# ONE-SIDED RESONANCE FOR QUASILINEAR PROBLEMS WITH ASYMMETRIC NONLINEARITIES

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## 1. Introduction

We consider the quasilinear elliptic boundary value problem,

$$-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1} + f(x,u), \quad u \in W_0^{1,p}(\Omega), \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $1 < p < \infty$, $u^\pm = \max\{\pm u, 0\}$, $\alpha_\pm \in L^\infty(\Omega)$, and $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying a growth condition,

$$|f(x,t)| \leq qV(x)^{p-q}|t|^{q-1} + W(x)^{p-1}, \tag{1.2}$$

with $1 \leq q < p$ and $V, W \in L^p(\Omega)$. We assume that (1.1) is resonant from one side in the sense that either

$$\lambda_l \leq \alpha_\pm(x) \leq \lambda_{l+1} - \varepsilon \tag{1.3}$$

or

$$\lambda_l + \varepsilon \leq \alpha_\pm(x) \leq \lambda_{l+1}, \tag{1.4}$$

for two consecutive variational eigenvalues, $\lambda_l < \lambda_{l+1}$ of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, and some $\varepsilon > 0$ (see Section 2 for the definition of the variational spectrum).

The special case where $\alpha_+(x) = \alpha_-(x) \equiv \lambda_l$ and $q = 1$ was recently studied by Arcoya and Orsina [1], Bouchala and Drábek [3], and Drábek and Robinson [8] (see also Cuesta et al. [6] and Dancer and Perera [7]). In the present paper, we prove a single existence theorem for the general case that includes all their results and much more.


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Denote by $N$ the set of nontrivial solutions of the asymptotic problem

$$-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1}, \quad u \in W_0^{1,p}(\Omega), \tag{1.5}$$

and set

$$F(x,t) := \int_0^t f(x,s)\,ds, \qquad H(x,t) := pF(x,t) - tf(x,t). \tag{1.6}$$

Our main result is the following theorem.

THEOREM 1.1. *Problem (1.1) has a solution in the following cases:*

(i) *equation (1.3) holds and* $\int_\Omega H(x,u_j) \to +\infty$,

(ii) *equation (1.4) holds and* $\int_\Omega H(x,u_j) \to -\infty$ *for every sequence* $(u_j)$ *in* $W_0^{1,p}(\Omega)$ *such that* $\|u_j\| \to \infty$ *and* $u_j/\|u_j\|$ *converges to some element of* $N$. *In particular, (1.1) is solvable when (1.3) or (1.4) holds and* $N$ *is empty.*

As is usually the case in resonance problems, the main difficulty here is the lack of compactness of the associated variational functional, which we will overcome by constructing a sequence of approximating nonresonance problems, finding approximate solutions for them using linking and min-max type arguments, and passing to the limit (see Rabinowitz [10] for standard details of the variational theory). But first we give some corollaries and deduce the results of [1, 3, 8]. In what follows, $(u_j)$ is as in the theorem, that is, $\rho_j := \|u_j\| \to \infty$ and $v_j := u_j/\rho_j \to v \in N$.

First, we give simple pointwise assumptions on $H$ that imply the limits in the theorem.

COROLLARY 1.2. *Problem (1.1) has a solution in the following cases:*

(i) *equation (1.3) holds,* $H(x,t) \to +\infty$ *a.e. as* $|t| \to \infty$, *and* $H(x,t) \geq -C(x)$,

(ii) *equation (1.4) holds,* $H(x,t) \to -\infty$ *a.e. as* $|t| \to \infty$, *and* $H(x,t) \leq C(x)$ *for some* $C \in L^1(\Omega)$.

Note that this corollary makes no reference to $N$.

