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Research Article

New Generalization of f-Best Simultaneous Approximation in Topological Vector Spaces

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Let K be a nonempty subset of a Hausdorff topological vector space X, and let f be a real-valued continuous function on X. If for each $x = (x_1, x_2, \dots, x_n) \in X^n$, there exists $k_0 \in K$ such that $F_K(x) = \sum_{i=1}^n f(x_i - k_0) = \inf\{\sum_{i=1}^n f(x_i - k) : k \in K\}$, then K is called f-simultaneously proximal and k_0 is called f-best simultaneous approximation for x in K. In this paper, we study the problem of f-simultaneous approximation for a vector subspace K in X. Some other results regarding f-simultaneous approximation in quotient space are presented.

1. Introduction

Let K be a closed subset of a Hausdorff topological vector space X and f a real-valued continuous function on X. For $x \in X$, set $F_K(x) = \inf_{k\in K} f(x - k)$. A point $k_0 \in K$ is called f-best approximation to x in K if $F_K(x) = f(x - k_0)$. The set $P_K^f(x) = \{k \in K : F_K(x) = f(x - \kappa)\}$ denotes the set of all f-best approximations to x in K. Note that this set may be empty. The set K is said to be f-proximal (f-Chebyshev) if for each $x \in X$, $P_K^f(x)$ is nonempty (singleton). The notion of f-best approximation in a vector space X was given by Breckner and Brosowski [1] and in a Hausdorff topological space X by Narang [2, 3]. For a Hausdorff locally convex topological vector space and a continuous sublinear functional f on X, certain results on best approximation relative to the functional f were proved in [1, 4]. By using the existence of elements of f-best approximation, certain results on fixed points were proved by Pai and Veermani in [5]. In addition, for a topological vector space X relative to upper semicontinuous functions, some results on best approximation were proved by Haddadi and Hamzenejad [6]. Moreover, Naidu [7] proved some results on best simultaneous approximation related to f-nearest point and topological vector space X.

Analogous to the problem of simultaneous approximation [8], we introduce the concept of best f-simultaneous approximation as follows.

Definition 1. Let K be a non-empty subset of a Hausdorff topological vector space X, and let f be a real-valued continuous function on X. A point $k_0 \in K$ is called f-best simultaneous approximation in K if there exists $x = (x_1, x_2, \dots, x_n) \in X^n$ such that

$$F_K(x) = \inf\left\{\sum_{i=1}^n f(x_i - k) : k \in K\right\} = \sum_{i=1}^n f(x_i - k_0). \quad (1)$$

The set of all f-best simultaneous approximations to $x = (x_1, x_2, \dots, x_n) \in X^n$ in K is denoted by

$$P_K^f(x) = \left\{k \in K : F_K(x) = \sum_{i=1}^n f(x_i - k)\right\}. \quad (2)$$

The set K is called f-simultaneously proximal (f-simultaneously Chebyshev) if for each $x = (x_1, x_2, \dots, x_n) \in X^n$, $P_K^f(x) \neq \phi$ (singleton). If $n = 1$, simultaneous f-proximal is precisely f-proximal.

We remark that if $f(x) = \|x\|$, then the concept of f-best approximation is precisely the best approximation.

A set K is said to be inf-compact at a point $x = (x_1, x_2, \ldots, x_n) \in X^n$ [5] if each minimizing sequence in K (i.e., $\sum_{i=1}^{n} f(x_i - k_n) \to F_K(x)$) has a convergent subsequence in K. The set K is called inf-compact if it is inf-compact at each $x = (x_1, x_2, \ldots, x_n) \in X^n$.

It is easy to see that if K is compact or inf-compact, then K is f-simultaneously proximal.

In this paper, we introduce the concept of f-simultaneous approximation and study the existence and uniqueness problem of f-simultaneous approximation of a subspace K of a Hausdorff topological vector space X. Certain results regarding f-simultaneous approximation in quotient spaces are obtained by generalizing some of the results in [9].

Throughout this paper, X is a Hausdorff topological vector space and f is a real-valued continuous function on X.

2. f-Simultaneous Approximation

In this section, we give some characterizations of f-proximal sets in X. We begin with the following definitions.

