Research Article

Stabilization and Controller Design of 2D Discrete Switched Systems with State Delays under Asynchronous Switching

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This paper is concerned with the problem of robust stabilization for a class of uncertain two-dimensional (2D) discrete switched systems with state delays under asynchronous switching. The asynchronous switching here means that the switching instants of the controller experience delays with respect to those of the system. The parameter uncertainties are assumed to be norm-bounded. A state feedback controller is proposed to guarantee the exponential stability. The dwell time approach is utilized for the stability analysis and controller design. A numerical example is given to illustrate the effectiveness of the proposed method.

1. Introduction

Two-dimensional (2D) systems have attracted considerable attention for several decades due to their numerous applications in many areas, such as multidimensional digital filtering, linear image processing, signal processing and process control [1–3]. It is well known that 2D systems can be represented by different models such as Roesser model, Fornasini-Marchesini model, and Attasi model. The issues of stability analysis and control synthesis of these systems have been studied in [4–8]. Considering that time delays frequently occur in practical systems and are often the source of instability, many authors have devoted their energies to studying time-delay systems. Recently, many results on delay systems have been reported in the literature. For example, the delay-fractional approach has been utilized to deal with discrete time-delay systems in [9–11]. The stability of 2D discrete systems with state delays has been investigated in [12–16].

On the other hand, because of their wide applications in many fields, such as mechanical systems, automotive industry, aircraft and air traffic control, and switched power converters, switched systems have also received considerable attention during the past few decades. A switched system is a hybrid system which consists of a finite number of continuous-time or discrete-time subsystems and a switching signal specifying the switch between these subsystems. The stability and stabilization problems have been extensively studied in [17–25].

In many modelling problems of physical processes, a 2D switching representation is needed. One can cite a 2D physically based model for advanced power bipolar devices [26] and heat flux switching, and modulating in a thermal transistor [27]. This class of systems can correspond to 2D state space or 2D time space switched systems. Recently, there are a few reports on 2D switched systems. Benzaouia et al. firstly studied the stabilization problem of 2D discrete switched systems with arbitrary switching sequences in [28, 29]. By using the common Lyapunov function method and multiple Lyapunov functions method, two different sufficient conditions for the existence of state feedback controllers were proposed. In [30], the authors first extended the concept of average dwell time to 2D switched systems and designed a switching signal to guarantee the exponential stability of delay-free 2D switched systems. It should be pointed out that a very common assumption in [30] is that the controllers are switched synchronously with the switching of system modes, which is quite unpractical. As stated in [31, 32], there inevitably exists asynchronous switching in actual operation that is, the switching instants of the controllers exceed or lag behind those of system modes. Thus, it is necessary to

consider asynchronous switching for efficient control design. Some results on the control synthesis for switched systems under asynchronous switching have been proposed in [33–36]. However, to the best of our knowledge, the stabilization problem for 2D switched systems under asynchronous switching has not been yet investigated to date, especially for 2D switched systems with state delays, which motivates our present study.

In this paper, we are interested in designing a stabilizing controller for 2D discrete switched delayed systems represented by a model of Roesser type under asynchronous switching such that the corresponding closed-loop systems are exponentially stable. The dwell time approach is utilized for the stability analysis and controller design. The main contributions of this paper can be summarized as follows: (i) the asynchronous stabilization problem is for the first time addressed in the paper; (ii) an exponential stability criterion is established for 2D switched systems with state delays; and (iii) an asynchronous switching controller design scheme is proposed to guarantee the exponential stability of the resulting closed-loop system.

This paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, based on the dwell time approach, stability and stabilization for 2D discrete switched systems with state delays are addressed. Then, a sufficient condition for the existence of a stabilizing controller for such 2D discrete switched systems under asynchronous switching is derived in terms of a set of matrix inequalities. A numerical example is provided to illustrate the effectiveness of the proposed approach in Section 4. Concluding remarks are given in Section 5.

Notations. Throughout this paper, the superscript "T" denotes the transpose, and the notation $X \geq Y$ $(X > Y)$ means that the matrix $X - Y$ is positive semidefinite (positive definite, resp.). $\| \cdot \|$ denotes the Euclidean norm. I represents the identity matrix with an appropriate dimension. I_h is the identity matrix with n_1 dimension and I_v is the identity matrix with n_2 dimension. diag$\{a_i\}$ denotes the diagonal matrix with the diagonal elements a_i, and $i = 1, 2, \ldots, n$. X^{-1} denotes the inverse of X. The asterisk $*$ in a matrix is used to denote the term that is induced by symmetry. The set of all nonnegative integers is represented by Z_+.

2. Problem Formulation and Preliminaries

Consider the following uncertain 2D discrete switched systems with state delays:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix}$$

$$= \widehat{A}^{\sigma(i,j)} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \widehat{A}_d^{\sigma(i,j)} \begin{bmatrix} x^h(i - d_h, j) \\ x^v(i, j - d_v) \end{bmatrix} \quad (1)$$

$$+ \widehat{B}^{\sigma(i,j)} u(i, j),$$

where $x^h(i, j) \in R^{n_1}$ and $x^v(i, j) \in R^{n_2}$ are the horizontal state and the vertical state, respectively, $x(i, j)$ is the whole state in

R^n with $n = n_1 + n_2$, and $u(i, j) \in R^m$ is the control input. i and j are integers in Z_+. $\sigma(i, j) : Z_+ \times Z_+ \rightarrow \underline{N} = \{1, 2, \ldots, N\}$ is the switching signal. N is the number of subsystems, and $\sigma(i, j) = k$ means that the kth subsystem is activated. d_h and d_v are constant delays along horizontal and vertical directions, respectively. $\widehat{A}^k$ and $\widehat{A}_d^k$ $(k \in \underline{N})$ are uncertain real-valued matrices with appropriate dimensions and are assumed to be of the form

$$\widehat{A}^k = A^k + H^k F^k(i, j) E_1^k,$$

$$\widehat{A}_d^k = A_d^k + H^k F^k(i, j) E_2^k, \quad (2)$$

$$\widehat{B}^k = B^k + H^k F^k(i, j) E_3^k,$$

with

$$A^k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \qquad A_d^k = \begin{bmatrix} A_{d11}^k & A_{d12}^k \\ A_{d21}^k & A_{d22}^k \end{bmatrix},$$

$$H^k = \begin{bmatrix} H_1^k \\ H_2^k \end{bmatrix}, \qquad E_1^k = \begin{bmatrix} E_{11}^k \\ E_{12}^k \end{bmatrix}, \quad (3)$$

$$E_2^k = \begin{bmatrix} E_{21}^k \\ E_{22}^k \end{bmatrix}, \qquad E_3^k = \begin{bmatrix} E_{31}^k \\ E_{32}^k \end{bmatrix},$$