*Proof.* If (i) holds, then $H(x,u_j(x)) = H(x,\rho_j v_j(x)) \to +\infty$ for a.e. $x$ such that $v(x) \neq 0$ and $H(x,u_j(x)) \geq -C(x)$, so

$$\int_\Omega H(x,u_j) \geq \int_{v\neq 0} H(x,u_j) - \int_{v=0} C(x) \longrightarrow +\infty \tag{1.7}$$

by Fatou's lemma. Similarly, $\int_\Omega H(x,u_j) \to -\infty$ if (ii) holds. □

Note that the above argument goes through as long as the limits in (i) and (ii) hold on subsets of $\{x \in \Omega : v(x) \neq 0\}$ with positive measure. Now, taking $w = v^\pm$ in

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w = \int_\Omega \left[\alpha_+(x)(v^+)^{p-1} - \alpha_-(x)(v^-)^{p-1}\right] w \tag{1.8}$$

gives

$$\begin{aligned} \|v^{\pm}\|^{p} &= \int_{\Omega_{\pm}} \alpha_{\pm}(x)(v^{\pm})^{p} \leq \|\alpha_{\pm}\|_{\infty} \|v^{\pm}\|_{p^*}^{p} \mu(\Omega_{\pm})^{p/n} \\ &\leq \|\alpha_{\pm}\|_{\infty} S^{-1} \|v^{\pm}\|^{p} \mu(\Omega_{\pm})^{p/n}, \end{aligned} \tag{1.9}$$

where $\Omega_{\pm} = \{x \in \Omega : v(x) \gtrless 0\}$, $p^* = np/(n-p)$ is the critical Sobolev exponent, $S$ is the best constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, and $\mu$ is the Lebesgue measure in $\mathbb{R}^n$. So

$$\mu(\Omega_{\pm}) \geq \left(S\|\alpha_{\pm}\|_{\infty}^{-1}\right)^{n/p}, \tag{1.10}$$

and hence

$$\mu(\{x \in \Omega : v(x) = 0\}) \leq \mu(\Omega) - S^{n/p}\left(\|\alpha_{+}\|_{\infty}^{-n/p} + \|\alpha_{-}\|_{\infty}^{-n/p}\right). \tag{1.11}$$

Thus, we have the following corollary.

Corollary 1.3. *Problem (1.1) has a solution in the following cases:*

(i) *equation (1.3) holds,* $H(x,t) \to +\infty$ *in* $\Omega'$ *as* $|t| \to \infty$, *and* $H(x,t) \geq -C(x)$,

(ii) *equation (1.4) holds,* $H(x,t) \to -\infty$ *in* $\Omega'$ *as* $|t| \to \infty$, *and* $H(x,t) \leq C(x)$ *for some* $\Omega' \subset \Omega$ *with* $\mu(\Omega') > \mu(\Omega) - S^{n/p}(\|\alpha_{+}\|_{\infty}^{-n/p} + \|\alpha_{-}\|_{\infty}^{-n/p})$ *and* $C \in L^1(\Omega)$.

Similar conditions on $H$ were recently used by Furtado and Silva [9] in the semilinear case $p = 2$.

Next, note that

$$\begin{aligned} &\underline{H}_{+}(x)(v^{+}(x))^{q} + \underline{H}_{-}(x)(v^{-}(x))^{q} \\ &\quad \leq \liminf \frac{H(x,u_j(x))}{\rho_j^{q}} \leq \limsup \frac{H(x,u_j(x))}{\rho_j^{q}} \\ &\quad \leq \overline{H}_{+}(x)(v^{+}(x))^{q} + \overline{H}_{-}(x)(v^{-}(x))^{q}, \end{aligned} \tag{1.12}$$

where

$$\underline{H}_{\pm}(x) = \liminf_{t \to \pm\infty} \frac{H(x,t)}{|t|^{q}}, \qquad \overline{H}_{\pm}(x) = \limsup_{t \to \pm\infty} \frac{H(x,t)}{|t|^{q}}. \tag{1.13}$$