Definition 2. A function $f : X \to \mathbb{R}$ is called absolutely homogeneous if $f(\alpha x) = |\alpha| f(x)$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

Definition 3. A subset K of X is called f-closed if for all sequences $\{k_m\}$ of K and for all $x = (x_1, x_2, \ldots, x_n) \in X^n$, such that $\sum_{i=1}^{n} f(x_i - k_m) \to 0$, we have $x \in K^n$.

Definition 4. A subset K of X is called f-compact if for every sequence $\{k_n\}$ in K there exist a subsequence $\{k_{n_k}\}$ of $\{k_n\}$ and $k_0 \in K$ such that $f(k_{n_k} - k_0) \to 0$.

Definition 5. For $x, y \in X$, where $x = (x_1, x_2, \ldots, x_n) \in X^n$ and $y = (y_1, y_2, \ldots, y_n) \in X^n$, x is said to be f-orthogonal to y denoted by $x \perp_f y$, if $\sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f(x_i + \alpha y_i)$ for every scalar $\alpha \in \mathbb{R}$. Also, x is said to be f-orthogonal to a set K if $x \perp_f k$, for all $k \in K$.

Definition 6. We say that K is w-compact if every net $\{k_\alpha\}$ in K has a convergent subnet.

Theorem 7. *Let K be a subset of X. Then, one has the following.*

(1) *$F_{K+y}(x+Y) = F_K(x)$, for all $x = (x_1, x_2, \ldots, x_n)$, where $Y = (y, y, \ldots, y) \in X^n$.*

(2) *$P^f_{K+y}(x + Y) = P^f_K(x) + y$, for all $x = (x_1, x_2, \ldots, x_n)$.*

(3) *K is f-simultaneously proximal (f-simultaneously Chebyshev) if and only if $K + y$ is f-simultaneously proximal (f-simultaneously Chebyshev) for every $y \in X$.*

Moreover, if f is absolutely homogeneous function, then one has the following.

(4) *$F_{\alpha K}(\alpha x) = |\alpha| F_K(x)$, for all $x = (x_1, x_2, \ldots, x_n) \in X^n$ and $\alpha \in \mathbb{R}$.*

(5) *$P^f_{\alpha K}(\alpha x) = \alpha P^F_K(x)$, for all $x = (x_1, x_2, \ldots, x_n) \in X^n$ and $\alpha \in \mathbb{R}$.*

(6) *K is f-simultaneously proximal (f-simultaneously Chebyshev) if and only if αK is f-simultaneously proximal (f-simultaneously Chebyshev), $\alpha \in \mathbb{R}$.*

(7) *If f is convex function and K is a convex set, then $P^f_K(x)$ is convex.*

Proof. (1) Let $x = (x_1, x_2, \ldots, x_n)$ and $Y = (y, y, \ldots, y) \in X^n$. Then

$$F_{K+y}(x + Y) = \inf_{k \in K} \sum_{i=1}^{n} f\left((x_i + y) - (\kappa + y)\right) = F_K(x). \tag{3}$$

(2) The equation

$$\begin{aligned}\sum_{i=1}^{n} f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^{n} f\left((x_i + y) - (k + y)\right) \\ &= \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k)\end{aligned} \tag{4}$$

implies that $k_0 + y \in P^f_{K+y}(x + Y)$ if and only if $k_0 \in P^f_K(x)$. Thus,

$$P^f_{K+y}(x + Y) = P^f_K(x) + y. \tag{5}$$

(3) The proof follows immediately from part (2) above.

(4) Let $x = (x_1, x_2, \ldots, x_n) \in X^n$, $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned}F_{\alpha K}(\alpha x) &= \inf_{k \in K} \sum_{i=1}^{n} f(\alpha x_i - \alpha k) \\ &= |\alpha| \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k) = |\alpha| F_K(x).\end{aligned} \tag{6}$$

(5) If $\alpha = 0$, then we are done. If $\alpha \neq 0$ and $k_0 \in P^f_{\alpha K}(\alpha x)$, then $k_0 \in \alpha K$ and

$$\sum_{i=1}^{n} f(\alpha x_i - k_0) = \inf_{k \in K} \sum_{i=1}^{n} f(\alpha x_i - \alpha k). \tag{7}$$

This implies that

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{\alpha} k_0\right) = F_K(x), \tag{8}$$

which implies that $(1/\alpha) k_0 \in P^f_K(x)$.

