Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2013, Article ID 832591, 6 pages http://dx.doi.org/10.1155/2013/832591



# Research Article

# **Bounds of the Neuman-Sándor Mean Using Power and Identric Means**

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Received 8 November 2012; Revised 4 January 2013; Accepted 11 January 2013

Academic Editor: Wenchang Sun

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In this paper we find the best possible lower power mean bounds for the Neuman-Sándor mean and present the sharp bounds for the ratio of the Neuman-Sándor and identric means.

## 1. Introduction

For  $p \in \mathbb{R}$  the pth power mean  $M_p(a,b)$ , Neuman-Sándor Mean M(a,b) [1], and identric mean I(a,b) of two positive numbers a and b are defined by

$$M_{p}(a,b) = \begin{cases} \left(\frac{a^{p} + b^{p}}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$
 (1)

$$M(a,b) = \begin{cases} \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases}$$
(2)

$$I(a,b) = \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases}$$
 (3)

respectively, where  $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$  is the inverse hyperbolic sine function.

The main properties for  $M_p(a,b)$  and I(a,b) are given in [2]. It is well known that  $M_p(a,b)$  is continuously and strictly increasing with respect to  $p \in \mathbb{R}$  for fixed a,b>0 with  $a \neq b$ . Recently, the power, Neuman-Sándor, and identric means have been a subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [3–26].

Let H(a,b) = 2ab/(a+b),  $G(a,b) = \sqrt{ab}$ ,  $L(a,b) = (b-a)/(\log b - \log a)$ ,  $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$ , A(a,b) = (a+b)/2,  $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ ,  $Q(a,b) = \sqrt{(a^2+b^2)/2}$ , and  $C(a,b) = (a^2+b^2)/(a+b)$  be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of two positive numbers a and b with  $a \neq b$ , respectively. Then, it is well known that the inequalities

$$\begin{split} H\left(a,b\right) &= M_{-1}\left(a,b\right) < G\left(a,b\right) = M_{0}\left(a,b\right) < L\left(a,b\right) \\ &< P\left(a,b\right) < I\left(a,b\right) < A\left(a,b\right) = M_{1}\left(a,b\right) < M\left(a,b\right) \\ &< T\left(a,b\right) < Q\left(a,b\right) = M_{2}\left(a,b\right) < C\left(a,b\right)\,, \end{split} \tag{4}$$

hold for all a, b > 0 with  $a \neq b$ .

The following sharp bounds for L, I,  $(IL)^{1/2}$ , and (I + L)/2 in terms of power means are presented in [27–32]:

$$\begin{split} M_{0}\left(a,b\right) &< L\left(a,b\right) < M_{1/3}\left(a,b\right)\,,\\ M_{2/3}\left(a,b\right) &< I\left(a,b\right) < M_{\log 2}\left(a,b\right)\,,\\ M_{0}\left(a,b\right) &< I^{1/2}\left(a,b\right)L^{1/2}\left(a,b\right) < M_{1/2}\left(a,b\right)\,,\\ &\frac{1}{2}\left[I\left(a,b\right) + L\left(a,b\right)\right] &< M_{1/2}\left(a,b\right)\,, \end{split} \tag{5}$$

for all a, b > 0 with  $a \neq b$ .

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Pittenger [31] found the greatest value  $r_1$  and the least value  $r_2$  such that the double inequality

$$M_{r_1}(a,b) \le L_p(a,b) \le M_{r_2}(a,b),$$
 (6)

holds for all a, b > 0, where  $L_r(a, b)$  is the rth generalized logarithmic means which is defined by

$$L_{r}(a,b) = \begin{cases} \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & a \neq b, \ r \neq -1, \ r \neq 0, \\ \frac{1}{e} \left(\frac{b^{b}}{a^{a}}\right)^{1/(b-a)}, & a \neq b, \ r = 0, \\ \frac{b-a}{\log b - \log a}, & a \neq b, \ r = -1, \\ a, & a = b. \end{cases}$$
(7)

The following sharp power mean bounds for the first Seiffert mean P(a, b) are given in [10, 33]:

$$M_{\log 2/\log \pi}(a,b) < P(a,b) < M_{2/3}(a,b),$$
 (8)

for all a, b > 0 with  $a \neq b$ .

