SINGULAR NONLINEAR ELLIPTIC EQUATIONS IN $\mathbf{R}^N$

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Abstract. This paper deals with existence, uniqueness and regularity of positive generalized solutions of singular nonlinear equations of the form $-\Delta u + a(x)u = h(x)u^{-\gamma}$ *in* $\mathbf{R}^N$ where a, h are given, not necessarily continuous functions, and γ is a positive number. We explore both situations where a, h are radial functions, with a being eventually identically zero, and cases where no symmetry is required from either a or h. Schauder's fixed point theorem, combined with penalty arguments, is exploited.

1. Introduction

This paper addresses existence, uniqueness and regularity questions on generalized solutions of the singular nonlinear elliptic problem

$$(*) \qquad \begin{cases} -\Delta u + a(x)u = h(x)u^{-\gamma} \;\; in \;\; \mathbf{R}^N \\ u > 0 \;\; \text{in} \;\; \mathbf{R}^N \end{cases}$$

where a, h are nonnegative L^∞_{loc} functions, $h \not\equiv 0$, (eventually we consider $a \equiv 0$), $\gamma > 0$ and $N \geq 3$. We point out that the search of positive solutions of the Dirichlet problem for the equation

$$-\Delta u + a(x)u = h(x)u^{-\gamma} \;\; in \;\; \Omega$$

where Ω is a smooth bounded domain has deserved the attention of many authors. Nowosad [1] studied a related Hammerstein equation, namely

$$u(x) = \int_0^1 K(x,y)(u(x))^{-\gamma} dy,$$

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where $\gamma = 1$, $\int_0^1 K(x,y)dy \geq \delta > 0$ and $K(x,y)$ is positive semidefinite. Nowosad's work was extended by Karlin and Nirenberg [2] where more general Hammerstein equations were considered including the case $\gamma > 0$ in the equation above. Crandall-Rabinowitz and Tartar [3] studied the Dirichlet problem

$$Lu = f(x,u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

where L is a linear second order elliptic operator and $f : \Omega \times (0, +\infty) \to \mathbf{R}$ is singular in the sense that $f(x,r) \to \infty$ as $r \to 0^+$. Examples such as $f(x,r) = r^{-\gamma}$ with $\gamma > 1$, $\gamma < 1$ or $\gamma = 1$ were covered.

There is by now an extensive literature on singular elliptic problems. With respect to the case of bounded domains $\Omega \subset \mathbf{R}^N$ we would like to further mention Gomes [4], Lazer and McKenna [5], Cac and Hernandez [8], Chen [9], Lair and Shaker [10], Shangbin [13] while for the case $\Omega = \mathbf{R}^N$ we recall Kuzano and and Swanson [11], Lair and Shaker [12,14]. This reference list is far from complete. In the earlier papers concerning $\Omega = \mathbf{R}^N$, $h(x)$ is assumed at least continuous and several techniques are developed such as the method of lower and upper solutions. In this paper we assume $h(x)$ only integrable and use the Schauder fixed point theorem and elliptic estimates. Singular equations appear in the theory of heat conduction in electrically conducting materials, (Fulks and Maybee [6]), in binary communications by signals (Nowosad [1]) and in the theory of pseudoplastic fluids (Nachman and Callegari [7]).

The following condition on a will be required in the first one of our main results stated below:

$$(a)_R \qquad a(x) \geq a_0 \text{ for } |x| \geq R \text{ for some } a_0, R > 0.$$

In what follows we take $\gamma, \alpha \in (0,1)$ and $h \in L^\theta \cap L^2$ where $\theta \equiv \frac{2}{2-(1-\gamma)}$.

Theorem 1. *Assume $(a)_R$. Then $(*)$ has a unique solution $u \in \mathcal{D}^{1,2} \cap W^{2,p}_{loc}$ where $1 < p < \infty$ with $\int a(x)u^2 < \infty$. If a, h are radial functions the solution is radial, as well, and in fact, $u(x) \to 0$ as $|x| \to \infty$. Moreover if $a, h \in C^\alpha_{loc}$ then $u \in C^{2,\alpha}_{loc}$.*

In our second result we take $a \equiv 0$ and h radially symmetric that is, we study the problem

$$(*)_o \qquad \begin{cases} -\Delta u = h(|x|)u^{-\gamma} \ \text{in } \mathbf{R}^N \\ u > 0 \ \text{ in } \mathbf{R}^N. \end{cases}$$

This problem shall be treated by first perturbing the equation by a radially symmetric term, then using the earlier result in the case a, h are radial functions and finally taking limits.