where matrices $A_{11}^k \in R^{n_1 \times n_1}$, $A_{12}^k \in R^{n_1 \times n_2}$, $A_{21}^k \in R^{n_2 \times n_1}$, $A_{22}^k \in R^{n_2 \times n_2}$, $A_{d11}^k \in R^{n_1 \times n_1}$, $A_{d12}^k \in R^{n_1 \times n_2}$, $A_{d21}^k \in R^{n_2 \times n_1}$, $A_{d22}^k \in R^{n_2 \times n_2}$, $H_1^k, H_2^k, E_{11}^k, E_{12}^k, E_{21}^k, E_{22}^k, E_{31}^k, E_{32}^k$ are constant matrices. $F^k(i, j)$ $(k \in \underline{N})$ is an unknown matrix representing parameter uncertainty and satisfies

$$F^{kT}(i, j) F^k(i, j) \leq I. \quad (4)$$

The boundary conditions are given by

$$x^h(i, j) = h_{ij}, \quad \forall 0 \leq j \leq z_1, \ -d_h \leq i \leq 0,$$

$$x^h(i, j) = 0, \quad \forall j > z_1, \ -d_h \leq i \leq 0,$$

$$x^v(i, j) = v_{ij}, \quad \forall 0 \leq i \leq z_2, \ -d_v \leq j \leq 0, \quad (5)$$

$$x^v(i, j) = 0, \quad \forall i > z_2, \ -d_v \leq j \leq 0,$$

where $z_1 < \infty$ and $z_2 < \infty$ are positive integers, and h_{ij} and v_{ij} are given vectors.

In this paper, it is assumed that (1) at each time only one subsystem is active; (2) the switching signal is not known a priori, but its value is available at each sampling period; (3) the switch occurs only at each sampling point of i or j. The switching sequence can be described as follows:

$$((i_0, j_0), \sigma(i_0, j_0)), ((i_1, j_1), \sigma(i_1, j_1)), \ldots,$$

$$((i_\kappa, j_\kappa), \sigma(i_\kappa, j_\kappa)), \ldots, \quad (6)$$

where (i_κ, j_κ) denotes the κth switching instant. It should be noted that the value of $\sigma(i, j)$ only depends on $i + j$ (see [29, 30]).

However, in actual operation, there inevitably exists asynchronous switching between the controller and the system. Without loss of generality, we only consider the case where the switching instants of the controller experience delays with respect to those of the system. Let $\sigma'(i,j)$ denote the switching signal of the controller. Denoting $m_\kappa = i_\kappa + j_\kappa$, $\Delta m_\kappa = \Delta i_\kappa + \Delta j_\kappa$, $\kappa = 1,2,\ldots$, then the switching points of the controller can be described as

$$(i_0, j_0), (i_1 + \Delta i_1, j_1 + \Delta j_1), \ldots, (i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa), \ldots, \tag{7}$$

where Δi_κ and Δj_κ represent the delayed period along horizontal and vertical directions, respectively. $\Delta m_\kappa < \inf(m_{\kappa+1} - m_\kappa)$ is said to be the mismatched period.

Remark 1. Similar to the one-dimensional switched system case [33–36], the mismatched period $\Delta m_\kappa < \inf(m_{\kappa+1} - m_\kappa)$ guarantees that there always exists a period in which the controller and the system operate synchronously. This period is said to be the matched period in the later section.

Remark 2. If there is only one subsystem in system (1), it will degenerate to the following 2D system in Roesser model with state delays [12]:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \widehat{A} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \widehat{A}_d \begin{bmatrix} x^h(i - d_h, j) \\ x^v(i, j - d_v) \end{bmatrix}, \tag{8}$$

Definition 3. System (1) is said to be exponentially stable under $\sigma(i,j)$ if, for a given $z \geq 0$, there exist positive constants c and η, such that

$$\sum_{i+j=D} \|x(i,j)\|^2 \leq \eta e^{-c(D-z)} \sum_{i+j=z} \|x(i,j)\|_C^2, \tag{9}$$

holds for all $D \geq z$, where

$$\sum_{i+j=z} \|x(i,j)\|_C^2$$
$$\triangleq \sup_{\substack{-d_h \leq \theta_h \leq 0, \\ -d_v \leq \theta_v \leq 0}} \sum_{i+j=z} \left(\|x^h(i - \theta_h, j)\|^2 + \|x^v(i, j - \theta_v)\|^2 \right). \tag{10}$$

Remark 4. From Definition 3, it is easy to see that when z is given, $\sum_{i+j=z} \|x(i,j)\|_C^2$ will be bounded and $\sum_{i+j=D} \|x(i,j)\|^2$ will tend to be zero exponentially as D goes to infinity, which implies that $\|x(i,j)\|$ tends to be zero.

Definition 5. Let (i_κ, j_κ) denote the κth switching point and $(i_{\kappa+1}, j_{\kappa+1})$ denote the $(\kappa+1)$th switching point. Denote $m_\kappa = i_\kappa + j_\kappa$, $m_{\kappa+1} = i_{\kappa+1} + j_{\kappa+1}$, $\tau = \inf(m_{\kappa+1} - m_\kappa)$, then τ is called the dwell time.

Definition 6 (see [30]). For any $i + j = D \geq z = i_z + j_z$, let $N_{\sigma(i,j)}(z, D)$ denote the switching number of $\sigma(i,j)$ on the interval $[z, D]$. If

$$N_{\sigma(i,j)}(z, D) \leq N_0 + \frac{D - z}{\tau_a} \tag{11}$$

holds for given $N_0 \geq 0$ and $\tau_a \geq 0$, then the constant τ_a is called the average dwell time and N_0 is the chatter bound.

Lemma 7 (see [37]). *For a given matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, where S_{11} and S_{22} are square matrices, the following conditions are equivalent:*

(i) $S < 0$,

(ii) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(iii) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 8 (see [38]). *Let U, V, W, and X be real matrices of appropriate dimensions with X satisfying $X = X^T$, then for all $V^T V \leq I$,*

$$X + UVW + W^T V^T U^T < 0, \tag{12}$$

if and only if there exists a scalar ε such that

$$X + \varepsilon UU^T + \varepsilon^{-1} W^T W < 0. \tag{13}$$

3. Main results

3.1. Stability Analysis. In this section, we first focus on the stability analysis for system (8).

Lemma 9. *Consider system (8) with the boundary conditions (5), suppose that there exists a C^1 function $V : R^n \to R$. For a given positive constant α, if there exist positive definite symmetric matrices $P = \mathrm{diag}\{P_h, P_v\}$ and $Q = \mathrm{diag}\{Q_h, Q_v\}$ with appropriate dimensions, such that the following inequality holds:*

$$\begin{bmatrix} Q - \alpha P & 0 & \widehat{A}^T P \\ * & -\Lambda_1 Q & \widehat{A}_d^T P \\ * & * & -P \end{bmatrix} < 0, \tag{14}$$

where $\Lambda_1 = \mathrm{diag}\{\alpha^{d_h} I_h, \alpha^{d_v} I_v\}$, then along the trajectory of systems (8), the following inequality holds for any $D \geq D'$:

$$\sum_{i+j=D} V(x(i,j)) < \alpha^{D-D'} \sum_{i+j=D'} V(x(i,j)), \tag{15}$$

where $D' \geq z$ and $z = \max\{z_1, z_2\}$.