In [17], the authors answered the question: for  $\alpha \in (0, 1)$ , what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b) < P^{\alpha}(a,b) G^{1-\alpha}(a,b) < M_q(a,b)$$
 (9)

holds for all a, b > 0 with  $a \neq b$ ?

Neuman and Sándor [1] established that

$$A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})},$$

$$\frac{\pi}{4}T(a,b) < M(a,b) < T(a,b),$$

$$M(a,b) < \frac{2A(a,b) + Q(a,b)}{3},$$
(10)

for all a, b > 0 with  $a \neq b$ .

Let  $0 < a, b \le 1/2$  with  $a \ne b$ , a' = 1 - a and b' = 1 - b. Then, the Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} 
< \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')}$$
(11)

were presented in [1].

In [24], Li et al. found the best possible bounds for the Neuman-Sándor mean M(a,b) in terms of the generalized

logarithmic mean  $L_r(a, b)$ . Neuman [25] and Zhao et al. [26] proved that the inequalities

$$\alpha Q(a,b) + (1-\alpha) A(a,b)$$

$$< M(a,b) < \beta Q(a,b) + (1-\beta) A(a,b),$$

$$\lambda C(a,b) + (1-\lambda) A(a,b) < M(a,b)$$

$$< \mu C(a,b) + (1-\mu) A(a,b),$$

$$\alpha_1 H(a,b) + (1-\alpha_1) Q(a,b) < M(a,b)$$

$$< \beta_1 H(a,b) + (1-\beta_1) Q(a,b),$$

$$\alpha_2 G(a,b) + (1-\alpha_2) Q(a,b) < M(a,b)$$

$$< \beta_2 G(a,b) + (1-\beta_2) Q(a,b)$$
(12)

hold for all a,b>0 with  $a\neq b$  if and only if  $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})], \ \beta \geq 1/3, \ \lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}), \ \mu \geq 1/6, \ \alpha_1 \geq 2/9, \ \beta_1 \leq 1-1/[\sqrt{2}\log(1+\sqrt{2})], \ \alpha_2 \geq 1/3, \ \text{and} \ \beta_2 \leq 1-1/[\sqrt{2}\log(1+\sqrt{2})].$  In [7], Sándor and Trif proved that the inequalities

$$e^{((a-b)^{2}/6(a+b)^{2})} < \frac{A(a,b)}{I(a,b)} < e^{((a-b)^{2}/24ab)},$$

$$e^{((a-b)^{2}/3(a+b)^{2})} < \frac{I(a,b)}{G(a,b)} < e^{((a-b)^{2}/12ab)},$$

$$e^{((a-b)^{4}/30(a+b)^{4})} < \frac{I(a,b)}{A^{2/3}(a,b) G^{1/3}(a,b)}$$

$$< e^{((a-b)^{4}/120ab(a+b)^{4})}$$

hold for all a, b > 0 with  $a \neq b$ .

Neuman and Sándor [15] and Gao [20] proved that  $\alpha_1 = 1$ ,  $\beta_1 = e/2$ ,  $\alpha_2 = 1$ ,  $\beta_2 = 2\sqrt{2}/e$ ,  $\alpha_3 = 1$ ,  $\beta_3 = 3/e$ ,  $\alpha_4 = e/\pi$ ,  $\beta_4 = 1$ ,  $\alpha_5 = 1$ , and  $\beta_5 = 2e/\pi$  are the best possible constants such that the double inequalities  $\alpha_1 < A(a,b)/I(a,b) < \beta_1$ ,  $\alpha_2 < I(a,b)/M_{2/3}(a,b) < \beta_2$ ,  $\alpha_3 < I(a,b)/He(a,b) < \beta_3$ ,  $\alpha_4 < P(a,b)/I(a,b) < \beta_4$ , and  $\alpha_5 < T(a,b)/I(a,b) < \beta_5$  hold for all a,b > 0 with  $a \ne b$ , where  $He(a,b) = (a + \sqrt{ab} + b)/3 = (2A(a,b) + G(a,b))/3$  is the Heronian mean of a and b.