(6) The proof follows immediately from part (5) above.

(7) Let $k_1, k_2 \in P^f_K(x)$. Since K is convex, then $\lambda k_1 - (1 - \lambda) k_2 \in K$. We must show that $\lambda k_1 - (1 - \lambda) k_2 \in P^f_K(x)$; that is,

$$\sum_{i=1}^{n} f\left(x_i - (\lambda k_1 - (1 - \lambda) k_2)\right) = \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k). \tag{9}$$

So,

$$\begin{aligned}
&\sum_{i=1}^{n} f\left(x_i-(\lambda k_1-(1-\lambda)k_2)\right)\\
&\quad=\sum_{i=1}^{n} f\left(\lambda\left(x_i-k_1\right)+(1-\lambda)\left(x_i-k_2\right)\right)\\
&\quad=\lambda\sum_{i=1}^{n} f\left(x_i-k_1\right)+(1-\lambda)\sum_{i=1}^{n} f\left(x_i-k_2\right)\\
&\quad=\lambda F_K(x)+(1-\lambda)F_K(x)\\
&\quad=F_K(x)=\sum_{i=1}^{n} f\left(x_i-k\right),
\end{aligned}\tag{10}$$

which implies that $P_K^f(x)$ is convex. □

Example 8. Let $X=\mathbb{R}^2$ and $K=\{(x_1,x_2)\in\mathbb{R}^2: x_1^2+x_2^2\le 4\}$, and let $f(x,y)=x^2-y^2$. If $z=((0,0),(0,1))\in X^2$, then one can show that $F_K(z)=f(0,1/2)=-1/4$.

Theorem 9. *Let f be an absolutely homogeneous real-valued function on X and M a vector subspace of X. Then,*

(1) *$F_M(\alpha x)=|\alpha|F_M(x)$, for all $x=(x_1,x_2,\ldots,x_n)\in X^n$, $\alpha\in\mathbb{R}-\{0\}$;*

(2) *$P_M^f(\alpha x)=\alpha P_M^f(x)$, for all $x=(x_1,x_2,\ldots,x_n)\in X^n$, $\alpha\in\mathbb{R}-\{0\}$.*

Proof. (1) Let $x=(x_1,x_2,\ldots,x_n)$. Then,

$$\begin{aligned}
F_M(\alpha x)&=\inf_{m\in M}\sum_{i=1}^{n} f\left(\alpha x_i-m\right)\\
&=|\alpha|\inf_{m'\in M}\sum_{i=1}^{n} f\left(x_i-m'\right)=|\alpha|F_M(x).
\end{aligned}\tag{11}$$

(2) Let $m_0\in P_M^f(\alpha x)$. Then,

$$\sum_{i=1}^{n} f\left(\alpha x_i-m_0\right)=\inf_{m\in M}\sum_{i=1}^{n} f\left(\alpha x_i-m\right)\tag{12}$$

if and only if

$$\sum_{i=1}^{n} f\left(x_i-\frac{1}{\alpha}m_0\right)=\inf_{m'\in M}\sum f\left(x_i-m'\right)=F_M(x),\tag{13}$$

for all $\alpha\in\mathbb{R}-\{0\}$, which implies that $(1/\alpha)m_0\in P_M^f(x)$, so, $m_0\in\alpha P_M^f(x)$. □

Theorem 10. *Let f be a positive real-valued function on X such that $x=0$ if and only if $f(x)=0$. Then, if K is f-simultaneously proximal, then K is f-closed.*

Proof. Since f is a positive function, then $\sum_{i=1}^{n} f(x_i)\ge 0$ for all $x=(x_1,x_2,\ldots,x_n)\in X^n$. Let $\{k_m\}$ be a sequence of K and $x=(x_1,x_2,\ldots,x_n)\in X^n$, such that $\sum_{i=1}^{n} f(x_i-k_m)\to 0$. This implies that

$$F_K(x)=\inf_{k\in K}\sum_{i=1}^{n} f\left(x_i-k\right)\le\sum_{i=1}^{n} f\left(x_i-k_m\right)\longrightarrow 0.\tag{14}$$