Theorem 2. *Let $a \equiv 0$ and let h be radially symmetric. Then $(*)_o$ has a unique radially symmetric solution $u \in \mathcal{D}^{1,2} \cap W^{2,p}_{loc}$, $1 < p < \infty$ and $u(x) \to 0$ as $|x| \to \infty$. Moreover, if $h \in C^\alpha_{loc}$ then $u \in C^{2,\alpha}_{loc}$.*

2. Preliminaries

The main goal in this section is to prove theorem 1. For that purpose let $\epsilon > 0$ and consider the problem

$$(2.1)\qquad \begin{cases} -\Delta u + a(x)u = \frac{h(x)}{(u+\epsilon)^\gamma} \ \ in \ \ \mathbf{R}^N \\ u > 0 \ \text{ in } \ \mathbf{R}^N. \end{cases}$$

We are going to show by applying the Schauder fixed point theorem that (2.1) has a solution $u_\epsilon \in W^{2,p}_{loc}, \ \ 1 < p < \infty$, and then by passing to the limit as $\epsilon \to 0$ we arrive at a solution of $(*)$.

In order to deal with a first step namely, existence of a solution of (2.1), let $f \in L^2$ and consider the linear equation

$$(2.2)\qquad -\Delta u + a(x)u = f(x) \ \ in \ \ \mathbf{R}^N.$$

Recalling that the Hilbert space $\mathcal{D}^{1,2}$ is defined as the closure of C_0^∞ with respect to the gradient norm $\|\varphi\|_1^2 = \int |\nabla\varphi|^2$ we introduce the space

$$E \equiv \left\{ u \in \mathcal{D}^{1,2} \ | \ \int au^2 < \infty \right\}$$

which endowed with the inner product and norm given respectively by

$$\langle u, v\rangle = \int (\nabla u.\nabla v + auv) \ \text{ and } \|u\|^2 = \langle u, u\rangle$$

is itself a Hilbert space. Under condition $(a)_R$ it follows that $u \in E$ iff $u \in W^{1,2}(\mathbf{R}^N)$.

Yet if $f \in L^2$ it follows by minimizing over E the energy functional associated with (2.2),

$$I(u) = \frac{1}{2}\|u\|^2 - \int fu$$

that (2.2) has a weak solution $u \in E$, that is,

$$\int (\nabla u \nabla\varphi + au\varphi) = \int f(x)\varphi, \ \ \varphi \in E.$$

The solution u is, in fact, unique. Letting $S : L^2 \to E$ be the solution operator associated to (2.2) that is $Sf = u$ for $f \in L^2$ it follows that S is linear and moreover

$$\|Sf\| \le C|f|_2, \ \ f \in L^2$$

for some $C > 0$. In addition, splitting u into $u^+ - u^-$ where $u^\pm$ are respectively the positive and negative parts of u, taking $\varphi = -u^-$ above and noticing that $u^- \in E$ we infer that

$$Sf \ge 0 \ \text{ whenever } \ f \ge 0.$$

Now let $u \in L^2$ with $u \ge 0$. Since

$$(2.3)\qquad 0 \le \frac{h(x)}{(u+\epsilon)^\gamma} \le \frac{h(x)}{\epsilon^\gamma}$$

and $\frac{h(x)}{\epsilon^\gamma} \in L^2$ the operator

$$Tu \equiv S\left[\frac{h(x)}{(u+\epsilon)^\gamma}\right]$$

is continuous in L^2, and as a matter of fact, letting $w \equiv T(0)$ we have

$$w = S\left[\frac{h(x)}{\epsilon^\gamma}\right].$$

Considering

$$K \equiv \left\{v \in L^2 \mid 0 \le v \le w \text{ a.e. in } \mathbf{R}^N\right\}$$

we shall prove that the following result holds true.