Proof. See appendix for the detailed proof, it is omitted here. □

Remark 10. Lemma 9 provides a method for the estimation of the C^1 function V which will be used to design the controller for system (1) under asynchronous switching. It is worth pointing out that when $0 < \alpha < 1$, (15) presents the decay estimation of the C^1 function V, and when $\alpha > 1$, (15) shows the growth estimation of the C^1 function V.

Remark 11. It is noted that the block diagonal matrices P and Q are often chosen as the matrices for Lyapunov functional analysis of 2D systems by the Roesser model in the existing literature (see, e.g., [12, 13, 15]). This is because 2D systems in Roesser model may be unstable when the block diagonal matrices are not chosen, which has been shown in the literature [4, 5].

3.2. Controller Design. Consider system (1), under the following asynchronous switching controller:

$$u(i,j) = K^{\sigma'(i,j)}x(i,j), \qquad K^{\sigma'(i,j)} = \begin{bmatrix} K_1^{\sigma'(i,j)} & K_2^{\sigma'(i,j)} \end{bmatrix}, \tag{16}$$

where $K_1^{\sigma'(i,j)} \in R^{m \times n_1}$ and $K_2^{\sigma'(i,j)} \in R^{m \times n_2}$, the corresponding closed-loop system is given by

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}$$

$$= \left(\widehat{A}^{\sigma(i,j)} + \widehat{B}^{\sigma(i,j)} K^{\sigma'(i,j)} \right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{17}$$

$$+ \widehat{A}_d^{\sigma(i,j)} \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix}.$$

Without loss of generality, we denote $\sigma(i_\kappa, j_\kappa) = k \in \underline{N}$ and $\sigma(i_{\kappa+1}, j_{\kappa+1}) = l \in \underline{N}$, then due to the existence of asynchronous switching, we can obtain from (7) that

$$\sigma'(i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa) = k,$$
$$\sigma'(i_{\kappa+1} + \Delta i_{\kappa+1}, j_{\kappa+1} + \Delta j_{\kappa+1}) = l. \tag{18}$$

In many actual applications, it is always difficult to verify each asynchronous period in advance, but the maximal asynchronous period can be easily predicted offline. Let $\Delta = \max_{\kappa=1,2,\dots}(\Delta m_\kappa)$ denote the maximal asynchronous period, then we can get the following result.

Theorem 12. *Consider system* (1), *for given positive constants* $\alpha < 1$ *and* $\beta > 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$, $Y^k = \mathrm{diag}\{Y_h^k, Y_v^k\}$, $X^{kl} = \mathrm{diag}\{X_h^{kl}, X_v^{kl}\}$, $Y^{kl} = \mathrm{diag}\{Y_h^{kl}, Y_v^{kl}\}$ *and* W^k *with appropriate dimensions, and positive scalars* ε_k *and* ε_{kl} *such that, for* $k,l \in \underline{N}$, $k \neq l$, *the following inequalities hold,*

$$\begin{bmatrix} -\alpha X^k & 0 & \left(A^k X^k + B^k W^k\right)^T & X^k & \left(E_1^k X^k + E_3^k W^k\right)^T \\ * & -\Lambda_1 Y^k & \left(A_d^k Y^k\right)^T & 0 & \left(E_2^k Y^k\right)^T \\ * & * & -X^k + \varepsilon_k H^k H^{kT} & 0 & 0 \\ * & * & * & -Y^k & 0 \\ * & * & * & * & -\varepsilon_k I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} -\beta X^{kl} & 0 & \left(A^l X^{kl} + B^l K^k X^{kl}\right)^T & X^{kl} & \left(E_1^l X^{kl} + E_3^l K^k X^{kl}\right)^T \\ * & -\Lambda_2 Y^{kl} & \left(A_d^l Y^{kl}\right)^T & 0 & \left(E_2^l Y^{kl}\right)^T \\ * & * & -X^{kl} + \varepsilon_{kl} H^l H^{lT} & 0 & 0 \\ * & * & * & -Y^{kl} & 0 \\ * & * & * & * & -\varepsilon_{kl} I \end{bmatrix} < 0, \tag{20}$$

where $\Lambda_2 = \mathrm{diag}\{\beta^{d_h} I_h, \beta^{d_v} I_v\}$, *then under the following switching controller:*

$$u(i,j) = K^{\sigma'(i,j)}x(i,j), \qquad K^k = W^k \left(X^k\right)^{-1}, \tag{21}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\Delta(\ln\beta - \ln\alpha) + \ln(\mu_1\mu_2)}{-\ln\alpha}, \tag{22}$$

the resulting closed-loop system (17) *is exponentially stable, where* $\mu = (\alpha/\beta)^{\bar{d}}$, $\bar{d} = \max\{d_h, d_v\}$, *and* $\mu_1\mu_2\mu \geq 1$ *satisfies*

$$X_l^{-1} \leq \mu_1 X_{kl}^{-1}, \qquad X_{kl}^{-1} \leq \mu_2 X_k^{-1},$$
$$Y_l^{-1} \leq \mu_1 Y_{kl}^{-1}, \qquad Y_{kl}^{-1} \leq \mu_2 \mu Y_k^{-1}. \tag{23}$$

Proof. See the appendix. □

Remark 13. In Theorem 12, we propose a sufficient condition for the existence of a state feedback controller such that the resulting closed-loop system (17) is exponentially stable. It is worth noting that this condition is obtained by using the average dwell time approach. Here, α plays a key role in

controlling the rate of decaying of the system in the matched period, and β plays a key role in controlling the rate of increasing of the system in the mismatched period. From (22), we know that the value of τ_a^* is proportional to the value of Δ, which also means that it needs much larger dwell time to guarantee the stability of the considered system when there exists much larger asynchronous period.

Remark 14. It is noticed that (19) and (20) are mutually dependent. Therefore, we can firstly solve (19) to obtain the solution of matrices X^k, Y^k, and W^k. Then, (20) can be transformed into the LMI by substituting $K^k = W^k(X^k)^{-1}$ into it. By adjusting the parameters α and β, we can find a feasible solution of X^k, Y^k, W^k, X^{kl}, and Y^{kl} such that (19) and (20) hold.

The procedure of the controller design for system (1) can be given as follows.

Step 1. Given system matrices and positive scalar $0 < \alpha < 1$, and by solving LMI (19), we can get the feasible solution of matrices X^k, Y^k, and W^k and positive scalar ε_k then, the controller can be obtained from (21).

