clearly see that both $M(a,b)$ and $I(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$. Let

$$f(x)=\frac{M(x,1)}{I(x,1)}=\frac{e(x-1)}{2x^{x/(x-1)}\sinh^{-1}((x-1)/(x+1))}.\tag{39}$$

Then, simple computations lead to

$$\frac{f'(x)}{f(x)}=\frac{\log x}{(x-1)^2\sinh^{-1}((x-1)/(x+1))}f_1(x),\tag{40}$$

where

$$\begin{aligned}
f_1(x)&=\sinh^{-1}\left(\frac{x-1}{x+1}\right)-\frac{\sqrt{2}(x-1)^2}{(x+1)\sqrt{1+x^2}\log x},\\
&\lim_{x\to1^+}f_1(x)=0,
\end{aligned}\tag{41}$$

$$f_1'(x)=\frac{\sqrt{2}f_2(x)}{x(x+1)^2(1+x^2)^{3/2}\log^2 x},\tag{42}$$

where

$$\begin{aligned}
f_2(x)=\;&x(x+1)\left(1+x^2\right)\log^2 x\\
&-x\left(3x^3-x^2+x-3\right)\log x\\
&+(x-1)^2(x+1)\left(1+x^2\right),\\
&f_2(1)=0,
\end{aligned}\tag{43}$$

$$f_2'(x) = \left(4x^3 + 3x^2 + 2x + 1\right)\log^2 x + 5\left(-2x^3 + x^2 + 1\right)\log x + 5x^4 - 7x^3 + x^2 - x + 2, \quad f_2'(1) = 0, \tag{44}$$

$$f_2''(x) = 2\left(6x^2 + 3x + 1\right)\log^2 x + 2\left(-11x^2 + 8x + 2 + x^{-1}\right)\log x + 20x^3 - 31x^2 + 7x - 1 + 5x^{-1}, \quad f_2''(1) = 0, \tag{45}$$

$$f_2'''(x) = 6(4x + 1)\log^2 x + 2\left(-10x + 14 + 2x^{-1} - x^{-2}\right)\log x + 60x^2 - 84x + 23 + 4x^{-1} - 3x^{-2}, \quad f_2'''(1) = 0, \tag{46}$$

$$f_2^{(4)}(x) = 24\log^2 x + 4\left(7 + 3x^{-1} - x^{-2} + x^{-3}\right)\log x + 120x - 104 + 28x^{-1} + 4x^{-3} > 0 \tag{47}$$

for $x > 1$.

From (46) and (47), we clearly see that $f_2''(x)$ is strictly increasing on $[1, +\infty)$. Then, (45) leads to the conclusion that $f_2'(x)$ is strictly increasing on $[1, +\infty)$.

Equations (43) and (44) together with the monotonicity of $f_2'(x)$ impliy that $f_2(x) > 0$ for $x > 1$. Then, (42) leads to the conclusion that $f_1(x)$ is strictly increasing on $[1, +\infty)$.

It follows from equations (40) and (41) together with the monotonicity of $f_1(x)$ that $f(x)$ is strictly increasing on $(1, +\infty)$.

Therefore, Theorem 3 follows from (39) and the monotonicity of $f(x)$ together with the facts that

$$\lim_{x\to+\infty} f(x) = \frac{e}{2\log\left(1+\sqrt{2}\right)}, \quad \lim_{x\to 1^+} f(x) = 1. \tag{48}$$

□

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grants nos. 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant no. T200924.

References

[1] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[2] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and its Applications*, D. Reidel Publishing, Dordrecht, The Netherlands, 1988.

[3] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 2-3, pp. 261–270, 1990.

[4] J. E. Pečarić, "Generalization of the power means and their inequalities," *Journal of Mathematical Analysis and Applications*, vol. 161, no. 2, pp. 395–404, 1991.

[5] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.