Lemma 3. *The set $K \subset L^2$ is closed, convex and bounded and moreover $T(K) \subset K$ and $\overline{T(K)}$ is a compact subset of L^2.*

Using lemma 3 and the Schauder fixed point theorem there is some $u_\epsilon \in K$ satisfying

$$u_\epsilon = S\left[\frac{h(x)}{(u_\epsilon+\epsilon)^\gamma}\right]$$

that is

$$\begin{cases} \int (\nabla u_\epsilon \nabla\varphi + a u_\epsilon \varphi) = \int \frac{h(x)\varphi}{(u_\epsilon+\epsilon)^\gamma}, & \varphi \in E \\ u_\epsilon \ge 0 \text{ a.e. in } \mathbf{R}^N, & u_\epsilon \in E. \end{cases}$$

Now since by (2.3)

$$\frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \in L^\infty_{loc}$$

it follows by the elliptic regularity theory that $u_\epsilon \in W^{2,p}_{loc}$, $1 < p < \infty$, and further if $B \subset \mathbf{R}^N$ is a ball, then

$$-\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \text{ a.e. in } B.$$

In fact, it follows by the maximum principle that $u_\epsilon > 0$ in B and so

$$\begin{cases} -\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} & \text{a.e. in } \mathbf{R}^N \\ u_\epsilon > 0 \text{ in } \mathbf{R}^N. \end{cases}$$

On the other hand, if $f \in L^2_{rad}$ we get by minimizing the functional I above over the space

$$E_{rad} \equiv \left\{u \in W^{1,2}_{rad} \mid \int a(r)u^2 < \infty\right\}$$

which endowed with the inner product and norm given above is also a Hilbert space, a weak solution $u \in E_{rad}$ of (2.2) that is

$$\int (\nabla u \nabla\varphi + a u\varphi) = \int f(x)\varphi, \quad \varphi \in E_{rad}.$$

The solution is also unique and as before the solution operator associated to (2.2), namely $S : L^2_{rad} \to E_{rad}$ satisfies

$$\|Sf\| \le C|f|_2$$

for $f \in L^2_{rad}$ and further

$$Sf \geq 0 \text{ whenever } f \geq 0.$$

Letting

$$K \equiv \left\{ v \in L^2_{rad} \mid 0 \leq v \leq w \ \text{ a.e. in } \ \mathbf{R}^N \right\}$$

we have a corresponding symmetric variant of lemma 3 and so there is some $u_\epsilon \in E_{rad}$ with

$$\int (\nabla u_\epsilon \nabla \varphi + a(r) u_\epsilon \varphi) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \varphi, \ \ \varphi \in E_{rad}.$$

Proof of Lemma 3.

It is easy to show that K is convex, closed and bounded. So we will only show that $T(K) \subset K$ and $\overline{T(K)}$ is compact in L^2. If $v \in K$ then

$$T(0) - T(v) = S\left[h\left(\frac{1}{\epsilon^\gamma} - \frac{1}{(v+\epsilon)^\gamma} \right) \right] \geq 0$$

that is $T(v) \leq w$ and hence $T(K) \subset K$.

In order to show that $\overline{T(K)} \subset L^2$ is compact let v_n be a sequence in $T(K)$ say $v_n = T(u_n)$ for some $u_n \in K$. By (2.3)

$$\frac{h(x)}{(u_n + \epsilon)^\gamma} \ \text{ is bounded in } \ L^2$$

so that

$$T(u_n) = S\left[\frac{h(x)}{(u_n + \epsilon)^\gamma} \right] \ \text{ is bounded in } \ E.$$

Thus, passing to subsequences,

$$T(u_n) \ \rightharpoonup v \text{ for some } \ v \in E$$

and

$$T(u_n) \to \ v \text{ a.e. in } \mathbf{R}^N.$$

On the other hand, since $0 \leq T(u_n) \leq w$ it follows by Lebesgue's theorem that

$$T(u_n) \to v \text{ in } L^2.$$

showing that $\overline{T(K)}$ is compact in L^2, ending the proof of lemma 3. The radial case is handled similarly. ∎

The next result states that the family u_ϵ increases when ϵ decreases.

Lemma 4. *If* $0 < \epsilon < \epsilon'$ *then* $u_{\epsilon'} \leq u_\epsilon$ *in* $\mathbf{R}^N$.

Proof of Lemma 4.