Moreover,

$$\frac{|H(x,u_j(x))|}{\rho_j^{q}} \leq (p+q)V(x)^{p-q}|v_j(x)|^{q} + \frac{(p+1)W(x)^{p-1}|v_j(x)|}{\rho_j^{q-1}} \tag{1.14}$$

by (1.2), so it follows that

$$\int_\Omega \underline{H}_+(v^+)^q + \underline{H}_-(v^-)^q \le \liminf \frac{\int_\Omega H(x,u_j)}{\rho_j^q} \le \limsup \frac{\int_\Omega H(x,u_j)}{\rho_j^q} \le \int_\Omega \overline{H}_+(v^+)^q + \overline{H}_-(v^-)^q. \tag{1.15}$$

Thus we have the following corollary.

Corollary 1.4. *Problem (1.1) has a solution in the following cases:*

(i) *equation (1.3) holds and* $\int_\Omega \underline{\underline{H}}_+(v^+)^q + \underline{\underline{H}}_-(v^-)^q > 0$ *for all* $v \in N$,

(ii) *equation (1.4) holds and* $\int_\Omega \overline{\overline{H}}_+(v^+)^q + \overline{\overline{H}}_-(v^-)^q < 0$ *for all* $v \in N$.

When $\alpha_+(x) = \alpha_-(x) \equiv \lambda_1$ and $q = 1$ this reduces to the result of Bouchala and Drábek [3].

Finally, we note that if

$$\frac{tf(x,t)}{|t|^q} \longrightarrow f_\pm(x) \quad \text{a.e. as } t \longrightarrow \pm\infty, \tag{1.16}$$

then

$$\frac{F(x,t)}{|t|^q} = \frac{1}{|t|^q}\int_0^t \left[\frac{sf(x,s)}{|s|^q} - f_\pm(x)\right]|s|^{q-2}s\,ds + \frac{f_\pm(x)}{q} \longrightarrow \frac{f_\pm(x)}{q} \tag{1.17}$$

and hence

$$\frac{H(x,t)}{|t|^q} \longrightarrow \left(\frac{p}{q} - 1\right) f_\pm(x), \tag{1.18}$$

so Corollary 1.4 implies the following corollary.

Corollary 1.5. *Problem (1.1) has a solution in the following cases:*

(i) *equation (1.3) holds and* $\int_\Omega f_+(v^+)^q + f_-(v^-)^q > 0$ *for all* $v \in N$,

(ii) *equation (1.4) holds and* $\int_\Omega f_+(v^+)^q + f_-(v^-)^q < 0$ *for all* $v \in N$.

This was proved in Arcoya and Orsina [1] and Drábek and Robinson [8] for the special case $\alpha_+(x) = \alpha_-(x) \equiv \lambda_l$ and $q = 1$.

## 2. Proof of Theorem 1.1

First we recall some facts about the variational spectrum of the $p$-Laplacian. It is easily seen from the Lagrange multiplier rule that the eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ correspond to the critical values of

$$J(u) = \int_\Omega |\nabla u|^p, \quad u \in S := \{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\}. \tag{2.1}$$

Moreover, $J$ satisfies the Palais-Smale condition (cf. Drábek and Robinson [8]). Thus, we can define an unbounded sequence of min-max eigenvalues by

$$\lambda_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} J(u), \quad l \in \mathbb{N}, \tag{2.2}$$

where

$$\mathcal{F}_l = \{A \subset S : \exists \text{ a continuous odd surjection } h : S^{l-1} \longrightarrow A\} \tag{2.3}$$

and $S^{l-1}$ is the unit sphere in $\mathbb{R}^l$.