[6] J. Sándor and I. Raşa, "Inequalities for certain means in two arguments," *Nieuw Archief voor Wiskunde*, vol. 15, no. 1-2, pp. 51–55, 1997.

[7] J. Sándor and T. Trif, "Some new inequalities for means of two arguments," *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 8, pp. 525–532, 2001.

[8] T. Trif, "On certain inequalities involving the identric mean in *n* variables," *Universitatis Babeş-Bolyai*, vol. 46, no. 4, pp. 105–114, 2001.

[9] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.

[10] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.

[11] S. Wu, "Generalization and sharpness of the power means inequality and their applications," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 637–652, 2005.

[12] O. Kouba, "New bounds for the identric mean of two arguments," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 3, article 71, 6 pages, 2008.

[13] L. Zhu, "New inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 2, pp. 229–235, 2008.

[14] L. Zhu, "Some new inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 3, pp. 443–448, 2008.

[15] E. Neuman and J. Sándor, "Companion inequalities for certain bivariate means," *Applicable Analysis and Discrete Mathematics*, vol. 3, no. 1, pp. 46–51, 2009.

[16] Y.-M. Chu and W.-F. Xia, "Two sharp inequalities for power mean, geometric mean, and harmonic mean," *Journal of Inequalities and Applications*, vol. 2009, Article ID 741923, 6 pages, 2009.

[17] Y.-M. Chu, Y.-F. Qiu, and M.-K. Wang, "Sharp power mean bounds for the combination of Seiffert and geometric means," *Abstract and Applied Analysis*, vol. 2010, Article ID 108920, 12 pages, 2010.

[18] B.-Y. Long and Y.-M. Chu, "Optimal power mean bounds for the weighted geometric mean of classical means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 905679, 6 pages, 2010.

[19] M.-K. Wang, Y.-M. Chu, and Y.-F. Qiu, "Some comparison inequalities for generalized Muirhead and identric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 295620, 10 pages, 2010.

[20] S. Gao, "Inequalities for the Seiffert's means in terms of the identric mean," *Journal of Mathematical Sciences*, vol. 10, no. 1-2, pp. 23–31, 2011.

[21] Y.-M. Chu, M.-K. Wang, and Z.-K. Wang, "A sharp double inequalities between harmonic and identric means," *Abstract and Applied Analysis*, vol. 2011, Article ID 657935, 7 pages, 2011.

[22] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu, and G.-D. Wang, "Two sharp inequalities for Lehmer mean, identric mean and logarithmic

mean," *Journal of Mathematical Inequalities*, vol. 5, no. 3, pp. 301–306, 2011.

[23] M.-K. Wang, Z.-K. Wang, and Y.-M. Chu, "An optimal double inequality between geometric and identric means," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 471–475, 2012.

[24] Y. M. Li, B. Y. Long, and Y. M. Chu, "Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 567–577, 2012.

[25] E. Neuman, "A note on a certain bivariate mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 637–643, 2012.

[26] T. H. Zhao, Y. M. Chu, and B. Y. Liu, "Optimal bounds for Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means," *Abstract and Applied Analysis*, vol. 2012, Article ID 302635, 9 pages, 2012.

[27] K. Baumgartner, "Zur Unauflösbarkeit für $(e/a)^a(b/e)^b$," *Elemente der Mathematik*, vol. 40, no. 5, p. 123, 1985.

[28] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[29] F. Burk, "Notes: the geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.

[30] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.

[31] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Univerzitet u Beogradu*, no. 678–715, pp. 15–18, 1980.

[32] A. O. Pittenger, "The symmetric, logarithmic and power means," *Univerzitet u Beogradu*, no. 678–715, pp. 19–23, 1980.

[33] A. A. Jagers, "Solution of problem 887," *Nieuw Archief voor Wiskunde*, vol. 12, pp. 230–231, 1994.

[34] J. Sándor, "On certain identities for means," *Universitatis Babeş-Bolyai*, vol. 38, no. 4, pp. 7–14, 1993.