Letting $\omega \equiv u_{\epsilon'} - u_\epsilon$ we get

$$-\Delta\omega + a(x)\omega = h(x)\left[\frac{1}{(u_{\epsilon'} + \epsilon')^\gamma} - \frac{1}{(u_\epsilon + \epsilon)^\gamma} \right] \ \text{ a.e. in } \ \mathbf{R}^N$$

which gives

$$\int |\nabla \omega^+|^2 + a(x)\omega^{+2} = \int h(x)\left[\frac{1}{(u_{\epsilon'}+\epsilon')^\gamma} - \frac{1}{(u_\epsilon+\epsilon)^\gamma}\right]\omega^+ \leq 0$$

showing that $\omega^+ = 0$ and thus $u_{\epsilon'} \leq u_\epsilon$ a.e. in $\mathbf{R}^N$, finishing the proof of lemma 4. ■

3. Proofs of Main Results

Proof of Theorem 1.

Step 1 (the non-symmetric case).

Let $\epsilon_n > 0$ be a decreasing sequence converging to 0 and set $u_n = u_{\epsilon_n}$. We claim that

$$\|u_n\| \text{ is bounded.}$$

Indeed,

$$\int \left(|\nabla u_n|^2 + a|u_n|^2\right) = \int \frac{h(x)u_n}{(u_n+\epsilon_n)^\gamma} \leq \int h(x)u_n^{1-\gamma} \leq C|h|_\theta \|u_n\|^{1-\gamma} \tag{3.1}$$

for some $C > 0$, showing that u_n is bounded in E. Hence, passing to subsequences, we have

$$u_n \rightharpoonup u \text{ in } E, \text{ and } u_n \to u \text{ a.e. in } \mathbf{R}^N.$$

Moreover since by lemma 4 $0 < u_1 \leq u_n$ in $\mathbf{R}^N$ we infer that if $\varphi \in E$ has compact support then $supp(\varphi) \subset B$ for some ball $B \subset \mathbf{R}^N$ and

$$\frac{|h(x)\varphi|}{(u_n+\epsilon_n)^\gamma} \leq H(x) \text{ for some } H \in L^1$$

which gives, by applying Lebesgue's theorem to

$$\int (\nabla u_n \nabla \varphi + a u_n \varphi) = \int \frac{h(x)\varphi}{(u_n+\epsilon_n)^\gamma}$$

that

$$\begin{cases} \int (\nabla u \nabla \varphi + a u \varphi) = \int \frac{h(x)\varphi}{u^\gamma} \\ u \geq u_1 > 0 \text{ in } \mathbf{R}^N. \end{cases}$$

Using the regularity theory again we arrive at

$$\begin{cases} -\Delta u + a(x)u = h(x)u^{-\gamma} \text{ a.e. in } \mathbf{R}^N \\ u \in W^{2,p}_{loc}, \ 1 < p < \infty \\ u > 0 \text{ in } \mathbf{R}^N. \end{cases}$$

In order to prove uniqueness let $M \in C_0^\infty$ such that

$$M(x) = 1 \text{ if } |x| \leq 1, \ M(x) = 0 \text{ if } |x| \geq 2 \text{ and } 0 \leq M \leq 1.$$

Given $\varphi \in E$, an integer $j \geq 1$ and letting

$$\varphi^j(x) \equiv M(\frac{x}{j})\varphi(x), \ x \in \mathbf{R}^N$$

it follows that $\varphi^j \in E$ and $supp(\varphi^j)$ is compact. Moreover as we will show in the Appendix

$$\varphi^j \to \varphi \quad \text{in} \ \ E. \tag{3.2}$$

Now assume u, v are two solutions of $(*)$ and let $w_j \equiv u^j - v^j$. Then

$$\begin{aligned} \langle u-v, u^j - v^j\rangle &= \int (\nabla(u-v)\nabla w_j + a(x)(u-v)w_j) \\ &= \int h(x)\left(\tfrac{1}{u^\gamma} - \tfrac{1}{v^\gamma}\right) w_j. \end{aligned}$$

Assuming, by contradiction, that $u \neq v$ and once

$$\left\langle u-v, u^j - v^j\right\rangle \to \|u-v\|^2$$

we have

$$\int h(x)\left(\frac{1}{u^\gamma} - \frac{1}{v^\gamma}\right) w_j > 0$$

for large values of j. On the other hand,

$$\int h(x)\left(\frac{1}{u^\gamma} - \frac{1}{v^\gamma}\right) w_j \leq \int_{\Omega_j} h(x)u^{1-\gamma} + \int_{\Omega_j} h(x)v^{1-\gamma}$$

where $\Omega_j \equiv B_{2j}\backslash B_j$. Therefore, passing to the limit as $j \to \infty$ and noticing that the two integrals in the right hand side tend to zero we get a contradiction, that is $u = v$.