Step 2. Substituting K^k into (20), we can find the feasible solution of X^{kl}, Y^{kl}, and ε_{kl} such that (20) holds by adjusting the parameter β.

Step 3. From (23), we can obtain μ_1 and μ_2 satisfying $\mu_1 \mu_2 \mu \geq 1$.

Step 4. Taking the value of Δ, we can compute the value of τ_a^* by (22).

Remark 15. From the procedure above, it can be seen that the proposed method is feasible. We can find the desire controller and switching signal according to the procedure. However, we would like to point out that there still exists the conservatism to some extent for this method because (19) and (20) are mutually dependent, which brings about the increase of the complex computation. The result can be improved by adopting the method presented in [31, 32].

When $\widehat{A}_d^{\sigma(i,j)} = 0$, system (17) degenerates to the following delay-free system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(\widehat{A}^{\sigma(i,j)} + \widehat{B}^{\sigma(i,j)} K^{\sigma'(i,j)} \right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{24}$$

Then, we can get the following result.

Corollary 16. *Consider system* (1) *with* $\widehat{A}_d^{\sigma(i,j)} = 0$, *for given positive constants* $\alpha < 1$ *and* $\beta > 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$ *and* $X^{kl} = \mathrm{diag}\{X_h^{kl}, X_v^{kl}\}$ *with appropriate dimensions, and positive scalars* ε_k *and* ε_{kl} *such that, for* $k, l \in \underline{N}$, $k \neq l$, *the following inequalities hold:*

$$\begin{bmatrix} -\alpha X^k & \left(A^k X^k + B^k W^k \right)^T & \left(E_1^k X^k + E_3^k W^k \right)^T \\ * & -X^k + \varepsilon_k H^k H^{kT} & 0 \\ * & * & -\varepsilon_k I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\beta X^{kl} & \left(A^l X^{kl} + B^l K^k X^{kl} \right)^T & \left(E_1^l X^{kl} + E_3^l K^k X^{kl} \right)^T \\ * & -X^{kl} + \varepsilon_{kl} H^l H^{lT} & 0 \\ * & * & -\varepsilon_{kl} I \end{bmatrix} < 0, \tag{25}$$

then under the following switching controller:

$$u(i,j) = K^{\sigma'(i,j)} x(i,j), \qquad K^k = W^k (X^k)^{-1}, \tag{26}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\Delta(\ln \beta - \ln \alpha) + \ln(\mu_1 \mu_2)}{-\ln \alpha}, \tag{27}$$

the resulting closed-loop system (24) *is exponentially stable, where* $\mu_1 \mu_2 \geq 1$ *satisfies*

$$X_l^{-1} \leq \mu_1 X_{kl}^{-1}, \qquad X_{kl}^{-1} \leq \mu_2 X_k^{-1}. \tag{28}$$

Furthermore, it should also be noted that if the criterion in Theorem 12 is satisfied when $\Delta = 0$, which means that the controller and the subsystem are synchronous, in other words, the results presented in Theorem 12 can be reduced to the synchronous case, then we can obtain the following corollary.

Corollary 17. *Consider system* (1) *under synchronous switching, for a given positive scalar* $\alpha < 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$, $Y^k = \mathrm{diag}\{Y_h^k, Y_v^k\}$, *and* W^k, *with appropriate dimensions, and a positive scalar* ε_k, *such that, for* $k \in \underline{N}$, *the following inequality holds:*

$$\begin{bmatrix} -\alpha X^k & 0 & \left(A^k X^k + B^k W^k \right)^T & X^k & \left(E_1^k X^k + E_3^k W^k \right)^T \\ * & -\Lambda_1 Y^k & \left(A_d^k Y^k \right)^T & 0 & \left(E_2^k Y^k \right)^T \\ * & * & -X^k + \varepsilon_k H^k H^{kT} & 0 & 0 \\ * & * & * & -Y^k & 0 \\ * & * & * & * & -\varepsilon_k I \end{bmatrix} < 0, \tag{29}$$

then under the following controller:

$$u(i,j) = K^k x(i,j), \qquad K^k = W^k (X^k)^{-1}, \tag{30}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\ln \mu_1}{-\ln \alpha}, \tag{31}$$

the resulting closed-loop system is exponentially stable, where $\mu_1 \geq 1$ *satisfies*

$$X_l^{-1} \leq \mu_1 X_k^{-1}, \qquad Y_l^{-1} \leq \mu_1 Y_k^{-1}. \tag{32}$$

Remark 18. In [30], by using the average dwell time approach, a criterion of exponential stability for a class of 2D discrete delay-free switched systems is developed. However, the focus of our work is on stability analysis and controller design under asynchronous switching, which is different from [30], and this is also the major contribution of our work. In fact, if we let $\Delta = 0$ and do not consider the uncertainties and state delays, then the closed-loop system (24) is the same as (36) in [30]. In this case, Corollary 17 can be reduced to Theorem 2 in [30].

4. Numerical Example

In this section, we present an example to illustrate the effectiveness of the proposed approach.

Consider system (1) with the following parameters:

$$A^1 = \begin{bmatrix} 1 & 1.5 \\ 1 & 0.5 \end{bmatrix}, \qquad A_d^1 = \begin{bmatrix} -0.15 & 0 \\ -0.1 & -0.12 \end{bmatrix}, \tag{33}$$

$$B^1 = \begin{bmatrix} -4.5 & 0 \\ 1 & -3 \end{bmatrix}, \qquad H^1 = \begin{bmatrix} 0.2 & 0.15 \\ 0.1 & 0.2 \end{bmatrix}, \tag{34}$$

$$E_1^1 = \begin{bmatrix} -0.2 & 0 \\ -0.2 & -0.2 \end{bmatrix}, \qquad E_2^1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix},$$

$$F^1 = \text{diag}\left\{\sin\left(0.5\pi\left(i+j\right)\right), \sin\left(0.5\pi\left(i+j\right)\right)\right\},$$

$$E_3^1 = \begin{bmatrix} 0.15 & 0 \\ 0.13 & 0.12 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$A_d^2 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.2 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} -5 & 1 \\ -1 & -3 \end{bmatrix}, \qquad (35)$$

$$H^2 = \begin{bmatrix} 0.2 & 0.25 \\ 0.2 & 0.3 \end{bmatrix}, \qquad E_1^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix},$$

$$E_2^2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \qquad E_3^2 = \begin{bmatrix} 0.12 & 0.15 \\ 0.12 & 0.1 \end{bmatrix},$$

$$F^2 = \text{diag}\left\{\cos\left(0.5\pi\left(i+j\right)\right), \cos\left(0.5\pi\left(i+j\right)\right)\right\},$$

$$d_h = 2, \qquad d_v = 3.$$

The boundary conditions are given as follows:

$$x^h(0, j) = \begin{cases} 5, & 0 \le j \le 20, \\ 0, & j > 20, \end{cases}$$

$$x^v(i, 0) = \begin{cases} 3, & 0 \le i \le 20, \\ 0, & i > 20, \end{cases} \qquad (36)$$

where the state dimensions are $n_1 = 1$ and $n_2 = 1$.