Lemma 2.1. *$\lambda_l$ is an eigenvalue of $-\Delta_p$ and $\lambda_l \to \infty$.*

*Proof.* If $\lambda_l$ is a regular value of $J$, then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta : S \to S$ such that $\eta(J^{\lambda_l+\varepsilon}) \subset J^{\lambda_l-\varepsilon}$ by [2, Theorem 2.5] (the standard first deformation lemma is not sufficient because the manifold $S$ is not of class $C^{1,1}$ when $p < 2$). But then taking $A \in \mathcal{F}_l$ with $\max J(A) \le \lambda_l + \varepsilon$ and setting $\tilde{A} = \eta(A)$, we get a set in $\mathcal{F}_l$ for which $\max J(\tilde{A}) \le \lambda_l - \varepsilon$, contradicting the definition of $\lambda_l$. Finally, denoting by $\mu_l \to \infty$ the usual Ljusternik-Schnirelmann eigenvalues, we have $\lambda_l \ge \mu_l$ since the genus of each $A$ in $\mathcal{F}_l$ is $l$, so $\lambda_l \to \infty$. □

It is not known whether this is a complete list of eigenvalues when $p \neq 2$ and $n \ge 2$. However, the variational structure provided by this portion of the spectrum is sufficient to show that the associated functional admits a linking geometry in the nonresonant case. We only consider (i) as the proof for (ii) is similar. Let

$$\alpha_\pm^j(x) = \begin{cases} \alpha_\pm(x), & \text{if } \alpha_\pm(x) \ge \lambda_l + \dfrac{1}{j}, \\ \lambda_l + \dfrac{1}{j}, & \text{if } \alpha_\pm(x) < \lambda_l + \dfrac{1}{j}, \end{cases} \tag{2.4}$$

so that

$$\lambda_l + \frac{1}{j} \le \alpha_\pm^j(x) \le \lambda_{l+1} - \varepsilon, \qquad |\alpha_\pm^j(x) - \alpha_\pm(x)| \le \frac{1}{j}, \tag{2.5}$$

and let

$$\Phi_j(u) = \int_\Omega |\nabla u|^p - \alpha_+^j(x)(u^+)^p - \alpha_-^j(x)(u^-)^p - pF(x,u), \quad u \in W_0^{1,p}(\Omega). \tag{2.6}$$

First, we show that there is a $u_j \in W_0^{1,p}(\Omega)$ such that

$$\|u_j\| \|\Phi_j'(u_j)\| \longrightarrow 0, \quad \inf \Phi_j(u_j) > -\infty. \tag{2.7}$$

By (2.2), there is an $A \in \mathcal{F}_l$ such that

$$J(u) \le \lambda_l + \frac{1}{2j}, \quad u \in A. \tag{2.8}$$

For $u \in A$ and $R > 0$,

$$\begin{aligned}\Phi_j(Ru) &= R^p \left[ J(u) - \int_\Omega \alpha_+^j(x)(u^+)^p + \alpha_-^j(x)(u^-)^p \right] - \int_\Omega pF(x,Ru) \\ &\le -\frac{R^p}{2j} + p\left( \|V\|_p^{p-q} R^q + \|W\|_p^{p-1} R \right)\end{aligned} \tag{2.9}$$

by (1.2), (2.5), and (2.8), so

$$\max_{u\in A} \Phi_j(Ru) \longrightarrow -\infty \quad \text{as } R \longrightarrow \infty. \tag{2.10}$$

Next, let

$$\mathcal{S} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \ge \lambda_{l+1} \int_\Omega |u|^p \right\}. \tag{2.11}$$

For $u \in \mathcal{S}$,

$$\Phi_j(u) \ge \varepsilon \|u\|_p^p - p\left( \|V\|_p^{p-q} \|u\|_p^q + \|W\|_p^{p-1} \|u\|_p \right), \tag{2.12}$$

so

$$\inf_{u\in\mathcal{S}} \Phi_j(u) \ge C := \min_{r\ge 0} \left[ \varepsilon r^p - p\left( \|V\|_p^{p-q} r^q + \|W\|_p^{p-1} r \right) \right] > -\infty. \tag{2.13}$$

Now use (2.10) to fix $R > 0$ so large that

$$\max \Phi_j(RA) < C, \tag{2.14}$$

where $RA = \{Ru : u \in A\}$.