Assume now, $h \in C^\alpha_{loc}$. Then by the elliptic regularity theory more precisely, interior elliptic estimates, we get $u \in C^{2,\alpha}_{loc}$. This proves theorem 1 (in the case of Step 1).

Step 2 (the symmetric case: a, h are radial).

From section 2 we have found by Schauder's Theorem some radial function $u_\epsilon \in K$, $u_\epsilon \neq 0$ satisfying $u_\epsilon = Tu_\epsilon$, which means

$$\int (\nabla u_\epsilon \nabla v + a(r)u_\epsilon v) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} v, \quad v \in E_{rad}. \tag{3.3}$$

We will show next that $u_\epsilon \in W^{2,p}_{loc}(\mathbf{R}^N\backslash\{0\})$ for $1 < p < \infty$, and

$$-\Delta u_\epsilon + a(r)u_\epsilon = \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \ \text{ a.e. in } \ \mathbf{R}^N\backslash\{0\}.$$

Indeed, changing variables we get from (3.3)

$$\int_S \int_0^\infty (u'_\epsilon v' + a(r)u_\epsilon v)\, r^{N-1} dr dS = \int_S \int_0^\infty \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} v r^{N-1} dr dS$$

where $S \subset \mathbf{R}^N$ is the unit sphere. Making

$$v \equiv r^{-(N-1)}\psi, \ r > 0, \ \psi \in C_0^\infty(0, \infty)$$

we have

$$\int_0^\infty \left[\left(r^{(N-1)}u_\epsilon'\right)\left(r^{-(N-1)}\psi\right)' + au_\epsilon\psi\right]dr = \int_0^\infty \frac{h(r)}{(u_\epsilon+\epsilon)^\gamma}\psi(r)dr,$$

for $\psi \in C_0^\infty(0,\infty)$, and labelling

$$\frac{h(r)}{(u_\epsilon+\epsilon)^\gamma} - a(r)u_\epsilon \equiv \widehat{H}(r), \ r > 0$$

we get

$$-\frac{1}{r^{N-1}}(r^{(N-1)}u_\epsilon')' = \widehat{H}(r) \text{ in } (0,\infty)$$

in the distribution sense. But since $a, h, u_\epsilon \in L^p_{loc}(0,\infty)$, $1 < p < \infty$ it follows that $\widehat{H} \in L^p_{loc}(0,\infty)$ and using the regularity theory we infer that $u_\epsilon \in W^{2,p}_{loc}(0,\infty)$ and

$$-\frac{1}{r^{N-1}}(r^{(N-1)}u_\epsilon')' = \widehat{H}(r) \text{ a.e. in } (0,\infty)\,.$$

By the maximum principle,

$$u_\epsilon > 0 \text{ in } (0,\infty)\,.$$

Since $u_\epsilon \in W^{2,p}_{loc}(\mathbf{R}^N\backslash\{0\})$ and

$$-\Delta u_\epsilon = -\frac{1}{r^{N-1}}(r^{(N-1)}u_\epsilon')'$$

we also have

$$-\Delta u_\epsilon + a(r)u_\epsilon = \frac{h(r)}{(u_\epsilon+\epsilon)^\gamma} \text{ a.e. in } \mathbf{R}^N\backslash\{0\}.$$

Now, let $\epsilon_n > 0$ such that $\epsilon_n \to 0$ and label $u_{\epsilon_n} \equiv u_n$. Following the proof of lemma 4 we have $u_n \geq u_1 > 0$. On the other hand we claim that

$$\|u_n\| \text{ is bounded.}$$

Indeed, as in (3.1) we have

$$\int \left(|\nabla u_n|^2 + a|u_n|^2\right) \leq C\ |h|_\theta \|u_n\|^{1-\gamma}$$

so that u_n is bounded in E_{rad}. Passing to subsequences we have

$$u_n \rightharpoonup u \text{ in } E_{rad}, \text{ and } u_n \to u \text{ a.e. in } \mathbf{R}^N.$$