Since K is f-simultaneously proximal, then there exists $k_0\in K$ such that

$$F_K(x)=\sum_{i=1}^{n} f\left(x_i-k_0\right)=0.\tag{15}$$

Hence, for all $i=1,2,\ldots,n$, $f(x_i-k_0)=0$. Using the assumption it follows that $x_i-k_0=0$, and, hence, $x_i=k_0\in K$. Consequently, $x\in K^n$ and K is f-closed. □

Theorem 11. *Let X be a topological vector space and K a vector subspace of X. Suppose that f is continuous function and K is w-compact; then, K is f-simultaneously proximal.*

Proof. Let $x=(x_1,x_2,\ldots,x_n)\in X^n$. Since

$$F_K(x)=\inf\sum_{i=1}^{n} f\left(x_i-k\right),\quad\text{where } k\in K,\tag{16}$$

then, for any constant α, there exists $\{k_\alpha\}$ such that

$$\sum_{i=1}^{n} f\left(x_i-k_\alpha\right)\le\sum_{i=1}^{n} f\left(x_i-k\right)+\frac{1}{\alpha}.\tag{17}$$

But K is w-compact; then, there exists a subnet $\{k_{\alpha_\beta}\}$ such that $k_{\alpha_\beta}\to k_0$. Thus,

$$x_i-k_{\alpha_\beta}\longrightarrow x_i-k_0,\quad\forall i=1,2,\ldots,n.\tag{18}$$

Since f is continuous, then

$$\sum_{i=1}^{n} f\left(x_i-k_{\alpha_\beta}\right)\le\sum_{i=1}^{n} f\left(x_i-k\right)+\frac{1}{\alpha}.\tag{19}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{n} f\left(x_i-k_0\right)&=\liminf\sum_{i=1}^{n} f\left(x_i-k_{\alpha_\beta}\right)\\
&\le\sum_{i=1}^{n} f\left(x_i-k\right).
\end{aligned}\tag{20}$$

Hence, $k_0\in P_K^f(x)$. □

For a subset K of X, let us define $\widehat{K_F}$ to be such that

$$\widehat{K_F}=\left\{x=(x_1,x_2,\ldots,x_n)\in X^n: F_K(x)=\sum_{i=1}^{n} f\left(x_i\right)\right\}.\tag{21}$$

Example 12. Consider $X=(\mathbb{R}^2)^2$ and $K=\{((x_1,y_1),(x_2,y_2)): x_i=y_i$, for all $i=1,2\}$. Let $f(x,y)=x^2+y^2$; then, one can see that

$$\widehat{K_F}=\left\{\left(\left(x_1,-x_1\right),\left(x_2,-x_2\right)\right)\right\}.\tag{22}$$

Using the previous definition of $\widehat{K_F}$, we prove the following theorem characterizing f-simultaneously proximal subspaces.

Theorem 13. *Let K be a vector subspace of X. Then, K is f-simultaneously proximal in X if and only if $X^n = D_k + \widehat{K_F}$, where $D_K = \{(k, k, \ldots, k) : k \in K\}$.*

Proof. Suppose that $X^n = D_k + \widehat{K_F}$. Then, for $x = (x_1, x_2, \ldots, x_n) \in X^n$, there exists $k_1 = (k_0, k_0, \ldots, k_0) \in D_K$ and $y = (y_1, y_2, \ldots, y_n) \in \widehat{K_F}$ such that $x = y + k_1$. Hence, $x - k_1 = y \in \widehat{K_F}$, and

$$F_K(y) = F_K(x - k_1) = \sum_{i=1}^{n} f(x_i - k_0), \tag{23}$$

and so

$$\begin{aligned} \sum_{i=1}^{n} f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k_0 - k) \\ &= \inf_{k' \in K} \sum_{i=1}^{n} f\left(x_i - k'\right) = F_K(x). \end{aligned} \tag{24}$$

So, K is f-simultaneously proximal.