Take $\alpha = 0.6$ and $\beta = 1.2$, then solving (19) in Theorem 12 gives rise to

$$X^1 = \begin{bmatrix} 36.3903 & 1.0247 \\ 1.0247 & 38.3909 \end{bmatrix}, \qquad X^2 = \begin{bmatrix} 40.0833 & -6.6640 \\ -6.6640 & 34.4516 \end{bmatrix},$$

$$Y^1 = \begin{bmatrix} 96.6213 & -4.5896 \\ -4.5896 & 96.9472 \end{bmatrix}, \qquad Y^2 = \begin{bmatrix} 99.6213 & -8.8472 \\ -8.8472 & 79.5315 \end{bmatrix},$$

$$W^1 = \begin{bmatrix} 8.5866 & 13.0604 \\ 15.5568 & 11.4091 \end{bmatrix}, \qquad W^2 = \begin{bmatrix} 6.9844 & 13.2930 \\ 8.3790 & 4.4234 \end{bmatrix},$$

$$\varepsilon_1 = 74.2107, \qquad \varepsilon_2 = 73.4149.$$

$$(37)$$

By (21), K^1 and K^2 can be obtained as follows:

$$K^1 = \begin{bmatrix} 0.2265 & 0.3341 \\ 0.4194 & 0.2860 \end{bmatrix}, \qquad K^2 = \begin{bmatrix} 0.2463 & 0.4335 \\ 0.2380 & 0.1744 \end{bmatrix}.$$

$$(38)$$

Substituting K^1 and K^2 into (20), and solving it, we get the following solution:

$$X^{12} = \begin{bmatrix} 0.5400 & -0.0934 \\ -0.0934 & 0.4811 \end{bmatrix}, \qquad X^{21} = \begin{bmatrix} 0.5805 & -0.0926 \\ -0.0926 & 0.5677 \end{bmatrix},$$

$$Y^{12} = \begin{bmatrix} 0.9398 & -0.0418 \\ -0.0418 & 0.8451 \end{bmatrix}, \qquad Y^{21} = \begin{bmatrix} 0.9436 & -0.0340 \\ -0.0340 & 0.8867 \end{bmatrix},$$

$$\varepsilon_{12} = 0.9011, \qquad \varepsilon_{21} = 0.7725.$$

$$(39)$$

Then, from (23), we can get that $\mu_1 = 0.0171$ and $\mu_2 = 917.7224$. Taking $\Delta = 2$, it is easy to obtain from (22) that $\tau_a^* = 7.9$. Choosing $\tau_a = 8$, the trajectories of the states $x^h(i, j)$ and $x^v(i, j)$ are shown in Figures 1 and 2, respectively. The
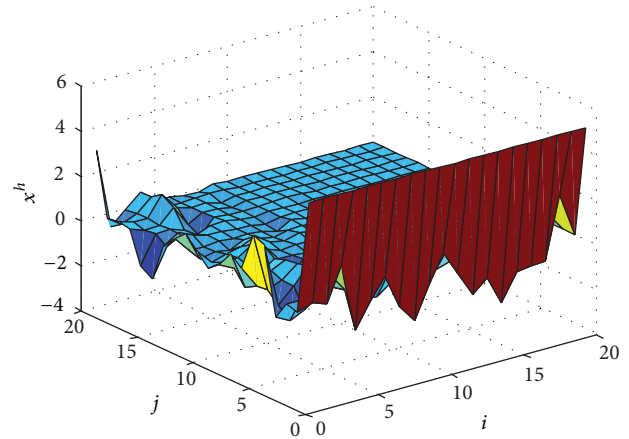

FIGURE 1: The trajectory of the state $x^h(i, j)$.

system switching signal $\sigma(i, j)$ and the controller switching signal $\sigma'(i, j)$ are shown in Figure 3. One can see that the states of the closed-loop system converge to zero under the asynchronous switching. This demonstrates the effectiveness of the proposed approach.

5. Conclusions

This paper has investigated the problem of stabilization for a class of 2D discrete switched systems with constant state delays under asynchronous switching. A state feedback controller is proposed to stabilize such system, and the dwell time approach is utilized for the stability analysis and controller design. A sufficient condition for the existence of such controller is formulated in terms of a set of LMIs. An example is also given to illustrate the applicability of the proposed approach. Our future work will focus on extending the proposed design method to other problems such as robust H_∞ control for 2D discrete switched systems with time-varying delays and fractional uncertainties under asynchronous switching.

Appendix

Proof of Lemma 9. Consider the following Lyapunov-Krasovskii functional candidate:

$$V(x(i, j)) = V^h\left(x^h(i, j)\right) + V^v\left(x^v(i, j)\right), \qquad (A.1)$$

where

$$\begin{aligned} &V^h\left(x^h(i, j)\right) \\ &= x^h(i, j)^T P_h x^h(i, j) \\ &\quad + \sum_{r=i-d_h}^{i-1} x^h(r, j)^T Q_h x^h(r, j)\, \alpha^{i-r-1}, \\ &V^v\left(x^v(i, j)\right) \\ &= x^v(i, j)^T P_v x^v(i, j) \\ &\quad + \sum_{t=j-d_v}^{j-1} x^v(i, t)^T Q_v x^v(i, t)\, \alpha^{j-t-1}. \end{aligned} \qquad (A.2)$$
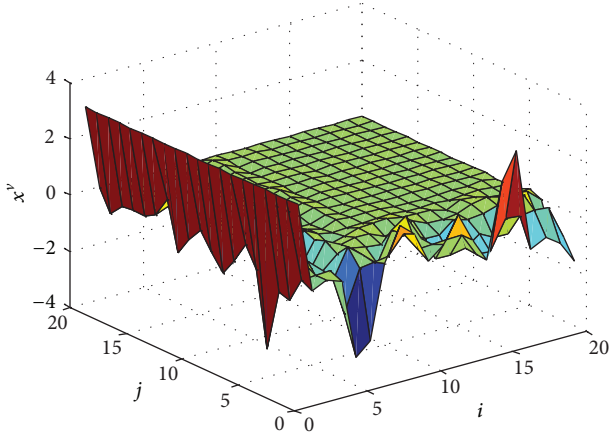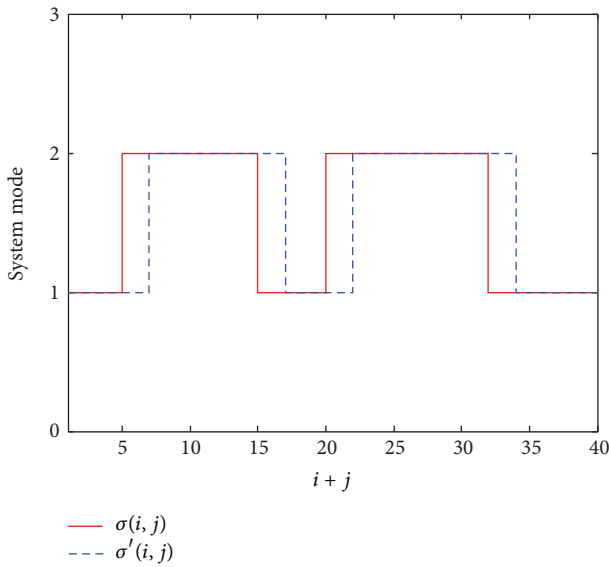
FIGURE 2: The trajectory of the state $x^v(i, j)$.