On the other hand, if $v \in E_{rad}$ has compact support then, as in section 1, applying Lebesgue's Theorem to

$$\int (\nabla u_n \nabla v + a(r)u_n v) = \int \frac{h(r)}{(u_n+\epsilon_n)^\gamma}v,$$

gives

$$\int (\nabla u \nabla v + a(r)uv) = \int \frac{h(r)}{u^\gamma}v.$$

Now changing variables, making again $v \equiv r^{-(N-1)}\psi$ where $r > 0$ and $\psi \in C_0^\infty(0,\infty)$ and arguing as above we obtain $u \in W^{2,p}_{loc}(\mathbf{R}^N\setminus \{0\})$ and

$$-\frac{1}{r^{N-1}}(r^{(N-1)}u')' + a(r)u = \frac{h(r)}{u^\gamma} \text{ a.e. in } (0,\infty)$$

and in addition,

$$-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \text{ a.e. in } \mathbf{R}^N\setminus\{0\}.$$

So, if $\varphi \in C_0^\infty(\mathbf{R}^N\setminus \{0\})$ then

$$\int (\nabla u \nabla\varphi + a(r)u\varphi) = \int \frac{h(r)}{u^\gamma}\varphi$$

that is

$$-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \text{ in } \mathbf{R}^N\setminus\{0\}$$

in the distribution sense. Next we show that $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ and

$$\int (\nabla u \nabla\varphi + a(r)u\varphi) = \int \frac{h(r)}{u^\gamma}\varphi, \ \varphi \in C_0^\infty(\mathbf{R}^N).$$

Indeed, let $\eta \in C^\infty(\mathbf{R}^N)$ such that

$$\eta(x) = 0 \text{ for } |x| \le 1, \text{ and } \eta(x) = 1 \text{ for } |x| \ge 2$$

and let

$$\psi_\epsilon(x) \equiv \eta(\frac{x}{\epsilon}), \ \epsilon > 0.$$

If $\varphi \in C_0^\infty(\mathbf{R}^N)$ then $\psi_\epsilon\varphi \in C_0^\infty(\mathbf{R}^N\setminus \{0\})$ and from above

$$\int (\nabla u \nabla(\psi_\epsilon\varphi) + a(r)u(\psi_\epsilon\varphi)) = \int \frac{h(r)}{u^\gamma}(\psi_\epsilon\varphi)$$

so that

$$\int (\psi_\epsilon\nabla u\nabla\varphi + \varphi\nabla u\nabla\psi_\epsilon + a(r)u\psi_\epsilon\varphi) = \int \frac{h(r)}{u^\gamma}\psi_\epsilon\varphi.$$

Making $\epsilon \to 0$ and using Lebesgues's Theorem we infer that

$$\int \psi_\epsilon\nabla u\nabla\varphi \to \int \nabla u\nabla\varphi,$$

$$\int a(r)u\psi_\epsilon\varphi \to \int a(r)u\varphi$$

and

$$\int \frac{h(r)}{u^\gamma}\psi_\epsilon\varphi \to \int \frac{h(r)}{u^\gamma}\varphi.$$

Claim.

$$\int \varphi\nabla u\nabla\psi_\epsilon \to 0.$$

Assuming the Claim has been proved we have

$$\int (\nabla u \nabla\varphi + a(r)u\varphi) = \int \frac{h(r)}{u^\gamma}\varphi$$

and since $a, h \in L^{\infty}_{loc}$ we get by the regularity theory that $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for $1 < p < \infty$ and

$$-\Delta u + a(r)u = \frac{h(r)}{u^{\gamma}} \text{ a.e. in } \mathbf{R}^N$$

and if in addition $a, h \in C^{\alpha}_{loc}$ then $u \in C^{2,\alpha}_{loc}$ by the interior Schauder estimates.

Verification of the Claim.