Conversely, suppose that K is f-simultaneously proximal and $x = (x_1, x_2, \ldots, x_n) \in X^n$. Then, there exists $k_0 \in K$ such that

$$\begin{aligned} \sum_{i=1}^{n} f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k) \\ &= \inf_{k \in K} \sum_{i=1}^{n} f\left(x_i - \left(k' + k_0\right)\right), \end{aligned} \tag{25}$$

where $k = k' + k_0$. If $k_1 = (k_0, k_0, \ldots, k_0) \in D_K$, then

$$\sum_{i=1}^{n} f(x_i - k_0) = F_K(x - k_1), \tag{26}$$

which implies that $x - k_1 = k_2 \in \widehat{K_F}$ and $X^n = D_k + \widehat{K_F}$. $\square$

Proposition 14. *Let X be a topological vector space and K f-simultaneous proximal subset of X. Then,*

(1) *$k_0 \in P_K^f(x)$ if and only if $x - k_0 \in \widehat{K_F}$;*

(2) *if f is symmetric (i.e., $f(-x) = f(x)$ for all $x \in X$), then $x \in \widehat{K_F}$ if and only if $-x \in \widehat{K_F}$;*

(3) *if $x \perp_F K$, then $x \in \widehat{K_F}$, where $x = (x_1, x_2, \ldots, x_n)$;*

(4) *if $x \in \widehat{K_F}$ and $\alpha K = K$, then $x \perp_F K$, where $x = (x_1, x_2, \ldots, x_n)$.*

Proof. (1) Let $k_0 \in P_K^f(x)$ if and only if $\sum_{i=1}^{n} f(x_i - k_0) = \inf\{\sum_{i=1}^{n} f(x_i - k) : k \in K\}$. Thus,

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i - k_0) \\ &\quad = \inf\left\{\sum_{i=1}^{n} f(x_i - k_0 + k_0 - k) : k \in K\right\} \\ &\quad = \inf\left\{\sum_{i=1}^{n} f\left(x_i - k_0 - k'\right) : k' \in K\right\}, \end{aligned} \tag{27}$$

which implies that $x - k_0 \in \widehat{K_F}$.

(2) Let $x = (x_1, x_2, \ldots, x_n) \in \widehat{K_F}$. Since f is symmetric, then

$$\begin{aligned} \sum_{i=1}^{n} f(-x_i) &= \sum_{i=1}^{n} f(x_i) \\ &= \inf\left\{\sum_{i=1}^{n} f(x_i - k) : k \in K\right\} \\ &= \inf\left\{\sum_{i=1}^{n} f(-(-x_i + k)) : -k \in K\right\} \\ &= \inf\left\{\sum_{i=1}^{n} f(-x_i + k) : -k \in K\right\}. \end{aligned} \tag{28}$$

Hence, $\sum_{i=1}^{n} f(-x_i) = \inf\{\sum_{i=1}^{n} f(-x_i + k) : -k \in K\}$, which implies that

$$-x = (-x_1, -x_2, \ldots, -x_n) \in \widehat{K_F}. \tag{29}$$

(3) Let $x = (x_1, x_2, \ldots, x_n)$. Since $x \perp_F K$, then

$$\begin{aligned} \sum_{i=1}^{n} f(x_i) &\le \sum_{i=1}^{n} f(x_i + \alpha k) \quad \forall \alpha \in \mathbb{R},\ k \in K. \\ &= \sum_{i=1}^{n} f(x_i - (-\alpha k)) \quad \forall \alpha \in \mathbb{R},\ k \in K. \\ &= \sum_{i=1}^{n} f\left(x_i - k'\right), \quad k' \in K. \end{aligned} \tag{30}$$

So,

$$\sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f\left(x_i - k'\right), \quad k' \in K. \tag{31}$$

Hence, $x = (x_1, x_2, \ldots, x_n) \in \widehat{K_F}$.

(4) Let $x \in \widehat{K_F}$ and $\alpha K = K$. Then,

$$\begin{aligned} \sum_{i=1}^{n} f(x_i) &= \inf_{k \in K} \sum_{i=1}^{n} f(x_i - k) \\ &= \inf_{\alpha k \in K} \sum_{i=1}^{n} f(x_i - \alpha k), \quad \text{since } \alpha K = K, \\ &= \inf \sum_{i=1}^{n} f(x_i + (-\alpha k)), \quad \forall k \in K. \end{aligned} \tag{32}$$

Thus,

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f\left(x_i + \alpha' k\right), \quad \forall \alpha' \in \mathbb{R}, \ \forall k \in K. \tag{33}$$

Hence, $x \perp_F K$. □

Theorem 15. *Let K be a vector subspace of X. If $\pi(\widehat{K_F}) = X^n/D_K$, then K is f-simultaneously proximal, where π is the canonical map $x \to x + D_k$.*