FIGURE 3: Switching signal.

Along the trajectory of system (8), we have

$$
V^h\left(x^h\left(i+1, j\right)\right) - \alpha V^h\left(x^h\left(i, j\right)\right)
$$

$$
= x^h(i+1, j)^T P_h x^h\left(i+1, j\right) - \alpha x^h(i, j)^T P_h x^h\left(i, j\right)
$$

$$
+ x^h(i, j)^T Q_h x^h\left(i, j\right)
$$

$$
- \alpha^{d_h} x^h(i-d_h, j)^T Q_h x^h\left(i-d_h, j\right),
$$

(A.3)

$$
V^v\left(x^v\left(i, j+1\right)\right) - \alpha V^v\left(x^v\left(i, j\right)\right)
$$

$$
= x^v(i, j+1)^T P_v x^v\left(i, j+1\right) - \alpha x^v(i, j)^T P_v x^v\left(i, j\right)
$$

$$
+ x^v(i, j)^T Q_v x^v\left(i, j\right)
$$

$$
- \alpha^{d_v} x^v(i, j-d_v)^T Q_v x^v\left(i, j-d_v\right).
$$

(A.4)

It follows that

$$
V^h\left(x^h\left(i+1, j\right)\right) - \alpha V^h\left(x^h\left(i, j\right)\right)
$$

$$
+ V^v\left(x^v\left(i, j+1\right)\right) - \alpha V^v\left(x^v\left(i, j\right)\right)
$$

$$
= \left[\begin{array}{c} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \\ x^h(i-d_h, j) \\ x^h(i, j-d_v) \end{array}\right]^T \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \left[\begin{array}{c} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \\ x^h(i-d_h, j) \\ x^h(i, j-d_v) \end{array}\right],
$$

(A.5)

where

$$
\Phi_{11} = Q - \alpha P + \widehat{A}^T P \widehat{A}, \qquad \Phi_{12} = \widehat{A}^T P \widehat{A}_d,
$$

$$
\Phi_{22} = \widehat{A}_d^T P \widehat{A}_d - \Lambda_1 Q, \qquad \Lambda_1 = \operatorname{diag}\left\{\alpha^{d_h} I_h, \alpha^{d_v} I_v\right\}.
$$

(A.6)

Applying Lemma 7, it can be obtained from (14) that

$$
\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0.
$$

(A.7)

For simplicity, we denote

$$
V^h(i, j) = V^h\left(x^h(i, j)\right), \qquad V^v(i, j) = V^v\left(x^v(i, j)\right),
$$

$$
V(i, j) = V\left(x(i, j)\right), \qquad V^h(i+1, j) = V^h\left(x(i+1, j)\right),
$$

$$
V^v(i, j+1) = V^v\left(x(i, j+1)\right).
$$

(A.8)

Thus, it is easy to get that

$$
V^h(i+1, j) + V^v(i, j+1) < \alpha\left(V^h(i, j) + V^v(i, j)\right).
$$

(A.9)

Notice that for any nonnegative integer $D > z = \max(z_1, z_2)$, it holds that $V^h(0, D) = V^v(D, 0) = 0$; then summing up both sides of (A.9) from $D - 1$ to 0 with respect to j and 0 to $D - 1$ with respect to i, for any nonnegative integer $D > D' \geq z = \max(z_1, z_2)$, one gets

$$
\sum_{i+j=D} V(i, j)
$$

$$
= V^h(0, D) + V^h(1, D-1) + V^h(2, D-2)
$$

$$
+ \cdots + V^h(D-1, 1) + V^h(D, 0)
$$

$$
+ V^v(0, D) + V^v(1, D-1) + V^v(2, D-2)
$$

$$
+ \cdots + V^v(D-1, 1) + V^v(D, 0)
$$

$$
< \alpha\left(V^h(0, D-1) + V^v(0, D-1)\right.
$$

$$
+ V^h(1, D-2) + V^v(1, D-2)
$$

$$
\left. + \cdots + V^h(D-1, 0) + V^v(D-1, 0)\right)
$$

$$
= \alpha \sum_{i+j=D-1} V(i, j) < \cdots < \alpha^{D-D'} \sum_{i+j=D'} V(i, j).
$$

(A.10)

The proof is completed. □

Proof of Theorem 12. When $D \in [m_\kappa + \Delta m_\kappa, m_{\kappa+1})$, the closed-loop system (17) can be written as

$$
\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}
= \left(\widehat{A}^k + \widehat{B}^k K^k\right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \widehat{A}_d^k \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix}.
$$

$$(A.11)$$

For the system, we consider the following Lyapunov function candidate:

$$
V_k\left(x(i,j)\right) = V_k^h\left(x^h(i,j)\right) + V_k^v\left(x^v(i,j)\right),
$$

$$(A.12)$$

where

$$
V_k^h\left(x^h(i,j)\right)
$$
$$
= x^h(i,j)^T P_h^k x^h(i,j) + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^k x^h(r,j)\, \alpha^{i-r-1},
$$
$$
V_k^v\left(x^v(i,j)\right)
$$
$$
= x^v(i,j)^T P_v^k x^v(i,j) + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^k x^v(i,t)\, \alpha^{j-t-1}.
$$

$$(A.13)$$

By Lemma 9, one gets that if there exist positive definite symmetric matrices $P^k = \mathrm{diag}\{P_h^k, P_v^k\}$ and $Q^k = \mathrm{diag}\{Q_h^k, Q_v^k\}$ with appropriate dimensions, such that

$$
\begin{bmatrix} Q^k - \alpha P^k & 0 & \left(\widehat{A}^k + \widehat{B}^k K^k\right)^T P^k \\ * & -\Lambda_1 Q^k & \widehat{A}_d^{kT} P^k \\ * & * & -P^k \end{bmatrix} < 0
$$

$$(A.14)$$

holds, then the following inequality holds for any $D \geq m_\kappa + \Delta m_\kappa \geq z$:

$$
\sum_{i+j=D} V_k(i,j) < \alpha^{D-m_k-\Delta m_k} \sum_{i+j=m_\kappa+\Delta m_\kappa} V_k(i,j).
$$

$$(A.15)$$

When $D \in [m_{\kappa+1}, m_{\kappa+1} + \Delta m_{\kappa+1})$, the closed-loop system (17) can be written as

$$
\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}
= \left(\widehat{A}^l + \widehat{B}^l K^k\right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \widehat{A}_d^l \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix}.
$$

$$(A.16)$$

Consider the following Lyapunov function candidate:

$$
V_{kl}\left(x(i,j)\right) = V_{kl}^h\left(x^h(i,j)\right) + V_{kl}^v\left(x^v(i,j)\right),
$$

$$(A.17)$$

where

$$
V_{kl}^h\left(x^h(i,j)\right)
$$
$$
= x^h(i,j)^T P_h^{kl} x^h(i,j)
$$
$$
+ \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{kl} x^h(r,j)\, \alpha^{i-r-1},
$$
$$
V_{kl}^v\left(x^v(i,j)\right)
$$
$$
= x^v(i,j)^T P_v^{kl} x^v(i,j)
$$
$$
+ \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{kl} x^v(i,t)\, \alpha^{j-t-1}.
$$

$$(A.18)$$

Similarly, by Lemma 9, we get that if there exist positive definite symmetric matrices $P^{kl} = \mathrm{diag}\{P_h^{kl}, P_v^{kl}\}$ and $Q^{kl} = \mathrm{diag}\{Q_h^{kl}, Q_v^{kl}\}$ with appropriate dimensions, such that

$$
\begin{bmatrix} Q^{kl} - \beta P^{kl} & 0 & \left(\widehat{A}^l + \widehat{B}^l K^k\right)^T P^{kl} \\ * & -\Lambda_2 Q^{kl} & \widehat{A}_d^{lT} P^{kl} \\ * & * & -P^{kl} \end{bmatrix} < 0
$$

$$(A.19)$$

holds, then the following inequality holds for any $D \geq m_{\kappa+1} \geq z$:

$$
\sum_{i+j=D} V_{kl}(i,j) < \beta^{D-m_{\kappa+1}} \sum_{i+j=m_{\kappa+1}} V_{kl}(i,j).
$$

$$(A.20)$$

Consider the following piecewise Lyapunov functional candidate for system (17):

$$
V(i,j) = \begin{cases}
\begin{aligned}
& x^h(i,j)^T P_h^{\sigma(i,j)} x^h(i,j) \\
& + x^v(i,j)^T P_v^{\sigma(i,j)} x^v(i,j) \\
& + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{\sigma(i,j)} x^h(r,j)\, \alpha^{i-r-1} \\
& + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{\sigma(i,j)} x^v(i,t)\, \alpha^{j-t-1}, \\
& \quad D \in [m_0, m_1) \cup [m_\pi + \Delta m_\pi, m_{\pi+1}), \\
& \qquad\qquad\qquad \pi = 1,2,\ldots,\kappa\ldots, \\
& x^h(i,j)^T P_h^{\sigma'(i,j)\sigma(i,j)} x^h(i,j) \\
& + x^v(i,j)^T P_v^{\sigma'(i,j)\sigma(i,j)} x^v(i,j) \\
& + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{\sigma'(i,j)\sigma(i,j)} x^h(r,j)\, \alpha^{i-r-1} \\
& + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{\sigma'(i,j)\sigma(i,j)} x^v(i,t)\, \alpha^{j-t-1}, \\
& \quad D \in [m_\pi, m_\pi + \Delta m_\pi), \quad \pi = 1,2,\ldots,\kappa\ldots.
\end{aligned}
\end{cases}
$$

$$(A.21)$$

Now, let $v = N_{\sigma(i,j)}(z, D)$ denote the switch number of $\sigma(i, j)$ on the interval $[z, D)$, and let $(i_{\kappa-v+1}, j_{\kappa-v+1})$, $(i_{\kappa-v+2}, j_{\kappa-v+2}), \ldots, (i_\kappa, j_\kappa)$ denote the switching points of $\sigma(i, j)$ over the interval $[z, D)$; then, the switching points of $\sigma'(i, j)$ can be denoted as follows:

$$(i_{\kappa-v+1} + \Delta i_{\kappa-v+1}, j_{\kappa-v+1} + \Delta j_{\kappa-v+1}),$$

$$(i_{\kappa-v+2} + \Delta i_{\kappa-v+2}, j_{\kappa-v+2} + \Delta j_{\kappa-v+2}), \ldots, (i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa). \tag{A.22}$$

Denoting $m_g = i_g + j_g$, $g = \kappa - v + 1, \ldots, \kappa$, then we can get from (23) and (A.21) that

$$\sum_{i+j=m_g+\Delta m_g} V(i,j) \le \mu_1 \sum_{i+j=(m_g+\Delta m_g)^-} V(i,j),$$

$$\sum_{i+j=m_g} V(i,j) \le \mu_2 \sum_{i+j=(m_g)^-} V(i,j), \tag{A.23}$$

where $(m_g)^-$ and $(m_g + \Delta m_g)^-$ satisfy the following conditions:

$$0 < m_g - (m_g)^- < \lambda,$$

$$0 < m_g + \Delta m_g - (m_g + \Delta m_g)^- < \lambda, \tag{A.24}$$

where λ is a sufficient small positive constant.

When $m_{\kappa-v+1} > z \ge m_{\kappa-v} + \Delta m_{\kappa-v}$, we have, for $D > m_\kappa + \Delta m_\kappa$,

$$\sum_{i+j=D} V(i,j) < \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=m_\kappa+\Delta m_\kappa} V(i,j) \le \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=(m_\kappa+\Delta m_\kappa)^-} V(i,j)$$

$$< \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=m_\kappa} V(i,j) \le \mu_1 \mu_2 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=(m_\kappa)^-} V(i,j) < \cdots$$

$$< (\mu_1 \mu_2)^v \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-v+2}-m_{\kappa-v+1}-\Delta m_{\kappa-v+1}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-v+1}} \sum_{i+j=(m_{k-v+1})^-} V(i,j) \tag{A.25}$$

$$< (\mu_1 \mu_2)^v \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-v+2}-m_{\kappa-v+1}-\Delta m_{\kappa-v+1}+m_{\kappa-v+1}-z} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-v+1}} \sum_{i+j=z} V(i,j)$$

$$\le (\mu_1 \mu_2)^v \alpha^{D-z-v\Delta} \beta^{v\Delta} \sum_{i+j=z} V(i,j) = \alpha^{D-z} e^{v \ln(\mu_1 \mu_2)+v\Delta(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i,j).$$