Using Schwarz inequality we have

$$\begin{aligned} |\textstyle\int \varphi \nabla u \nabla \psi_\epsilon| &\leq |\varphi|_\infty \left(\int_{|x|\leq 2\epsilon} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{|x|\leq 2\epsilon} |\nabla \psi_\epsilon|^2\right)^{\frac{1}{2}} \\ &\leq |\varphi|_\infty |\nabla \eta|_2 \left(\int_{|x|\leq 2\epsilon} |\nabla u|^2\right)^{\frac{1}{2}} \epsilon^{\frac{N-2}{2}} \end{aligned}$$

where $N \geq 3$. Letting $\epsilon \to 0$ shows the Claim.

As for the uniqueness the argument in the proof of theorem 1 (Step 1) applies ending the proof of theorem 1 (in case of Step 2). The proof of theorem 1 is finished. ∎

Proof of Theorem 2.

In order to solve $(*)_0$ we consider the family of problems

$$(3.4) \qquad \begin{cases} -\Delta u + \frac{1}{k}u = h(|x|)u^{-\gamma} \;\; in \;\; \mathbf{R}^N \\ u > 0 \;\text{ in }\; \mathbf{R}^N. \end{cases}$$

where $k \geq 1$ is an integer. Making $a(x) \equiv \frac{1}{k}$ in theorem 1 (radial case), it follows that (3.4) has a solution $u_k \in H^1_{rad} \cap W^{2,p}_{loc}$, $1 < p < \infty$ satisfying

$$\int |\nabla u_k|^2 + \frac{1}{k}u_k^2 = \int h(r)u_k^{1-\gamma}.$$

Using both Hölder's inequality and the continuous embedding $\mathcal{D}^{1,2} \to L^{2^*}$ in the equality above we infer that

$$(3.5) \qquad \int |\nabla u_k|^2 \leq C_1 \text{ for some } C_1 > 0.$$

By a well known property of radial functions $u \in D^{1,2}$, namely

$$|u(x)| \leq \frac{C_2}{|x|^{\frac{N-2}{2}}} \|u\|_{D^{1,2}}, \; x \neq 0 \text{ for some } C_2 > 0$$

we get

$$(3.6) \qquad 0 \leq u_k(x) \leq \frac{C}{|x|^{\frac{N-2}{2}}}, \; x \neq 0 \text{ for some } C > 0.$$

We shall need the following result which says that the sequence u_k increases with k.

Lemma 5. *If $k < k'$ then $u_k \leq u_{k'}$, a.e. in $\mathbf{R}^N$.*

By the boundedness of u_k in $D^{1,2}$ and lemma 5 there is some radial function $u \in D^{1,2}$ such that

$$u_k \rightharpoonup u \text{ in } D^{1,2},\ u_k \to u \text{ a.e. in } \mathbf{R}^N$$

and

$$u_1 \le u_2 \le, ..., \le u_k \le, ..., \le u \text{ a.e. in } \mathbf{R}^N.$$

Now if $\varphi \in C_0^\infty(\mathbf{R}^N)$ then

$$\int \left(\nabla u_k \nabla \varphi + \frac{1}{k} u_k \varphi \right) = \int h u_k^{-\gamma} \varphi. \tag{3.7}$$

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain such that $supp(\varphi) \subset \Omega$. Then

$$|h u_k^{-\gamma} \varphi| \le h u_1^{-\gamma} |\varphi| \in L^p(\Omega),\ 1 \le p < \infty$$

and

$$\int h u_k^{-\gamma} \varphi \to \int h u^{-\gamma} \varphi.$$

On the other hand, using (3.6) we get

$$\frac{1}{k} \int u_k \varphi \to 0.$$

Passing to the limit in (3.7) gives

$$\int \nabla u \nabla \varphi = \int h u^{-\gamma} \varphi.$$

Since $0 < u_1 \le u$ and $u_1 \in W^{2,p}_{loc}(\mathbf{R}^N)$ it follows that $hu^{-\gamma} \in L^p_{loc}(\mathbf{R}^N)$ and by the regularity theory $u \in W^{2,p}_{loc}(\mathbf{R}^N)$. In addition $u \in C^{2,\alpha}_{loc}$ when $h \in C^\alpha_{loc}$. This proves Theorem 2. ∎

Proof of Lemma 5.