Proof. Let $\pi(\widehat{K_F}) = X^n/D_K$ and $x = (x_1, x_2, \ldots, x_n) \in X^n$. Then, $x + D_K = y + D_K$ for some $y \in \widehat{K_F}$. Hence, $x - y = k_0$ for some $k_0 \in D_K$. Thus, $x = y + k_0 \in \widehat{K_F} + D_k$. Therefore, $\widehat{K_F} + D_k = X^n$. By Theorem 15, K is f-simultaneously proximal. □

3. f-Simultaneous Approximation in Quotient Space

Definition 16. Let K and M be two vector subspaces of X such that M is closed and $M \subset K$. Suppose that f is a positive real-valued function defined on X. Then, a function $\widetilde{f} : (X/M)^n \to \mathbb{R}$ can be defined as follows:

$$\begin{aligned}\widetilde{f}&(x_1 + M, x_2 + M, \ldots, x_n + M) \\ &= \inf\left\{\sum_{i=1}^{n} f(x_i + y) : y \in M\right\}\end{aligned} \tag{34}$$

for each $(x_1, x_2, \ldots, x_n) \in X^n$.

Theorem 17. *Let K and M be two vector subspaces of X such that $M \subset K$. If k_0 is a point of f-best simultaneous approximation to $(x_1, x_2, \ldots, x_n)$ in K, then $k_0 + M$ is an $\widetilde{f}$-best simultaneous approximation to $(x_1, x_2, \ldots, x_n) + M$ in K/M.*

Proof. Suppose that $k_0 + M$ is not $\widetilde{f}$-best simultaneous approximation to $(x_1 + M, x_2 + M, \ldots, x_n + M)$ in K/M. Then,

$$\widetilde{f}\left((x_i - k_0 + M)_{i=1}^n\right) \nleq \widetilde{f}\left((x_i - k + M)_{i=1}^n\right) \tag{35}$$

for at least $k \in K$, say $k_1 \in K$, such that

$$\widetilde{f}\left((x_i - k_1 + M)_{i=1}^n\right) < \widetilde{f}\left((x_i - k_0 + M)_{i=1}^n\right). \tag{36}$$

Since

$$\begin{aligned}\widetilde{f}\left((x_i - k_0 + M)_{i=1}^n\right) &= \inf\left\{\sum_{i=1}^{n} f(x_i - k_0 + y) : y \in M\right\} \\ &\leq \sum_{i=1}^{n} f(x_i - k_0),\end{aligned} \tag{37}$$

we have

$$\widetilde{f}\left((x_i - k_1 + M)_{i=1}^n\right) < \sum_{i=1}^{n} f(x_i - k_0). \tag{38}$$

Thus, for some $m_0 \in M$, we have

$$\sum_{i=1}^{n} f(x_i - k_1 + m_0) < \sum_{i=1}^{n} f(x_i - k_0), \tag{39}$$

so,

$$\sum_{i=1}^{n} f(x_i - (k_1 - m_0)) < \sum_{i=1}^{n} f(x_i - k_0). \tag{40}$$

Since $M \subset K$ implies that $k_1 - m_0 \in K$, therefore, k_0 is not f-best simultaneous approximation to $(x_1, x_2, \ldots, x_n)$ in K, which is a contradiction. □

Corollary 18. *Let K and M be two vector subspaces of X such that $M \subset K$. Then, if K is f-simultaneously proximal in X, then K/M is $\widetilde{f}$-simultaneously proximal in X/M.*

Proof. If K is f-simultaneously proximal in X, then there exists at least $k_0 \in K$ such that k_0 is f-best simultaneous approximation to $(x_1, x_2, \ldots, x_n)$ in K. Thus by Theorem 11, $k_0 + M$ is an $\widetilde{f}$-best simultaneous approximation to $(x_1, x_2, \ldots, x_n) + M$ in K/M, so, K/M is $\widetilde{f}$-simultaneously proximal in X/M. □

Theorem 19. *Let K and M be two vector subspaces of X such that $M \subset K$. If M is f-simultaneously proximal in X and K/M is $\widetilde{f}$-simultaneously proximal in X/M, then K is f-simultaneously proximal in X.*