Since $A \in \mathcal{F}_l$, there is a continuous odd surjection $h : S^{l-1} \to A$. Let

$$\Gamma = \left\{ \varphi \in C\left(D^l, W_0^{1,p}(\Omega)\right) : \varphi|_{S^{l-1}} = Rh \right\}, \tag{2.15}$$

where $D^l$ is the unit disk in $\mathbb{R}^l$ with boundary $S^{l-1}$. We claim that $RA$ links $\mathcal{S}$ with respect to $\Gamma$, that is,

$$\varphi(D^l) \cap \mathcal{S} \ne \emptyset \quad \forall \varphi \in \Gamma. \tag{2.16}$$

To see this, first note that the proof is done if $0 \in \varphi(D^l)$. Otherwise, denoting by $\pi$ the radial projection onto $S$, $\tilde{A} := \pi(\varphi(D^l)) \cup -\pi(\varphi(D^l)) \in \mathcal{F}_{l+1}$, and hence

$$\max_{u\in\pi(\varphi(D^l))} J(u) = \max_{u\in\tilde{A}} J(u) \ge \lambda_{l+1}, \tag{2.17}$$

so $\pi(\varphi(D^l)) \cap \mathcal{S} \ne \emptyset$, which implies that $\varphi(D^l) \cap \mathcal{S} \ne \emptyset$.

Now it follows from a deformation argument of Cerami [5] that there is a $u_j$ such that

$$\|u_j\| \|\Phi_j'(u_j)\| \longrightarrow 0, \qquad |\Phi_j(u_j) - c_j| \longrightarrow 0, \tag{2.18}$$

where

$$c_j := \inf_{\varphi\in\Gamma} \max_{u\in\varphi(D^l)} \Phi_j(u) \geq C, \tag{2.19}$$

from which (2.7) follows.

We complete the proof by showing that a subsequence of $(u_j)$ converges to a solution of (1.1). It is easy to see that this is the case if $(u_j)$ is bounded, so suppose that $\rho_j := \|u_j\| \to \infty$. Setting $v_j := u_j/\rho_j$ and passing to a subsequence, we may assume that $v_j \to v$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in $\Omega$. Then

$$\begin{aligned}&\int_\Omega |\nabla v_j|^{p-2}\nabla v_j\cdot\nabla(v_j-v)\\ &= \frac{(\Phi_j'(u_j),v_j-v)}{p\rho_j^{p-1}} + \int_\Omega\left[\alpha_+^j(x)(v_j^+)^{p-1} - \alpha_-^j(x)(v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}}\right](v_j-v) \longrightarrow 0,\end{aligned} \tag{2.20}$$

and we deduce that $v_j \to v$ strongly in $W_0^{1,p}(\Omega)$ (cf. Browder [4]). In particular, $\|v\| = 1$, so $v \neq 0$. Moreover, for each $w \in W_0^{1,p}(\Omega)$, passing to the limit in

$$\begin{aligned}\frac{(\Phi_j'(u_j),w)}{p\rho_j^{p-1}} &= \int_\Omega |\nabla v_j|^{p-2}\nabla v_j\cdot\nabla w\\ &\quad - \left[\alpha_+^j(x)(v_j^+)^{p-1} - \alpha_-^j(x)(v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}}\right]w\end{aligned} \tag{2.21}$$

gives that

$$\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla w - \left[\alpha_+(x)(v^+)^{p-1} - \alpha_-(x)(v^-)^{p-1}\right]w = 0, \tag{2.22}$$

so $v \in N$. Thus,

$$\frac{(\Phi_j'(u_j),u_j)}{p} - \Phi_j(u_j) = \int_\Omega H(x,u_j) \longrightarrow +\infty, \tag{2.23}$$

contradicting (2.7).

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