Letting $\omega = u_k - u_{k'}$ we have

$$\begin{aligned} \int |\nabla \omega^+|^2 + \tfrac{1}{k'} (\omega^+)^2 &\le \int \nabla \omega \nabla \omega^+ + \tfrac{1}{k'} \omega \omega^+ \\ &\le \int h \left(\tfrac{1}{u_k^\gamma} - \tfrac{1}{u_{k'}^\gamma} \right) \omega^+ \end{aligned}$$

showing that $\omega^+ = 0$ and so $\omega \le 0$, ending the proof of lemma 5. ∎

4. Appendix

Verification of (3.2).

Indeed,

$$a|\varphi^j - \varphi|^2 \le 4a\varphi^2 \in L^1 \text{ and } a|\varphi^j - \varphi|^2 \to 0 \text{ a.e. } in\ \mathbf{R}^N$$

so that by Lebesgue's theorem

$$\int a|\varphi^j - \varphi|^2 \to 0.$$

Now

$$\frac{\partial \varphi^j}{\partial x_i} = \frac{1}{j}\frac{\partial}{\partial x_i} M\left(\frac{x}{j}\right)\varphi + M\left(\frac{x}{j}\right)\frac{\partial \varphi}{\partial x_i}.$$

Hence

$$\begin{aligned} \int |\tfrac{\partial \varphi^j}{\partial x_i} - \tfrac{\partial \varphi}{\partial x_i}|^2 &= \int |\tfrac{1}{j}\tfrac{\partial}{\partial x_i} M\left(\tfrac{x}{j}\right)\varphi + M\left(\tfrac{x}{j}\right)\tfrac{\partial \varphi}{\partial x_i} - \tfrac{\partial \varphi}{\partial x_i}|^2 \\ &\leq C\int |\tfrac{1}{j^2}\tfrac{\partial}{\partial x_i} M\left(\tfrac{x}{j}\right)\varphi|^2 + \int |M\left(\tfrac{x}{j}\right)\tfrac{\partial \varphi}{\partial x_i} - \tfrac{\partial \varphi}{\partial x_i}|^2. \end{aligned}$$

Arguing as above we infer that

$$M\left(\frac{x}{j}\right)\frac{\partial \varphi}{\partial x_i} \to \frac{\partial \varphi}{\partial x_i} \ in\ L^2.$$

It remains to show that

$$\int |\frac{1}{j^2}\frac{\partial}{\partial x_i} M\left(\frac{x}{j}\right)\varphi|^2 \to 0.$$

At first we remark that

$$\begin{aligned} \int |\tfrac{1}{j^2}\tfrac{\partial}{\partial x_i} M\left(\tfrac{x}{j}\right)\varphi|^2 &= \int_{B_{2j}\setminus B_j} |\tfrac{1}{j^2}\tfrac{\partial}{\partial x_i} M\left(\tfrac{x}{j}\right)\varphi|^2 \\ &\leq \tfrac{C}{j^2}\int_{B_{2j}\setminus B_j}\varphi^2. \end{aligned}$$

Now using Hölder inequality with exponents $\frac{N}{N-2}$ and $\frac{N}{2}$ in the last integral we obtain

$$\begin{aligned} \int |\tfrac{1}{j^2}\tfrac{\partial}{\partial x_i} M\left(\tfrac{x}{j}\right)\varphi|^2 &\leq \tfrac{C}{j^2}\left(\int_{B_{2j}\setminus B_j} 1dx\right)^{\frac{2}{N}}\left(\int_{B_{2j}\setminus B_j}|\varphi|^{2^*}\right)^{\frac{1}{2^*}} \\ &\leq \tfrac{C}{j^2}\left(\int_{B_{2j}} 1dx\right)^{\frac{2}{N}}\left(\int_{B_{2j}^c}|\varphi|^{2^*}\right)^{\frac{1}{2^*}} \\ &\leq \tfrac{C\omega_N{}^{\frac{2}{N}}(2j)^2}{j^2}\left(\int_{B_{2j}^c}|\varphi|^{2^*}\right)^{\frac{1}{2^*}} \end{aligned}$$

where ω_N denotes the volume of the unit sphere of $\mathbf{R}^N$.

Next passing to the limit we get

$$\int |\frac{1}{j^2}\frac{\partial}{\partial x_i} M\left(\frac{x}{j}\right)\varphi|^2 \to 0.$$

This shows that $\varphi^j \to \varphi$ in E proving (3.2).

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