Proof. Since K/M is $\widetilde{f}$-simultaneously proximal in X/M, then there exists $k_0 \in K$ such that $k_0 + M$ is $\widetilde{f}$-best simultaneous approximation to $(x_1, x_2, \ldots, x_n) + M$ from K/M, so,

$$\begin{gathered}\widetilde{f}\left((x_i - k_0 + M)_{i=1}^n\right) \leq \widetilde{f}\left((x_i - k + M)_{i=1}^n\right), \quad \forall k \in K, \\ \Downarrow \\ \begin{aligned}\widetilde{f}\left((x_i - k_0 + M)_{i=1}^n\right) &= \inf_{m \in M} \sum_{i=1}^{n} f(x_i - k_0 + m) \\ &\leq \inf_{m \in M} \sum_{i=1}^{n} f(x_i - k + m),\end{aligned}\end{gathered} \tag{41}$$

for all $k \in K$. Note that

$$\begin{aligned}\inf_{m \in M} &\sum_{i=1}^{n} f(x_i - k_0 + m) \\ &= F_M(x_1 - k_0, x_2 - k_0, \ldots, x_n - k_0) \\ &\leq F_M(x_1 - k, x_2 - k, \ldots, x_n - k).\end{aligned} \tag{42}$$

Since M is f-simultaneously proximal in X, then there exists $m_0 \in M$ such that

$$\begin{aligned} F_M(x_1 - k_0, x_2 - k_0, \ldots, x_n - k_0) &= \sum_{i=1}^{n} f(x_i - k_0 - m_0) \\ &\leq \sum_{i=1}^{n} f(x_i - k - m), \end{aligned} \quad (43)$$

for all $m \in M$ and $k \in K$. So,

$$\sum_{i=1}^{n} f(x_i - (k_0 + m_0)) \leq \sum_{i=1}^{n} f(x_i - (k + m)), \quad (44)$$

for all $m \in M$ and $k \in K$. Hence,

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i - (k_0 + m_0)) \\ &\quad = \inf\left\{\sum_{i=1}^{n} f(x_i - (k + m)) : m \in M, k \in K\right\}. \end{aligned} \quad (45)$$

So, $k_0 + m_0$ is an f-best simultaneous approximation to $(x_1, x_2, \ldots, x_n)$ from K and K is f-simultaneously proximal in X. □

Theorem 20. *Let K and M be two vector subspaces of X such that $M \subset K$. If M is f-simultaneously proximal in X and K is f-simultaneously Chebyshev in X, then K/M is $\tilde{f}$-simultaneously Chebyshev in X/M.*

Proof. Suppose not, then there exists $(x_1, x_2, \ldots, x_n) + M \in X/M$, and $k_1 + M, k_2 + M \in P^{\tilde{f}}_{K/M}((x_1, x_2, \ldots, x_n) + M)$ such that $k_1 + M \neq k_2 + M$. Thus, $k_1 - k_2 \notin M$. Since M is f-simultaneously proximal in X, then

$$\begin{aligned} &P^f_M(x_1 - k_1, x_2 - k_1, \ldots, x_n - k_1) \neq \phi, \\ &P^f_M(x_1 - k_2, x_2 - k_2, \ldots, x_n - k_2) \neq \phi. \end{aligned} \quad (46)$$

Let $m_1 \in P^f_M(x_1 - k_1, x_2 - k_1, \ldots, x_n - k_1)$ and $m_2 \in P^f_M(x_1 - k_2, x_2 - k_2, \ldots, x_n - k_2)$. By Theorem 13, $k_1 + m_1$ and $k_2 + m_2$ are f-best simultaneous approximation to $(x_1, x_2, \ldots, x_n)$ from K. Since K is f-simultaneously Chebyshev in X, then $k_1 + m_1 = k_2 + m_2$, and, hence, $k_1 - k_2 = m_1 - m_2 \in M$, which is a contradiction. □

Theorem 21. *Let K and M be two vector subspaces of a topological vector space X. If M is f-simultaneously Chebyshev in X, then the following assertions are equivalent:*