In [34], Sándor established that

$$He(a,b) < M_{2/3}(a,b),$$
 (14)

for all a, b > 0 with  $a \neq b$ .

It is not difficult to verify that the inequality

$$\frac{2A(a,b) + Q(a,b)}{3} < \left[He(a^2,b^2)\right]^{1/2}$$
 (15)

holds for all a, b > 0 with  $a \neq b$ .

From inequalities (10), (14), and (15), one has

$$M(a,b) < \left[M_{2/3}(a^2,b^2)\right]^{1/2} = M_{4/3}(a,b),$$
 (16)

for all a, b > 0 with  $a \neq b$ .

It is the aim of this paper to find the best possible lower power mean bound for the Neuman-Sándor mean M(a,b) and to present the sharp constants  $\alpha$  and  $\beta$  such that the double inequality

$$\alpha < \frac{M(a,b)}{I(a,b)} < \beta \tag{17}$$

holds for all a, b > 0 with  $a \neq b$ .

#### 2. Main Results

**Theorem 1.**  $p_0 = (\log 2)/\log [2\log(1 + \sqrt{2})] = 1.224...$  is the greatest value such that the inequality

$$M(a,b) > M_{p_0}(a,b)$$
 (18)

holds for all a, b > 0 with  $a \neq b$ .

*Proof.* From (1) and (2), we clearly see that both M(a,b) and  $M_p(a,b)$  are symmetric and homogenous of degree one. Without loss of generality, we assume that b=1 and a=x>1.

Let  $p_0 = (\log 2) / \log [2 \log(1 + \sqrt{2})]$ , then from (1) and (2) one has

$$\log M(x,1) - \log M_{p_0}(x,1)$$

$$= \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0} \log \frac{x^{p_0}+1}{2}.$$
 (19)

Let

$$f(x) = \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log \frac{x^{p_0}+1}{2}.$$
 (20)

Then, simple computations lead to

$$\lim_{x \to 1^{+}} f(x) = 0, \tag{21}$$

$$\lim_{x \to +\infty} f(x) = \frac{1}{p_0} \log 2 - \log \left[ 2\sinh^{-1} (1) \right] = 0, \quad (22)$$

$$f'(x) = \frac{\left(1 + x^{p_0 - 1}\right) f_1(x)}{(x - 1)(x^{p_0} + 1)\sinh^{-1}((x - 1)/(x + 1))}, \quad (23)$$

where

$$f_1(x) = -\frac{\sqrt{2}(x-1)(x^{p_0}+1)}{(x+1)(x^{p_0-1}+1)\sqrt{1+x^2}} + \sinh^{-1}\left(\frac{x-1}{x+1}\right),$$

$$f_1(1) = 0, (24)$$

$$\lim_{x \to 100} f_1(x) = -\sqrt{2} + \sinh^{-1}(1) = -0.5328 \dots < 0, \quad (25)$$

$$f_1'(x) = \frac{\sqrt{2}(x-1)f_2(x)}{(x+1)^2(x^{p_0-1}+1)^2(1+x^2)^{3/2}},$$
 (26)

where

$$f_{2}(x) = 1 + x + 2x^{2} + (p_{0} - 1)x^{p_{0}-2} - x^{p_{0}-1} + x^{p_{0}+1}$$
$$-(p_{0} - 1)x^{p_{0}+2} - 2x^{2p_{0}-2} - x^{2p_{0}-1} - x^{2p_{0}}, \quad (27)$$
$$f_{2}(1) = 0,$$