When $m_{\kappa-v} < z < m_{\kappa-v} + \Delta m_{\kappa-v}$, we can also get that

$$\sum_{i+j=D} V(i,j) < \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=m_\kappa+\Delta m_\kappa} V(i,j) < \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=(m_\kappa+\Delta m_\kappa)^-} V(i,j)$$

$$< \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=m_\kappa} V(i,j) < \mu_1 \mu_2 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=(m_\kappa)^-} V(i,j) < \cdots$$

$$< (\mu_1 \mu_2)^v \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-v+2}-m_{\kappa-v+1}-\Delta m_{\kappa-v+1}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-v+1}} \sum_{i+j=(m_{k-v+1})^-} V(i,j)$$

$$< (\mu_1 \mu_2)^v \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-v+2}-m_{\kappa-v+1}-\Delta m_{\kappa-v+1}+m_{\kappa-v+1}-m_{\kappa-v}-\Delta m_{\kappa-v}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-v+1}+\Delta m_{\kappa-v}+m_{\kappa-v}-z} \sum_{i+j=z} V(i,j)$$

$$\le \mu_1 (\mu_1 \mu_2)^v \alpha^{D-z-v\Delta} \beta^{v\Delta} \sum_{i+j=z} V(i,j) = \mu_1 \alpha^{D-z} e^{v \ln(\mu_1 \mu_2)+v\Delta(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i,j).$$

$$\tag{A.26}$$

From (A.25) and (A.26), we can obtain

$$\sum_{i+j=D} V(i,j)$$

$$< \max\{\mu_1, 1\} \alpha^{D-z} e^{v \ln(\mu_1 \mu_2)+\Delta v(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i,j). \tag{A.27}$$

According to Definition 6, one has

$$v = N_{\sigma(i,j)}(z, D) \le N_0 + \frac{D-z}{\tau_a}. \tag{A.28}$$

From condition (22), the following inequality can be obtained:

$$-\ln\alpha > \frac{\Delta(\ln\beta - \ln\alpha) + \ln(\mu_1 \mu_2)}{\tau_a}. \tag{A.29}$$

Thus, it is easy to get the following inequality:

$$\sum_{i+j=D} V(i,j)$$

$$< \max\{\mu_1, 1\} (\mu_1\mu_2)^{N_0}$$

$$\times \left(\frac{\beta}{\alpha}\right)^{N_0\Delta} e^{((\ln(\mu_1\mu_2)+\Delta(\ln\beta-\ln\alpha))/\tau_a+\ln\alpha)(D-z)} \sum_{i+j=z} V(i,j). \tag{A.30}$$

It follows that

$$\sum_{i+j=D} \|x(i,j)\|^2$$

$$< \frac{\zeta_1}{\zeta_2} \max\{\mu_1, 1\} (\mu_1\mu_2)^{N_0}$$

$$\times \left(\frac{\beta}{\alpha}\right)^{N_0\Delta} e^{((\ln(\mu_1\mu_2)+\Delta(\ln\beta-\ln\alpha))/\tau_a+\ln\alpha)(D-z)} \sum_{i+j=z} \|x(i,j)\|^2, \tag{A.31}$$

where

$$\zeta_1 = \max_{k,l\in\underline{N},k\neq l} \{\lambda_{\max}(P^k) + \max(d_h, d_v)\lambda_{\max}(Q^k),$$

$$\lambda_{\max}(P^{kl}) + \max(d_h, d_v)\lambda_{\max}(Q^{kl})\},$$

$$\zeta_2 = \min_{k,l\in\underline{N},k\neq l} \{\lambda_{\min}(P^k), \lambda_{\min}(P^{kl})\},$$

$$\sum_{i+j=z} \|x(i,j)\|_C^2$$

$$\triangleq \sup_{\substack{-d_h\leq\theta_h\leq 0, \\ -d_v\leq\theta_v\leq 0}} \sum_{i+j=z} \{\|x^h(i-\theta_h,j)\|^2 + \|x^v(i,j-\theta_v)\|^2\}, \tag{A.32}$$

Thus, it can be obtained from (22) that the closed-loop system (17) is exponentially stable.

Denote $X^k = (P^k)^{-1}$ and $Y^k = (Q^k)^{-1}$, then it is easy to get $(X^k)^T = X^k$ and $(Y^k)^T = Y^k$. Using $\mathrm{diag}\{X^k, Y^k, X^k\}$ to pre- and postmultiply the left of (A.14), respectively, and applying Lemma 7, it follows that (A.33) and (A.14) are equivalent:

$$\begin{bmatrix} -\alpha X^k & 0 & (\widehat{A}^k X^k + \widehat{B}^k W^k)^T & X^k \\ * & -\Lambda_1 Y^k & (\widehat{A}_d^k Y^k)^T & 0 \\ * & * & -X^k & 0 \\ * & * & * & -Y^k \end{bmatrix} < 0, \tag{A.33}$$

where $W^k = K^k X^k$.

By substituting (2) into (A.33), we can get the following inequality

$$T = T_0 + T_1 < 0, \tag{A.34}$$

where

$$T_0 = \begin{bmatrix} -\alpha X^k & 0 & (A^k X^k + B^k W^k)^T & X^k \\ * & -\Lambda_1 Y^k & (A_d^k Y^k)^T & 0 \\ * & * & -X^k & 0 \\ * & * & * & -Y^k \end{bmatrix},$$

$$T_1 = \begin{bmatrix} (E_1^k X^k + E_3^k W^k)^T \\ (E_2^k Y^k)^T \\ 0 \\ 0 \end{bmatrix} F^{kT} \begin{bmatrix} 0 \\ 0 \\ H^k \\ 0 \end{bmatrix}^T \tag{A.35}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ H^k \\ 0 \end{bmatrix} F^k \begin{bmatrix} (E_1^k X^k + E_3^k W^k)^T \\ (E_2^k Y^k)^T \\ 0 \\ 0 \end{bmatrix}^T.$$

By Lemma 8, we get

$$T_0 + \varepsilon_k^{-1} \begin{bmatrix} (E_1^k X^k + E_3^k W^k)^T \\ (E_2^k Y^k)^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (E_1^k X^k + E_3^k W^k)^T \\ (E_2^k Y^k)^T \\ 0 \\ 0 \end{bmatrix}^T$$

$$+ \varepsilon_k \begin{bmatrix} 0 \\ 0 \\ H^k \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^k \\ 0 \end{bmatrix}^T < 0. \tag{A.36}$$

Applying Lemma 7 again, we obtain that (A.33) holds if (19) is satisfied.

Similarly, substitute (2) into (A.19) and denote $X^{kl} = (P^{kl})^{-1}$ and $Y^{kl} = (Q^{kl})^{-1}$, then using $\mathrm{diag}\{X^{kl}, Y^{kl}, X^{kl}\}$ to pre- and postmultiply the left of (A.19), respectively, and applying Lemmas 7 and 8, it is easy to get that (A.19) holds if (20) is satisfied.

The proof is completed. □

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