(i) *K/M is $\tilde{f}$-simultaneously Chebyshev in X/M;*

(ii) *$K + M$ is simultaneously Chebyshev in X.*

Proof. (i $\Rightarrow$ ii) By hypothesis, $(K + M)/M = K/M$ is $\tilde{f}$-simultaneous Chebyshev. Assume that $K + M$ is not f-simultaneous Chebyshev in X. Then, there exists $x = (x_1, \ldots, x_n) \in X^n$ which has two distinct f-best simultaneous approximations, say ℓ_0 and $\ell_1 \in K + M$. Thus, we have ℓ_0 and $\ell_1 \in P^f_{K+M}(x)$. Since $M \subseteq K + M$, we have that $\ell_0 + M$ and $\ell_1 + M \in P^f_{(K+M)/M}(x + M) = P^f_{K/M}(x + M)$. By hypothesis, K/M is $\tilde{f}$-simultaneous Chebyshev, and so $\ell_0 + M = \ell_1 + M$. Then, there exists $m_0 \in M \setminus \{0\}$ such that $\ell_1 = \ell_0 + m_0$. Thus, we conclude that

$$\begin{aligned} &\sum_{i=1}^{n} f((x_i - \ell_0) - m_0) \\ &\quad = \sum_{i=1}^{n} f(x_i - \ell_1) \\ &\quad = \inf_{m \in M}\left\{\sum_{i=1}^{n} f(x_i - (\ell_0 + m))\right\} \\ &\quad \leq \left\{\sum_{i=1}^{n} f((x_i - \ell_0) - m)\right\}, \quad \forall m \in M \\ &= F_M(x - \ell_0). \end{aligned} \quad (47)$$

So, m_0 and 0 are f-best simultaneous approximations to $x - \ell_0$ from M. Hence, M is not f-simultaneously Chebyshev. This is a contradiction.

(ii $\Rightarrow$ i) Assume that (i) does not hold. Then, there exists $x + M \in K/M$ which has two distinct $\tilde{f}$-best simultaneous approximations, say $k + M$ and $k' + M \in K/M$; thus, $k - k' \notin M$. Since M is f-simultaneously proximal, so there exist f-best simultaneous approximations m and m' to $x - k$ and $x - k'$ from M, respectively. Therefore, we have $m \in P^f_M(x - k)$ and $m' \in P^f_M(x - k')$. Since $M \subseteq K + M$, $k + M$ and $k' + M \in P^f_{K/M}(x + M) = P^f_{(K+M)/M}(x + M)$, so $k + m$ and $k' + m' \in P^f_{K+M}(x)$. But $K + M$ is f-simultaneously Chebyshev. Thus we get $k + m = k' + m'$, and therefore $k - k' \in M$. This is a contradiction. □

Definition 22. A subset K of X is called f-quasisimultaneously Chebyshev if $P^f_K(x)$ is non-empty and f-compact set in X, for all $x = (x_1, x_2, \ldots, x_n) \in X^n$.

Theorem 23. *Let f be a positive function, M an f-simultaneously proximal vector subspace of X, and K f-quasisimultaneously Chebyshev of X such that $M \subset K$. Then, K/M is $\tilde{f}$-quasi-simultaneously Chebyshev in X^n/M.*

Proof. Since K is f-simultaneously proximal in X, then by Corollary 12, K/M is $\tilde{f}$-simultaneously proximal in X/M. Let $x = (x_1, x_2, \ldots, x_n) \in X^n$ and $(k_n + M)$ a sequence in $P^{\tilde{f}}_{K/M}(x + M)$. For every n, there exists $m_n \in M$ such that $k_n + m_n = k'_n \in P^f_K(x)$. But since M is a vector subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M. \quad (48)$$

Since K is f-quasi-simultaneously Chebyshev of X, the sequence $\{k_n\}$ has a subsequence $\{k_{n_i}\}$ which is f-convergent to $k_0 \in P_K^f(x)$, meaning that

$$f\left(k_{n_i} - k_0\right) \longrightarrow 0. \tag{49}$$

But

$$\tilde{f}\left(k_{n_i} - k_0 + M\right) \leq f\left(k_{n_i} - k_0\right) \longrightarrow 0. \tag{50}$$

Hence,

$$\tilde{f}\left(k_{n_i} - k_0 + M\right) \longrightarrow 0. \tag{51}$$

Consequently, $P_{K/M}^{\hat{f}}(x + M)$ is $\tilde{f}$-compact and K/M is $\tilde{f}$-quasi-simultaneously Chebyshev. This completes the proof. □

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