$$\lim_{x \to +\infty} f_2(x) = -\infty, \tag{28}$$

$$f_{2}'(x) = 1 + 4x + (p_{0} - 1)(p_{0} - 2)x^{p_{0} - 3} - (p_{0} - 1)x^{p_{0} - 2} + (p_{0} + 1)x^{p_{0}} - (p_{0} - 1)(p_{0} + 2)x^{p_{0} + 1} - 4(p_{0} - 1)x^{2p_{0} - 3} - (2p_{0} - 1)x^{2p_{0} - 2} - 2p_{0}x^{2p_{0} - 1},$$

$$f_{2}'(1) = 4(4 - 3p_{0}) > 0,$$
(29)

$$\lim_{x \to +\infty} f_2'(x) = -\infty,\tag{30}$$

$$f_{2}^{"}(x) = 4 + (p_{0} - 1) (p_{0} - 2) (p_{0} - 3) x^{p_{0} - 4}$$

$$- (p_{0} - 1) (p_{0} - 2) x^{p_{0} - 3} + p_{0} (p_{0} + 1) x^{p_{0} - 1}$$

$$- (p_{0} - 1) (p_{0} + 2) (p_{0} + 1) x^{p_{0}}$$

$$- 4 (p_{0} - 1) (2p_{0} - 3) x^{2p_{0} - 4}$$

$$- 2 (2p_{0} - 1) (p_{0} - 1) x^{2p_{0} - 3}$$

$$- 2p_{0} (2p_{0} - 1) x^{2p_{0} - 2},$$

$$f_{2}^{"}(1) = 4 (2p_{0} - 1) (4 - 3p_{0}) > 0,$$
(31)

$$\lim_{x \to +\infty} f_2''(x) = -\infty,\tag{32}$$

$$f_2'''(x) = (p_0 - 1) x^{p_0 - 5} f_3(x),$$
 (33)

where

$$f_{3}(x) = -(2 - p_{0})(3 - p_{0})(4 - p_{0}) - (2 - p_{0})(3 - p_{0})x$$

$$+ p_{0}(p_{0} + 1)x^{3} - p_{0}(p_{0} + 1)(p_{0} + 2)x^{4}$$

$$- 8(3 - 2p_{0})(2 - p_{0})x^{p_{0}} + 2(2p_{0} - 1)(3 - 2p_{0})x^{p_{0} + 1}$$

$$- 4p_{0}(2p_{0} - 1)x^{p_{0} + 2}$$

$$< -(2 - p_{0})(3 - p_{0})(4 - p_{0})$$

$$-(2 - p_{0})(3 - p_{0})x + p_{0}(p_{0} + 1)x^{4}$$

$$- p_{0}(p_{0} + 1)(p_{0} + 2)x^{4} - 8(3 - 2p_{0})(2 - p_{0})x^{p_{0}}$$

$$+ 2(2p_{0} - 1)(3 - 2p_{0})x^{p_{0} + 2} - 4p_{0}(2p_{0} - 1)x^{p_{0} + 2}$$

$$= -(2 - p_{0})(3 - p_{0})(4 - p_{0}) - (2 - p_{0})(3 - p_{0})x$$

$$- p_{0}(p_{0} + 1)^{2}x^{4} - 8(3 - 2p_{0})(2 - p_{0})x^{p_{0}}$$

$$- 2(2p_{0} - 1)(4p_{0} - 3)x^{p_{0} + 2} < 0,$$
(34)

for x > 1.

Equation (33) and inequality (34) imply that  $f_2''(x)$  is strictly decreasing on  $[1, +\infty)$ . Then, the inequality (31) and (32) lead to the conclusion that there exists  $x_1 > 1$ , such that  $f_2'(x)$  is strictly increasing on  $[1, x_1]$  and strictly decreasing on  $[x_1, +\infty)$ .

From (29) and (30) together with the piecewise monotonicity of  $f_2'(x)$ , we clearly see that there exists  $x_2 > x_1 > 1$ , such that  $f_2(x)$  is strictly increasing on  $[1, x_2]$  and strictly decreasing on  $[x_2, +\infty)$ .

It follows from (26)–(28) and the piecewise monotonicity of  $f_2(x)$  that there exists  $x_3 > x_2 > 1$ , such that  $f_1(x)$ , is strictly increasing on  $[1, x_3]$  and strictly decreasing on  $[x_3, +\infty)$ .

From (23)–(25) and the piecewise monotonicity of  $f_1(x)$  we see that there exists  $x_4 > x_3 > 1$ , such that f(x) is strictly increasing on  $(1, x_4]$  and strictly decreasing on  $[x_4, +\infty)$ .

Therefore,  $M(x, 1) > M_{p_0}(x, 1)$  for x > 1 follows easily from (19)–(22) and the piecewise monotonicity of f(x).

Next, we prove that  $p_0 = (\log 2)/\log \left[2\log(1+\sqrt{2})\right] = 1.224...$  is the greatest value such that  $M(x,1) > M_{p_0}(x,1)$  for all x > 1.

For any  $\varepsilon > 0$  and x > 1, from (1) and (2), one has

$$\lim_{x \to +\infty} \frac{M_{p_0 + \varepsilon}(x, 1)}{M(x, 1)}$$

$$= \lim_{x \to +\infty} \left[ \left( \frac{1 + x^{p_0 + \varepsilon}}{2} \right)^{1/(p_0 + \varepsilon)} \frac{2\sinh^{-1}((x - 1) / (x + 1))}{x - 1} \right]$$

$$= 2^{-1/(p_0 + \varepsilon)} \times 2\sinh^{-1}(1)$$

$$= 2^{\varepsilon/p_0(p_0 + \varepsilon)} > 1.$$
(35)

Inequality (35) implies that for any  $\varepsilon > 0$ , there exists  $X = X(\varepsilon) > 1$ , such that  $M(x, 1) < M_{p_0 + \varepsilon}(x, 1)$  for  $x \in (X, +\infty)$ .

*Remark 2.* 4/3 is the least value such that inequality (16) holds for all a, b > 0 with  $a \neq b$ , namely,  $M_{4/3}(a, b)$  is the best possible upper power mean bound for the Neuman-Sándor mean M(a, b).

In fact, for any  $\varepsilon \in (0, 4/3)$  and x > 0, one has

$$M_{4/3-\varepsilon} (1+x,1) - M (1+x,1)$$

$$= \left[ \frac{(1+x)^{4/3-\varepsilon} + 1}{2} \right]^{1/(4/3-\varepsilon)} - \frac{x}{2\sinh^{-1} (x/(2+x))}.$$

Letting  $x \to 0$  and making use of Taylor expansion, we get

$$\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)} - \frac{x}{2\sinh^{-1}(x/(2+x))}$$

$$= \left[1 + \frac{4-3\varepsilon}{6}x + \frac{(4-3\varepsilon)(1-3\varepsilon)}{36}x^2 + o\left(x^2\right)\right]^{1/(4/3-\varepsilon)}$$

$$-\frac{x}{x-(1/2)x^2 + (5/24)x^3 + o\left(x^3\right)}$$

$$= \left[1 + \frac{1}{2}x + \frac{1 - 3\varepsilon}{24}x^2 + o\left(x^2\right)\right]$$
$$-\left[1 + \frac{1}{2}x + \frac{1}{24}x^2 + o\left(x^2\right)\right] = -\frac{\varepsilon}{8}x^2 + o\left(x^2\right). \tag{37}$$

Equations (36) and (37) imply that for any  $\varepsilon \in (0, 4/3)$  there exists  $\delta = \delta(\varepsilon) > 0$ , such that  $M(1+x, 1) > M_{(4/3)-\varepsilon}(1+x, 1)$  for  $x \in (0, \delta)$ .

**Theorem 3.** For all a, b > 0 with  $a \neq b$ , one has

$$1 < \frac{M(a,b)}{I(a,b)} < \frac{e}{2\log(1+\sqrt{2})},$$
 (38)

with the best possible constants 1 and  $e/[2\log(1 + \sqrt{2})] = 1.5419...$ 

*Proof.* From (2) and (3), we clearly see that both M(a,b) and I(a,b) are symmetric and homogenous of degree one. Without loss of generality, we assume that b=1 and a=x>1. Let

$$f(x) = \frac{M(x,1)}{I(x,1)} = \frac{e(x-1)}{2x^{x/(x-1)}\sinh^{-1}((x-1)/(x+1))}.$$
(39)

Then, simple computations lead to

$$\frac{f'(x)}{f(x)} = \frac{\log x}{(x-1)^2 \sinh^{-1}((x-1)/(x+1))} f_1(x), \quad (40)$$

where

$$f_{1}(x) = \sinh^{-1}\left(\frac{x-1}{x+1}\right) - \frac{\sqrt{2}(x-1)^{2}}{(x+1)\sqrt{1+x^{2}}\log x},$$

$$\lim_{x \to 1^{+}} f_{1}(x) = 0,$$
(41)

$$f_1'(x) = \frac{\sqrt{2}f_2(x)}{x(x+1)^2(1+x^2)^{3/2}\log^2 x},$$
 (42)

where

$$f_{2}(x) = x(x+1)(1+x^{2})\log^{2}x$$

$$-x(3x^{3}-x^{2}+x-3)\log x$$

$$+(x-1)^{2}(x+1)(1+x^{2}),$$

$$f_{2}(1) = 0,$$
(43)

$$f_2'(x) = (4x^3 + 3x^2 + 2x + 1)\log^2 x$$

$$+ 5(-2x^3 + x^2 + 1)\log x + 5x^4$$

$$- 7x^3 + x^2 - x + 2,$$

$$f_2'(1) = 0,$$
(44)

$$f_2''(x) = 2(6x^2 + 3x + 1)\log^2 x$$

$$+ 2(-11x^2 + 8x + 2 + x^{-1})\log x + 20x^3$$

$$- 31x^2 + 7x - 1 + 5x^{-1},$$

$$f_2''(1) = 0,$$
(45)

$$f_2'''(x) = 6(4x+1)\log^2 x$$

$$+ 2(-10x+14+2x^{-1}-x^{-2})\log x$$

$$+ 60x^2 - 84x + 23 + 4x^{-1} - 3x^{-2},$$

$$f_2'''(1) = 0,$$
(46)

$$f_2^{(4)}(x) = 24\log^2 x + 4\left(7 + 3x^{-1} - x^{-2} + x^{-3}\right)\log x + 120x - 104 + 28x^{-1} + 4x^{-3} > 0$$
(47)

for x > 1.

From (46) and (47), we clearly see that  $f_2''(x)$  is strictly increasing on  $[1, +\infty)$ . Then, (45) leads to the conclusion that  $f_2'(x)$  is strictly increasing on  $[1, +\infty)$ .

Equations (43) and (44) together with the monotonicity of  $f_2'(x)$  implies that  $f_2(x) > 0$  for x > 1. Then, (42) leads to the conclusion that  $f_1(x)$  is strictly increasing on  $[1, +\infty)$ .

It follows from equations (40) and (41) together with the monotonicity of  $f_1(x)$  that f(x) is strictly increasing on  $(1, +\infty)$ .

Therefore, Theorem 3 follows from (39) and the monotonicity of f(x) together with the facts that

$$\lim_{x \to +\infty} f(x) = \frac{e}{2\log\left(1 + \sqrt{2}\right)},$$

$$\lim_{x \to 1^+} f(x) = 1.$$

## Acknowledgments

This research was supported by the Natural Science Foundation of China under Grants nos. 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant no. T200924.

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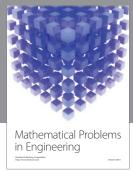
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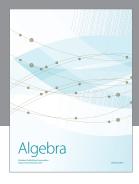
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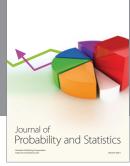
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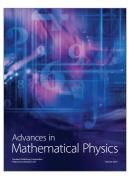


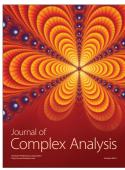




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